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A Generalization of Eneström-Kakeya Theorem and a Zero Free Region of a Polynomial

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Abstract

For the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$, $a_j \geq a_{j-1}$, $a_0 > 0$, $j=1,2,\cdots,n$, $a_n > 0$, a classical result of Eneström-Kakeya says that all the zeros of $P(z)$ lie in $|z| \leq 1$. This result was generalised by A. Joyall and G. Labelle, where they relaxed the non-negativity condition on coefficients. It was further generalized by M.A Shah by relaxing the monotonicity of some coefficients. In this paper, we use some known techniques and provide some more generalizations of the above results by giving more relaxation to the conditions.

Keywords

Polynomial, Zeros, Eneström-Kakeya Theorem

1. Introduction

If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree *n*. Then Eneström-Kakeya [1] [2] proved the following interesting result.

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree *n* such that $a_n \geq a_{n-1} \geq \cdots \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$.

For example: The Polynomial $10z^6+8z^5+7z^4+7z^3+6z^2+2z+1$ has all zeros in $|z| \leq 1$.

In the literature, there exist several extensions and generalizations of this theorem. Joyal *et al.* [3] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In fact, they proved the following result.

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that $a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0$, then $P(z)$ has all its zeros in the disk

$$|z| \le \frac{1}{|a_n|}\left(|a_n| - a_0 + |a_0|\right). \tag{1}$$

For example: Consider the Polynomial $4z^6 + 3z^5 + 2z^4 - z^2 - z - 3$

Here $n=6, a_n = 4$ and $a_0 = -3$

Then the zeros of this polynomial lie in

$$|z| \le \frac{4-(-3)+3}{4} = \frac{4+3+3}{4} = \frac{10}{4} = \frac{5}{2}$$

i.e. $|z| \le 2.5$

The above results were generalised by M.A. Shah [4]. In fact he proved the following result.

Theorem C: Let $P(z)=a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n satisfying

$$a_p \ge a_{p-1} \ge \cdots \ge a_1 \ge a_0,\ p = 0,1,2,\cdots,n \text{ and } M_p = \sum_{j=p+1}^{n}|a_j - a_{j-1}|,$$

then all the zeros of $P(z)$ lie in the disc

$$|z| \le \frac{M_p + a_p - a_0 + |a_0|}{a_n}. \tag{2}$$

Theorem D: Let $P(z)=a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n satisfying

$$a_p \ge a_{p-1} \ge \cdots \ge a_1 \ge a_0,\ p = 0,1,2,\cdots,n \text{ and } M_p = \sum_{j=p+1}^{n}|a_j - a_{j-1}|,$$

then $P(z)$ does not vanish in

$$|z| < \min\left[1, \frac{|a_0|}{|a_n| + M_p + a_p - a_0}\right]. \tag{3}$$

In literature [5]-[12], there exist several other generations and extensions of Eneström-Kakeya Theorem. Our main purpose is to relax some conditions on the monotonicity of coefficients and obtain some interesting generalizations of known results.

2. Main Results

This paper provides some further generalizations of the Eneström-Kakeya theorem and the above results. In this direction, we first prove the following result.

Theorem 1. *Let* $P(z)=a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots + a_q z^q + a_{q-1}z^{q-1} + \cdots + a_1 + a_0$ *be a polynomial of degree n satisfying*

$$a_p \ge a_{p-1} \ge \cdots \ge a_q,\ p \ge q.$$

$$M_p = \sum_{j=p+1}^{n} \left|a_j - a_{j-1}\right| \text{ and } M_q = \sum_{j=1}^{q} \left|a_j - a_{j-1}\right|,$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \le \frac{M_p + M_q + a_p - a_q + |a_0|}{|a_n|}. \tag{4}$$

Proof. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)\left(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + a_{p-1}z^{p-1} + \cdots \right. \\
&\quad \left. + a_q z^q + a_{q-1}z^{q-1} + \cdots + a_1 z + a_0\right) \\
&= -a_n z^{n+1} + \left[(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} \right. \\
&\quad + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + \cdots + (a_{q+1} - a_q)z^{q+1} \\
&\quad \left. + (a_q - a_{q-1})z^q + (a_{q-1} - a_{q-2})z^{q-1} + \cdots + (a_1 - a_0)z + a_0\right]
\end{aligned}$$

This gives

$$\begin{aligned}
|F(z)| &\ge |a_n||z|^{n+1} - \left[\left|(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1}\right.\right. \\
&\quad + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + \cdots + (a_{q+1} - a_q)z^{q+1} \\
&\quad \left.\left. + (a_q - a_{q-1})z^q + (a_{q-1} - a_{q-2})z^{q-1} + \cdots + (a_1 - a_0)z + a_0\right|\right] \\
&\ge |a_n||z|^{n+1} - \left[|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{p+1} - a_p||z|^{p+1}\right. \\
&\quad + |a_p - a_{p-1}||z|^p + |a_{p-1} - a_{p-2}||z|^{p-1} + \cdots + |a_{q+1} - a_q||z|^{q+1} \\
&\quad \left. + |a_q - a_{q-1}||z|^q + |a_{q-1} - a_{q-2}||z|^{q-1} + \cdots + |a_1 - a_0||z| + |a_0|\right] \\
&\ge |z|^n \left[|a_n||z| - \left(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_p - a_{p-1}|}{|z|^{n-p}} + \cdots \right.\right. \\
&\quad \left.\left. + \frac{a_q - a_{q-1}}{|z|^{n-q}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right)\right]
\end{aligned}$$

Now let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, 0 \le j \le n$, then we have

$$\begin{aligned}
|F(z)| &> |z|^n \left[|a_n||z| - \left(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_p - a_{p-1}| + \cdots \right.\right. \\
&\quad \left.\left. + |a_q - a_{q-1}| + \cdots + |a_1 - a_0| + |a_0|\right)\right] \\
&= |z|^n \left[|a_n||z| - \left(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| \right.\right. \\
&\quad \left.\left. + a_p - a_{p-1} + \cdots + a_{q+1} - a_q + |a_q - a_{q-1}| + \cdots + |a_1 - a_0| + |a_0|\right)\right] \\
&= |z|^n \left[|a_n||z| - \left(\sum_{j=p+1}^{n} |a_j - a_{j-1}| + \sum_{j=1}^{q} |a_j - a_{j-1}| + a_p - a_q + |a_0|\right)\right] \\
&= |z|^n [|a_n||z| - \left(M_p + M_q + a_p - a_q + |a_0|\right) \\
&> 0, \text{ if } |z||a_n| > \left(M_p + M_q + a_p - a_q + |a_0|\right)
\end{aligned}$$

i.e. if

$$|z| > \frac{M_p + M_q + a_p - a_q + |a_0|}{|a_n|}$$

where $M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}|$ and $M_q = \sum_{j=1}^{q} |a_j - a_{j-1}|$.

Thus all the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \le \frac{M_p + M_q + a_p - a_q + |a_0|}{|a_n|}$$

But the zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \le \frac{M_p + M_q + a_p - a_q + |a_0|}{|a_n|}$$

This completes the proof of the Theorem.

For example: Consider the polynomial

$$10z^{10} - z^9 + 2z^8 - 3z^7 + 4z^6 + 3z^5 + 2z^4 - z^3 + 3z^2 - 2z + 1$$

Here $n = 10$, $a_n = 10$, $p = 6$, $q = 3$, $a_p = 4$, $a_q = -1$, $a_0 = 1$, $M_p = 26$ and $M_q = 12$

$$|z| \le \frac{26 + 12 + 4 + 1 + 1}{10}$$

$$|z| \le 4.4$$

Remark. For $p = n$ and $q = 0$, theorem 1 reduces to theorem B.

Applying theorem 1 to the polynomial $p(tz)$, we get the following result

Corollary. Let

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_q z^q + a_{q-1} z^{q-1} + \cdots + a_1 + a_0$ be a polynomial of degree *n* such that for any $t > 0$,

$$t^p a_p \ge t^{p-1} a_{p-1} \ge \cdots \ge t^{q+1} a_{q+1} \ge t^q a_q$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \le \sum_{j=p+1}^{n} \frac{|ta_j - a_{j-1}|}{|a_n| t^{n-j+1}} + \sum_{j=1}^{q} \frac{|ta_j - a_{j-1}|}{|a_n| t^{n-j+1}} + \frac{t^P a_p - t^q a_q + |a_0|}{t^n |a_n|} \tag{5}$$

Remark. for $q = 0$ the above theorem reduces to theorem C.

Next, we prove the following result concerning the zero-free region of a polynomial. In fact we prove the following:

Theorem 2. *Let*

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_q z^q + a_{q-1} z^{q-1} + \cdots + a_1 + a_0$ *be a polynomial of degree n satisfying*

$$a_p \ge a_{p-1} \ge \cdots \ge a_q, p \ge q.$$

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \text{ and } M_q = \sum_{j=1}^{q} |a_j - a_{j-1}|$$

then $P(z)$ does not vanish in

$$|z| < \min\left[1, \frac{|a_0|}{|M_p + M_q + a_p - a_q + |a_n||}\right] \tag{6}$$

Proof. Consider the reciprocal polynomial

$$R(z) = z^n p\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_p z^{n-p} + \cdots + a_{n-1} z + a_n.$$

Let

$$\begin{aligned} S(z) &= (1-z)R(z) \\ &= (1-z)\left[a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_p z^{n-p} + \cdots + a_{n-1} z + a_n\right] \\ &= -a_0 z^{n+1} + (a_0 - a_1) z^n + \cdots + (a_{q+1} - a_q) z^{n-q} + \cdots + (a_p - a_{p+1}) z^{n-p} \\ &\quad + \cdots + (a_{n-1} - a_n) z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} |S(z)| &\geq |a_0||z|^{n+1} - \left[\left\{|a_0 - a_1||z|^n + \cdots + |a_{q+1} - a_q||z|^{n-q} + \cdots \right.\right. \\ &\quad \left.\left. + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n|\right\}\right] \\ &= |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + \cdots + \frac{a_{q-1} - a_q}{|z|^{q-1}} + \frac{|a_{q+1} - a_q|}{|z|^q} + \cdots \right.\right. \\ &\quad \left.\left. + \frac{|a_{p-1} - a_p|}{|z|^{p-1}} + \frac{|a_p - a_{p+1}|}{|z|^p} + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\right)\right]. \end{aligned}$$

Now let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, 0 \leq j \leq n$, then we have

$$\begin{aligned} |S(z)| &\geq |z|^n \left[|a_0||z| - \left(|a_0 - a_1| + \cdots + |a_{q-1} - a_q| + |a_{q+1} - a_q| + \cdots \right.\right. \\ &\quad \left.\left. + |a_{p-1} - a_p| + |a_p - a_{p+1}| + \cdots + |a_{n-1} - a_n| + |a_n|\right)\right] \\ &= |z|^n \left[|a_0||z| - \sum_{j=p+1}^{n} |a_j - a_{j-1}| + \sum_{j=1}^{q} |a_j - a_{j-1}| + a_{q+1} - a_q \right. \\ &\quad \left. + a_{q+2} - a_{q+1} + \cdots + a_{p+1} - a_{p-2} + a_p - a_{p-1} + |a_n|\right] \\ &= |z|^n \left[|a_0||z| - \left(M_p + M_q + |a_n| + a_p - a_q\right)\right] \\ &> 0, \text{ if } |z||a_0| > \left(M_p + M_q + a_p - a_q + |a_n|\right) \end{aligned}$$

i.e. if

$$|z| > \frac{M_p + M_q + a_p - a_q + |a_n|}{|a_0|}$$

where $M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}|$, and $M_q = \sum_{j=1}^{q} |a_j - a_{j-1}|$.

Thus all the zeros of $S(z)$ whose modulus is greater than 1 lie in

$$|z| \le \frac{M_p + M_q + a_p - a_q + |a_n|}{|a_0|}$$

Hence all the zeros of $S(z)$ and hence of $R(z)$ lie in

$$|z| \le \max\left[1, \frac{M_p + M_q + a_p - a_q + |a_n|}{|a_0|}\right]$$

Therefore all the zeros of $P(z)$ lie in

$$|z| \ge \min\left[1, \frac{|a_0|}{M_p + M_q + a_p - a_q + |a_n|}\right]$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \min\left[1, \frac{|a_0|}{M_p + M_q + a_p - a_q + |a_n|}\right]$$

This completes the proof of the Theorem.

For example: Consider the polynomial $2z^8 - 5z^7 + 7z^6 + 2z^5 - 2z^3 + z^2 - 3z + 10$

Here $n = 8$, $a_n = 2$, $p = 5$, $q = 3$, $a_p = 2$, $a_q = -2$, $a_0 = 10$, $M_p = 24$ and $M_q = 20$

$$|z| < \min\left[1, \frac{|a_0|}{|M_p + M_q + a_p - a_q + |a_n||}\right]$$

$$i.e., \quad |z| < \min\left[1, \frac{10}{24 + 20 + 2 + 2 + 2}\right]$$

$$i.e., \quad |z| < \min\left[1, \frac{10}{50}\right]$$

$$i.e., \quad |z| < \min(1, 0.2)$$

$$i.e., \quad |z| < 0.2$$

3. Conclusion and Suggestions

We can obtain several known results from the above results as special cases. If we apply monotonicity to all the coefficients, we can easily obtain all the previous known results in addition to Eneström-Kakeya theorem.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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