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Asymptotic Stability of Singular Solution for Camassa-Holm Equation

Yuetian Gao

School of Mathematics and Computer Science, Yunnan Minzu University, Kunming, China
Email: sweetgyt@126.com

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Abstract

The aim of this paper is to study singular dynamics of solutions of Camassa-Holm equation. Based on the semigroup theory of linear operators and Banach contraction mapping principle, we prove the asymptotic stability of the explicit singular solution of Camassa-Holm equation.

Keywords

Asymptotic Stability, Camassa-Holm Equation, Explicit Solution, Semigroup Theory, Banach Contraction Mapping Principle

1. Introduction and Main Results

1.1. Introduction

Consider the well-known Camassa-Holm equation as follows (see [1]):

$$m_t + c_0 u_x + u m_x + 2 m u_x = 0, \tag{1.1}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $u = u(t,x)$ is the velocity of fluid, m is the momentum given by

$$m = m(t,x) = u(t,x) - \alpha^2 u_{xx}(t,x),$$

$c_0 \in \mathbb{R}$ is the critical speed and $\alpha \in \mathbb{R}$ relates to the length scale. Thus,

$$u_t - \alpha^2 u_{txx} + c_0 u_x + 3 u u_x = \alpha^2 \left(2 u_x u_{xx} + u u_{xxx} \right). \tag{1.2}$$

Given the initial value as $u(0,x) = u_0(x)$ for $x \in \mathbb{R}$.

The Camassa-Holm equation describes unidirectional propagation of surface water waves in shallow water area. For the global well-posedness and stability of solutions, we recommend that the reader refers to [2]-[9], etc. For the wave breaking analysis, we refer the reader to [6] [10]-[15], etc. When $c_0 = 0$ and $\alpha = 1$, the Camassa-Holm equation becomes to the classical Camassa-Holm equation, which admits a bi-Hamiltonian structure [1] [5]. Moreover, the explicit peakon solution and its stability have been established in [12] [16] [17] [18] [19], etc.

Since it is rare to see the explicit stable blowup solutions of Camassa-Holm equation, in this paper, we study the stability of the explicit solution of (1.2) as follows (see [20]):

$$\bar{u}(t,x) = -\frac{1}{3}\left(c_0 + \frac{x}{T-t} + \frac{1}{T-t}\right), \tag{1.3}$$

where $T>0$ is a constant.

1.2. Main Results

Now, we state our main result of this paper.

Theorem 1.1. *Let* $s>2$ *be an integer and* δ *is a sufficiently small constant. Then the explicit solution* (1.3) *of the Camassa-Holm Equation* (1.2) *is asymptotic stable, i.e., if the initial data* $u_0(x)$ *satisfies*

$$\left\| u_0(x) + \frac{1}{3}\left(c_0 + \frac{x}{T} + \frac{1}{T}\right)\right\|_{\mathbb{H}^{s+1}(\mathbb{R})} \le \delta,$$

then there is a solution $u(t,x)$ of (1.2) satisfying

$$\left\| u(t,x) - \bar{u}(t,x)\right\|_{\mathbb{H}^{s}(\mathbb{R})} \le \frac{\tilde{C}(T-t)}{\alpha^2\left(1 + C\ln(T-t)\right)}, \quad (t,x)\in(0,T)\times\mathbb{R},$$

where C and $\tilde{C}$ are positive constants that depend on *s*.

1.3. Notations

Denote $\mathbb{L}^2(\mathbb{R}) = \mathbb{L}^2$ and $\mathbb{H}^s(\mathbb{R}) = \mathbb{H}^s$ by the Lebesgue spaces and Sobolev spaces with norms $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$, respectively. $*$ denotes the convolution. $[A,B]$ stands for the commutator.

2. Proof of Theorem 1.1

Let

$$u(t,x) = v(t,x) + \bar{u}(t,x), \tag{2.1}$$

be the solution of (1.2), where $\bar{u}(t,x) = -\frac{1}{3}\left(c_0 + \frac{x}{T-t} + \frac{1}{T-t}\right)$ is the explicit solution. Substituting (2.1) into (1.2), we get

$$\begin{aligned}
&v_t - \alpha^2 v_{txx} + \left[\frac{\alpha^2}{3}\left(c_0 + \frac{x}{T-t} + \frac{1}{T-t}\right)\right] v_{xxx} + \frac{2\alpha^2}{3(T-t)} v_{xx} \\
&-\left(\frac{x}{T-t} + \frac{1}{T-t}\right) v_x - \frac{1}{T-t} v + 3 v v_x \\
&= \alpha^2\left(2 v_x v_{xx} + v v_{xxx}\right), \quad \forall (t,x)\in(0,T)\times\mathbb{R}
\end{aligned} \tag{2.2}$$

with the initial condition $v(0,x) = v_0(x) = u_0(x) + \frac{1}{3}\left(c_0 + \frac{x}{T} + \frac{1}{T}\right)$ for $x\in\mathbb{R}$.

For the singular coefficients in (2.2), let $v(t,x)=\psi(\tau,\rho)$ by $\tau=-\ln(T-t)$ and $\rho=\frac{x}{T-t}$, then (2.2) becomes to

$$\begin{aligned}&\psi_\tau+\rho\psi_\rho-\alpha^2e^{2\tau}\left(\psi_{\tau\rho\rho}+2\psi_{\rho\rho}+\rho\psi_{\rho\rho}\right)+e^{2\tau}\left[\frac{\alpha^2}{3}\left(c_0+\rho+e^\tau\right)\right]\psi_{\rho\rho\rho}\\&+\frac{2\alpha^2}{3}e^{2\tau}\psi_{\rho\rho}-\left(\rho+e^\tau\right)\psi_\rho-\psi+3\psi\psi_\rho=\alpha^2e^{2\tau}\left(2\psi_\rho\psi_{\rho\rho}+\psi\psi_{\rho\rho\rho}\right).\end{aligned}\tag{2.3}$$

Let $\kappa=e^{-\tau}\rho$ and $\bar v(\tau,\kappa)=e^{-\tau}\psi(\tau,\rho)$. Then (2.3) becomes to

$$\begin{aligned}&\bar v_\tau-\alpha^2\bar v_{\tau\kappa\kappa}-\frac{\alpha^2}{3}\bar v_{\kappa\kappa}+e^{-\tau}\left[\gamma+\frac{\alpha^2}{3}\left(c_0+\kappa e^\tau+e^\tau\right)\right]\bar v_{\kappa\kappa\kappa}\\&-(\kappa+1)\bar v_\kappa+3\bar v\bar v_\kappa=\alpha^2\left(2\bar v_\kappa\bar v_{\kappa\kappa}+\bar v\bar v_{\kappa\kappa\kappa}\right).\end{aligned}\tag{2.4}$$

Let the operator $\mathcal{A}=\left(1-\alpha^2\partial_{\kappa\kappa}\right)^{\frac12}$. Since $1-\alpha^2\partial_{\kappa\kappa}$ admits a fundamental solution $\wp(x)=\frac{1}{2\alpha}e^{-\left|\frac{\kappa}{\alpha}\right|}$, we have $\mathcal{A}^{-2}\bar v=\wp(\kappa)*\bar v$ for all $\bar v\in\mathbb{L}^2$. Let $w(\tau,\kappa)=\bar v(\tau,\kappa)-\alpha^2\bar v_{\kappa\kappa}(\tau,\kappa)$, then $\bar v(\tau,\kappa)=\wp*w$, where $\kappa\in\mathbb{R}$. Furthermore, we have $(\rho*w)_{\kappa\kappa}=\alpha^{-2}(\rho*w-w)$, $\bar v_\kappa=(\wp*w)_\kappa$ and $\bar v_{\kappa\kappa\kappa}=\alpha^{-2}\left((\wp*w)_\kappa-w_\kappa\right)$. Then (2.3) can be rewritten as

$$\begin{aligned}&w_\tau+\frac13w-e^{-\tau}\left[\frac13\left(c_0+e^\tau\kappa+e^\tau\right)\right]w_\kappa-\frac13\wp*w\\&+\left\{e^{-\tau}\left[\frac13\left(c_0+e^\tau\kappa+e^\tau\right)\right]-(\kappa+1)\right\}(\wp*w)_\kappa+3(\wp*w)(\wp*w)_\kappa\\&=2(\wp*w)_\kappa(\wp*w-w)+(\wp*w)\left[(\wp*w)_\kappa-w_\kappa\right]\end{aligned}\tag{2.5}$$

with the initial data

$$w_0(\kappa)=u_0(x)-\alpha^2u_0''(x)+\frac13\left(\frac{x}{T}+\frac1T+c_0\right),\tag{2.6}$$

and the boundary condition

$$\lim_{|\kappa|\to+\infty}w(\tau,\kappa)=0,\quad \lim_{|\kappa|\to+\infty}w_\kappa(\tau,\kappa)=0.\tag{2.7}$$

Before making a priori estimate of the solutions to problems (2.5)-(2.7). We recall the following commutator estimate.

Lemma 2.1 ([21]). *Let* $s>0$. *Then it holds*

$$\left\|\left[\mathcal{A}^s,u\right]v\right\|_{\mathbb{L}^2}\le C\left(\left\|\partial_xu\right\|_{\mathbb{L}^\infty}\left\|\mathcal{A}^{s-1}v\right\|_{\mathbb{L}^2}+\left\|\mathcal{A}^su\right\|_{\mathbb{L}^2}\left\|v\right\|_{\mathbb{L}^\infty}\right),\tag{2.8}$$

where C is a positive constant that depends on *s*.

Now, we derive a priori estimate of the solutions for (2.5).

Lemma 2.2. *Let* $s>2$ *and* $\alpha\ne0$. *Assume that w be a solution of* (2.5), *then*

$$\|w\|_{\mathbb{H}^s}\le\frac{1}{\|w_0\|_{\mathbb{H}^s}^{-1}-C\tau},\tag{2.9}$$

where C is a positive constant depending upon *s*.

Proof. Applying $\mathcal{A}^s$ to both sides of (2.5) and taking the $\mathbb{L}^2$-inner product with $\mathcal{A}^s w$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w\|_{\mathbb{H}^s}^2+\frac{1}{3}\|w\|_{\mathbb{H}^s}^2-\frac{1}{3}\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s(\wp*w)\mathrm{d}\kappa\\&-\mathrm{e}^{-\tau}\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)w_\kappa\right]\mathrm{d}\kappa\\&+\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left\{\left[\mathrm{e}^{-\tau}\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)-(\kappa+1)\right](\wp*w)_\kappa\right\}\mathrm{d}\kappa\\&+3\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[(\wp*w)(\wp*w)_\kappa\right]\mathrm{d}\kappa\\&=2\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[(\wp*w)_\kappa(\wp*w-w)\right]\mathrm{d}\kappa\\&+\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[(\wp*w)\left((\wp*w)_\kappa-w_\kappa\right)\right]\mathrm{d}\kappa.\end{aligned}\tag{2.10}$$

Next, we estimate each of terms in (2.10).

$$-\frac{1}{3}\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s(\wp*w)\mathrm{d}\kappa=-\frac{1}{3}\|w\|_{\mathbb{H}^{s-1}}^2,\tag{2.11}$$

$$\begin{aligned}&-\mathrm{e}^{-\tau}\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)w_\kappa\right]\mathrm{d}\kappa\\&=\mathrm{e}^{-\tau}\int_{\mathbb{R}}\left[\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)\mathcal{A}^{2s}w\right]_\kappa w\mathrm{d}\kappa\\&=\frac{1}{3}\int_{\mathbb{R}}\mathcal{A}^{2s}w\cdot w\mathrm{d}\kappa-\frac{1}{2}\times\frac{1}{3}\int_{\mathbb{R}}\left(\mathcal{A}^s w\right)^2\mathrm{d}\kappa=\frac{1}{6}\|w\|_{\mathbb{H}^s}^2,\end{aligned}\tag{2.12}$$

$$\begin{aligned}&\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left\{\left[\mathrm{e}^{-\tau}\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)-(\kappa+1)\right](\wp*w)_\kappa\right\}\mathrm{d}\kappa\\&=-\int_{\mathbb{R}}\left\{\mathcal{A}^{2s}w\left[\mathrm{e}^{-\tau}\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)-(\kappa+1)\right]\right\}_\kappa(\wp*w)\mathrm{d}\kappa\\&=-\int_{\mathbb{R}}\mathcal{A}^{2s}w_\kappa\left[\mathrm{e}^{-\tau}\left(\frac{1}{3}\left(c_0+\mathrm{e}^{\tau}\kappa+\mathrm{e}^{\tau}\right)\right)-(\kappa+1)\right](\wp*w)\mathrm{d}\kappa\\&\quad-\left(\frac{1}{3}-1\right)\int_{\mathbb{R}}\mathcal{A}^{2s}w(\wp*w)\mathrm{d}\kappa\\&=\frac{1}{2}\times\left(\frac{1}{3}-1\right)\int_{\mathbb{R}}\left(\mathcal{A}^{s-1}w\right)^2\mathrm{d}\kappa+\frac{2}{3}\int_{\mathbb{R}}\mathcal{A}^{2s}w(\wp*w)\mathrm{d}\kappa=\frac{1}{3}\|w\|_{\mathbb{H}^{s-1}}^2,\end{aligned}\tag{2.13}$$

$$\begin{aligned}&3\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left((\wp*w)(\wp*w)_\kappa\right)\mathrm{d}\kappa\\&=-\frac{3}{2}\int_{\mathbb{R}}w_\kappa\mathcal{A}^{2s}\left((\wp*w)^2\right)\mathrm{d}\kappa\le\frac{3}{2}\|w_\kappa\|_{\mathbb{L}^\infty}\|w\|_{\mathbb{H}^{s-1}}^2\le\frac{3}{2}\|w\|_{\mathbb{H}^{s-1}}^3,\end{aligned}\tag{2.14}$$

In addition, using (2.8), we have

$$\begin{aligned}&2\left|\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left((\wp*w)_\kappa(\wp*w-w)\right)\mathrm{d}\kappa\right|\\&=2\left|\int_{\mathbb{R}}\left[\mathcal{A}^s,(\wp*w-w)\right](\wp*w)_\kappa\mathcal{A}^s w\mathrm{d}\kappa\right|\\&\quad+2\left|\int_{\mathbb{R}}(\wp*w-w)\mathcal{A}^s(\wp*w)_\kappa\mathcal{A}^s w\mathrm{d}\kappa\right|\\&\le C\left(\|\wp*w-w\|_{\mathbb{L}^\infty}\left\|\mathcal{A}^{s-1}(\wp*w)_\kappa\right\|_{\mathbb{L}^2}+\left\|\mathcal{A}^s(\wp*w-w)\right\|_{\mathbb{L}^2}\left\|(\wp*w)_\kappa\right\|_{\mathbb{L}^\infty}\right)\|w\|_{\mathbb{H}^s}\\&\quad+2\left(\left\|(\wp*w-w)\right\|_{\mathbb{L}^\infty}+\left\|(\wp*w-w)_\kappa\right\|_{\mathbb{L}^\infty}\right)\|w\|_{\mathbb{H}^2}^2\\&\le C\|w\|_{\mathbb{H}^s}^3,\end{aligned}\tag{2.15}$$

similarly,

$$\left|\int_{\mathbb{R}}\mathcal{A}^s w\mathcal{A}^s\left[(\wp * w)\left((\wp * w)_\kappa - w_\kappa\right)\right]\mathrm{d}\kappa\right| \le C\|w\|_{\mathbb{H}^s}^3, \tag{2.16}$$

where C is a positive constant depending upon s.

Substituting (2.11)-(2.16) into (2.10), we get $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w\|_{\mathbb{H}^s}^2 \le C\|w\|_{\mathbb{H}^s}^3$, and then $-\frac{\mathrm{d}}{\mathrm{d}\tau}\|w\|_{\mathbb{H}^s}^{-1} \le C$. Integrating this inequality above with respect to τ from 0 to τ, we get

$$\|w\|_{\mathbb{H}^s} \le \frac{1}{\|w_0\|_{\mathbb{H}^s}^{-1} - C\tau}. \tag{2.17}$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Now, we study the well-posedness for (2.5)-(2.7). Define the linear operator L as

$$\begin{aligned} L[w] = &-\frac{1}{3}w + \frac{1}{3}\wp * w + \mathrm{e}^{-\tau}\left[\frac{1}{3}\left(c_0 + \mathrm{e}^\tau\kappa + \mathrm{e}^\tau\right)\right]w_\kappa \\ &-\left[\mathrm{e}^{-\tau}\left(\frac{1}{3}\left(c_0 + \mathrm{e}^\tau\kappa + \mathrm{e}^\tau\right)\right) - (\kappa + 1)\right](\wp * w)_\kappa, \end{aligned} \tag{2.18}$$

then (2.5) becomes to

$$w_t = L[w] + f(w), \tag{2.19}$$

where f is the nonlinear terms:

$$\begin{aligned} f(w) = &-3(\wp * w)(\wp * w)_\kappa + 2(\wp * w)(\wp * w - w) \\ &+ (\wp * w)\left[(\wp * w)_\kappa - w_\kappa\right]. \end{aligned} \tag{2.20}$$

Lemma 2.3. *Let* $s > 2$. *Then*

- $L[w] \in \mathbb{H}^s$ for $\forall w \in \mathcal{D}(L)$.
- L is a closed and densely defined linear operator in $\mathbb{H}^s$.

Proof. It is a direct verification by the definition of L. □

Lemma 2.4. *Let* $s > 2$. *Then* L *is a dissipative operator in* $\mathbb{H}^s$, *i.e.,* $(L[w], w)_s \le 0$.

Proof. Using (2.11)-(2.14), a direct calculation shows that

$$\begin{aligned} \int_{\mathbb{R}}\left(\mathcal{A}^s L[w]\right)\mathcal{A}^s w\mathrm{d}\kappa &= -\frac{1}{3}\|w\|_{\mathbb{H}^s}^2 + \frac{1}{3}\|w\|_{\mathbb{H}^{s-1}}^2 - \frac{1}{6}\|w\|_{\mathbb{H}^s}^2 - \frac{1}{3}\|w\|_{\mathbb{H}^{s-1}}^2 \\ &= -\frac{1}{2}\|w\|_{\mathbb{H}^s}^2 \le 0. \end{aligned} \tag{2.21}$$

This completes the proof. □

Lemma 2.5 (Young inequality with ε, see [22]). *Let* $a, b > 0$ *and* $\varepsilon > 0$. *If* $p, q \in (1, \infty)$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$ab \le \varepsilon a^p + C(\varepsilon)b^q, \tag{2.22}$$

where $C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}}q^{-1}$.

Lemma 2.6. *Let* $s>2$ *. Then the operator L is invertible in* $\mathbb{H}^s$ *. Furthermore, it generates a* $\mathbb{C}_0$ *-semigroup* $(\mathbf{S}(t))_{\tau\geq 0}$ *in* $\mathbb{H}^s$ *.*

Proof. Firstly, we show that the existence of L^{-1}. Indeed, we need to prove L is injective and surjective. On the one hand, let $w\in\mathcal{D}(L)$ such that $L[w]=0$, then

$$\int_{\mathbb{R}}\mathcal{A}^s L[w]\mathcal{A}^s w\mathrm{d}\kappa=-\frac{1}{2}\|w\|_{\mathbb{H}^s}^2=0. \tag{2.23}$$

This combining with the boundary condition (2.7) gives that $w=0$. So the operator L is injective. On the other hand, for all $g\in\mathbb{H}^1$, put

$$L[w]=g. \tag{2.24}$$

Applying $\mathcal{A}^s$ to (2.24) and multiplying the result by $\mathcal{A}^s w$, and then integrating over $\mathbb{R}$, we get

$$\|w\|_{\mathbb{H}^s}^2=-2\int_{\mathbb{R}}\mathcal{A}^s g\mathcal{A}^s w\mathrm{d}\kappa. \tag{2.25}$$

It follows from the Young inequality with ε in Lemma 2.5 that

$$\|w\|_{\mathbb{H}^s}\leq C\|g\|_{\mathbb{H}^s}. \tag{2.26}$$

Note that $s>2$, then by the standard theory of elliptic equations (see [22]), there exists a unique weak solution $w\in\mathbb{H}^1$, moreover, we have $w\in\mathbb{H}^{s+1}$ if $g\in\mathbb{H}^s$. Thus, the operator L is surjective. Secondly, by the Lumer-Phillips theorem (see [23]), the operator L generates a $\mathbb{C}_0$-semigroup $(\mathbf{S}(t))_{\tau\geq 0}$ in $\mathbb{H}^s$. This completes the proof. □

As a consequence, we have

Proposition 2.7. *Let* $s>2$ *. Then the Cauchy problem*

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau}w=Lw,\\ w(0)=w_0\end{cases} \tag{2.27}$$

with zero boundary condition exists a unique solution $w(\tau)=\mathbf{S}(\tau)w_0$, where w_0 is the initial data defined in (2.6).

Using the Duhamel's principle, the solutions of (2.19) satisfies the integral equation:

$$w(\tau)=\mathbf{S}(\tau)w_0+\int_0^\tau\mathbf{S}(\tau-s)f(w(s))\mathrm{d}s. \tag{2.28}$$

To show this integral equation exists a solution, we define the solution space as

$$B_\delta=\left\{w\in\mathbb{H}^s:\|w\|_{\mathbb{H}^s}<\delta\ll 1\right\}, \tag{2.29}$$

and the map $\mathcal{T}$ as

$$\mathcal{T}w(\tau)=\mathbf{S}(\tau)w_0+\int_0^\tau\mathbf{S}(\tau-s)f(w(s))\mathrm{d}s. \tag{2.30}$$

We need to prove that $\mathcal{T}$ has a fixed point in the space B_δ.

Lemma 2.8 ([21]). *Let* $s>2$ *. Then* B_δ *is an algebra, and*

$$\|uv\|_{\mathbb{H}^s} \le C\left(\|u\|_{\mathbb{L}^\infty}\|v\|_{\mathbb{H}^s} + \|u\|_{\mathbb{H}^s}\|v\|_{\mathbb{L}^\infty}\right), \tag{2.31}$$

where C is a positive constant depending upon s.

Lemma 2.9. *Let* $s>2$ *be an integer. Assume that* $\|w_0\|_{\mathbb{H}^{s+1}} < \delta$ *for some sufficiently small* $\delta > 0$. *Then* $\mathcal{T}$ *is a self-mapping on* B_δ. *Moreover,* $\mathcal{T}$ *is a contraction mapping.*

Proof. By Lemma 2.8, we have

$$\begin{aligned}\|f(w)\|_{\mathbb{H}^s} &\le 3\|(\wp*w)(\wp*w)_\kappa\|_{\mathbb{H}^s} + 2\|(\wp*w)_\kappa(\wp*w-w)\|_{\mathbb{H}^s} \\ &\quad + \|(\wp*w)((\wp*w)_\kappa - w_\kappa)\|_{\mathbb{H}^s} \\ &\le C_1\Big(\|\wp*w\|_{\mathbb{H}^s}\|(\wp*w)_\kappa\|_{\mathbb{L}^\infty} + \|(\wp*w)_\kappa\|_{\mathbb{L}^\infty}\|(\wp*w-w)\|_{\mathbb{H}^s} \\ &\quad + \|\wp*w\|_{\mathbb{H}^s}\|(\wp*w)_\kappa - w_\kappa\|_{\mathbb{L}^\infty}\Big),\end{aligned} \tag{2.32}$$

where C_1 is a positive constant.

Note that $\mathbb{H}^s \subset \mathbb{L}^\infty$ and $w = \mathcal{A}^2(\wp(\kappa)*\overline{v})$, then using Lemma 2.2, we have

$$\|f(w)\|_{\mathbb{H}^s} \le C_1\|w\|_{\mathbb{H}^s}^2 < \frac{C_1}{\|w\|_{\mathbb{H}^s}^{-1} - C\tau} < \frac{C_1}{\delta^{-1} - C\tau} < \delta \tag{2.33}$$

for sufficiently small δ. Thus, $\mathcal{T}$ is a self-mapping on B_δ.

To show $\mathcal{T}$ is a contraction mapping, we choose $w, \overline{w} \in B_\delta$, by Lemma 2.8 and a direct calculation show that

$$\begin{aligned}&\|f(w) - f(\overline{w})\|_{\mathbb{H}^s} \\ &= \Big\|-3(\wp*w)(\wp*w)_\kappa + 2(\wp*w)_\kappa(\wp*w) \\ &\quad - 2(\wp*w)_\kappa w + (\wp*w)((\wp*w)_\kappa - w_\kappa) \\ &\quad + 3(\wp*\overline{w})(\wp*\overline{w})_\kappa - 2(\wp*\overline{w})_\kappa(\wp*\overline{w}) \\ &\quad + 2(\wp*\overline{w})_\kappa \overline{w} - (\wp*\overline{w})((\wp*\overline{w})_\kappa - \overline{w}_\kappa)\Big\|_{\mathbb{H}^s} \\ &\le \Big\|3\{(\wp*\overline{w})[\wp*(\overline{w}-w)]_\kappa + (\wp*\overline{w})_\kappa[\wp*(\overline{w}-w)]\} \\ &\quad + 2\{(\wp*w)_\kappa[\wp*(\overline{w}-w)] + (\wp*\overline{w})[\wp*(\overline{w}-w)]_\kappa\} \\ &\quad + 3\{(\wp*\overline{w})[\wp*(\overline{w}-w)]_\kappa + (\wp*\overline{w})_\kappa[\wp*(\overline{w}-w)]\} \\ &\quad + \{(\wp*\overline{w})[\wp*(\overline{w}-w)]_\kappa + (\wp*\overline{w})_\kappa[\wp*(\overline{w}-w)]\} \\ &\quad + \{[\wp*(\overline{w}-w)]\overline{w}_\kappa + (\wp*w)(\overline{w}-w)_\kappa\}\Big\|_{\mathbb{H}^s} \\ &\le C\delta\|w-\overline{w}\|_{\mathbb{H}^s}.\end{aligned} \tag{2.34}$$

Thus,

$$\|\mathcal{T}w(\tau) - \mathcal{T}\overline{w}(\tau)\|_{\mathbb{H}^s} \le C\delta\|w-\overline{w}\|_{\mathbb{H}^s}. \tag{2.35}$$

Since $\delta > 0$ is sufficiently small, $\mathcal{T}$ is a contraction mapping. □

Thus, we have the following existence results.

Proposition 2.10. *Let* $s>2$ *be a fixed integer and* $\delta > 0$ *is a sufficiently small constant. Then*

- if $\|w_0\|_{\mathbb{H}^{s+1}} < \delta$, there exists a unique solution $w \in B_\delta$ to (2.5) with the initial data (2.6) and the boundary condition (2.7).

- there exists a global solution $\psi(\tau,\rho)\in\mathbb{H}^s$ to (2.3) with the initial data (2.6) and the boundary condition (2.7). Moreover, if the initial data ψ_0 satisfies $\|\psi_0\|_{\mathbb{H}^{s+1}}<\delta$, then

$$\|\psi\|_{\mathbb{H}^s}\le\frac{\tilde{C}}{\alpha^2\mathrm{e}^{\tau}(1-C\tau)}. \tag{2.36}$$

Here C and $\tilde{C}$ are two positive constants that depend on *s*.

Proof. By Lemma 2.9 and the Banach fixed point theorem, the map $\mathcal{T}$ has a fixed point in B_δ, which is a solution of Equation (2.5). Thus, there exists a global solution of (2.3) as

$$\psi(\tau,\rho)=\mathrm{e}^{\tau}\overline{v}(\tau,\mathrm{e}^{-\tau}\rho)=\mathrm{e}^{\tau}\left(\left(\wp*w\right)\left(\tau,\mathrm{e}^{-\tau}\rho\right)\right). \tag{2.37}$$

Furthermore, we have

$$v_{\rho\rho}=\psi_{\rho\rho}=\left(\wp*w\right)_{\kappa\kappa}\mathrm{e}^{-\tau}=\alpha^{-2}\mathrm{e}^{-\tau}\left(\wp*w-w\right). \tag{2.38}$$

Thus, by Lemma 2.2, we get

$$\begin{aligned}\left\|\psi_{\rho\rho}\right\|_{\mathbb{H}^{s-2}}&\le\alpha^{-2}\mathrm{e}^{-\tau}\left\|\wp*w-w\right\|_{\mathbb{H}^{s-2}}\le\tilde{C}\alpha^{-2}\mathrm{e}^{-\tau}\left\|w\right\|_{\mathbb{H}^{s-2}}\\&\le\frac{\tilde{C}}{\alpha^2\mathrm{e}^{\tau}\left(\left\|w_0\right\|_{\mathbb{H}^s}^{-1}-C\tau\right)}\le\frac{\tilde{C}}{\alpha^2\mathrm{e}^{\tau}(1-C\tau)},\end{aligned} \tag{2.39}$$

where we have used $\delta<1$ in the last inequality. This completes the proof. □

As a consequence, we obtain that the global well-posedness of the initial value problem (2.2). This implies that the asymptotic stability of the explicit singular solution (1.3) for the Camassa-Holm Equation (1.2). Hence, we complete the proof of Theorem 1.1.

3. Conclusion

In this paper, the Semigroup theory of linear operators has been used to study the asymptotic stability of the explicit blowup solution of Camassa-Holm equation. This result shows that the explicit solution is a meaningful physical solution. However, this explicit solution does not depend on the wavelength (*i.e.*, it does not depend on α). Thus, further studies are needed to construct the explicit solutions that depend on α, and then prove their stability.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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