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Special Values for the Riemann Zeta Function

John H. Heinbockel

Old Dominion University, Norfolk, Virginia, USA
Email: jheinboc@odu.edu

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Abstract

The purpose for this research was to investigate the Riemann zeta function at odd integer values, because there was no simple representation for these results. The research resulted in the closed form expression

$$\zeta(2n+1)=\frac{(-4)^n \pi^{2n+1}E_{2n}-2\psi^{(2n)}(3/4)}{2^{2n+1}\left(2^{2n+1}-1\right)(2n)!},\quad n=1,2,3,\cdots$$

for representing the zeta function at the odd integer values $2n+1$ for n a positive integer. The above representation shows the zeta function at odd positive integers can be represented in terms of the Euler numbers E_{2n} and the polygamma functions $\psi^{(2n)}(3/4)$. This is a new result for this study area. For completeness, this paper presents a review of selected properties of the Riemann zeta function together with how these properties are derived. This paper will summarize how to evaluate zeta (n) for all integers n different from 1. Also as a result of this research, one can obtain a closed form expression for the Dirichlet beta series evaluated at positive even integers. The results presented enable one to construct closed form expressions for the Dirichlet eta, lambda and beta series evaluated at odd and even integers. Closed form expressions for Apéry's constant zeta (3) and Catalan's constant beta (2) are also presented.

Keywords

Riemann Zeta Function, Zeta (2n), Zeta (2n + 1), Apéry's Constant, Catalan Constant

1. Introduction

If you do not know about the Riemann zeta function, then do an internet search to observe the extensive research that has been done investigating various properties of this function. A more detailed introduction to the Riemann zeta function can be found in the references [1] [2]. One way of defining this function is to express it as an infinite series having the form

$$\zeta(\sigma)=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}}=\frac{1}{1^{\sigma}}+\frac{1}{2^{\sigma}}+\frac{1}{3^{\sigma}}+\cdots,\quad \sigma>1 \tag{1}$$

where σ is a real number greater than 1 in order for the infinite series to converge. Observing that for $\sigma=1$, the series becomes the harmonic series which slowly diverges. The zeta function was introduced by Leonhard Euler (1707-1783) who considered $\zeta(\sigma)$ to be a function of a real variable.

Another form for representing the zeta function is the integral representation

$$\Gamma(\sigma)\zeta(\sigma)=\int_0^{\infty}\frac{r^{\sigma-1}}{\mathrm{e}^{r}-1}\mathrm{d}r,\quad \sigma>1 \tag{2}$$

where $\Gamma(\sigma)$ is the gamma function.

Bernhard Riemann (1826-1866) studied the zeta function and changed the independent real variable σ to the complex variable $s=\sigma+it$. This notation is still used in current studies of the zeta function. By doing this, Riemann made $\zeta(s)$ a function of a complex variable. Riemann discovered that the zeta function satisfied the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{3}$$

where $\Gamma(s)$ is the gamma function. Several proofs of the above result can be found in the Titchmarsh reference [3]. Various forms for the functional equation are derived later in this paper. The equation allowed the zeta function to be defined for values $Re(\sigma)<1$. The point $s=1$ is a singular point. Using properties of the gamma function, the functional equation can be expressed in the alternative form

$$\zeta(s)=2(2\pi)^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s),\quad Re(s)<1 \tag{4}$$

derived later in this paper. The above results can be used to extend the definition of the zeta function to the whole of the complex plane.

Euler also showed that the zeta function can also be expressed using prime numbers

$$\zeta(s)=\prod_{p}\left(1-\frac{1}{p^{s}}\right)^{-1},\quad Re(s)>1 \tag{5}$$

where the product runs through all primes $p=2,3,5,7,\cdots$. The equation (5) is known as the Euler product formula.

The Euler-Riemann function $\zeta(s)$ is an important function in number theory where it is related to the distribution of prime numbers. It also can be found in such diverse study areas as probability and statistics, physics, Diophantine equations, modular forms and in many tables of integrals. The Euler-Riemann zeta function evaluated at special integer values for s occurs quite frequently in tables of integrals and in many areas of science and engineering.

2. Bernoulli and Euler Numbers

In later sections we need knowledge of the Bernoulli numbers B_n and Euler numbers E_n. Representation of these numbers can be obtained from reference [4] (24.2), where one finds the generating functions

$$\begin{aligned}\frac{x}{e^x-1}&=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=1-\frac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}+\cdots\\ \frac{2e^x}{e^{2x}+1}&=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}=1-\frac{x^2}{2!}+5\frac{x^4}{4!}-61\frac{x^6}{6!}+1385\frac{x^8}{8!}+\cdots\end{aligned}\tag{6}$$

Note that the first few values are

$$\begin{aligned}&B_0=1,\ B_1=-1/2,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\cdots\\ &E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\cdots\end{aligned}$$

Note that for m,n positive integers with $n\ge 1$, $B_{2n+1}=0$ and for $m\ge 0$, $E_{2m+1}=0$.

3. Calculation of $\zeta(2n)$ for $n=1,2,3,\cdots$

Leonhard Euler discovered values for the zeta function at $\zeta(2),\zeta(4),\zeta(6),\cdots$ In general for s an even integer, say $s=2n$, for $n=1,2,3,\cdots$, the zeta function $\zeta(2n)$, evaluated at positive even integers takes on the values given by

$$\begin{aligned}&\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \zeta(8)=\frac{\pi^8}{9450}\\ &\text{and in general}\quad \zeta(2n)=\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n},\quad n=1,2,3,\cdots\end{aligned}\tag{7}$$

where B_n are the Bernoulli numbers. These results were discovered by Leonhard Euler (1707-1783) sometime around 1724 and are well known. Observe that $\zeta(2n)$ is proportional to π^{2n}.

The above results can be derived from the following observations. The function $g(x)=x\cot(x)$ can be expressed in different forms. For example,

$$g(x)=x\frac{\cos(x)}{\sin(x)}=x\frac{\frac{e^{ix}+e^{-ix}}{2}}{\frac{e^{ix}-e^{-ix}}{2i}}=ix\frac{e^{2ix}+1}{e^{2ix}-1}=ix+\frac{2ix}{e^{2ix}-1}$$

Now compare the last term of the above equation with the previous equation (6) involving the Bernoulli numbers, to obtain

$$\begin{aligned}g(x)&=ix+\sum_{m=0}^{\infty}B_m\frac{(2ix)^m}{m!}\\ g(x)&=ix+B_0+B_1(2ix)+\sum_{m=2}^{\infty}B_m\frac{(2ix)^m}{m!}=1+\sum_{m=2}^{\infty}B_m\frac{(2ix)^m}{m!}\end{aligned}\tag{8}$$

One can examine the zeros of the denominator in $g(x)=x\frac{\cos(x)}{\sin(x)}$ and express $g(x)$ in the alternative form

$$g(x)=1-2\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2-x^2}$$
$$g(x)=1-2\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2}\frac{1}{1-\left(\frac{x}{n\pi}\right)^2}$$

The last term of the above equation can be expanded in a series to obtain

$$g(x)=1-2\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2}\sum_{j=0}^{\infty}\left(\frac{x}{n\pi}\right)^{2j}$$

One can interchange the order of summation and write

$$\begin{aligned} g(x)&=1-2\sum_{j=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2j+2}}\right)\frac{x^{2j+2}}{\pi^{2j+2}} \\ g(x)&=1-2\sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2j}}\right)\frac{x^{2j}}{\pi^{2j}} \end{aligned} \tag{9}$$

Now by comparing the coefficients of powers x^n in the equations (9) and (8) one obtains the well known result

$$\zeta(2n)=\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n},\quad n=1,2,3,\cdots$$

as previously given in equation (7). A similar derivation can be found in the reference [5].

4. The Zeta Function $\zeta(2k+1), k=1,2,3,\cdots$

Note that the reference [2] points out that there is no known formula for the zeta function evaluated at odd positive integers greater than or equal to three. This paper will provide such a formula.

It will be demonstrated that for odd positive integers *s*, say $s=2n+1$, for $n=1,2,3,\cdots$ that

$$\zeta(3)=-\frac{\pi^3E_2}{28}-\frac{\psi^{(2)}(3/4)}{56}=1.20205\cdots,$$
$$\zeta(5)=\frac{\pi^5E_4}{1488}-\frac{\psi^{(4)}(3/4)}{11904}=1.03692\cdots$$
$$\zeta(7)=-\frac{\pi^7E_6}{182880}-\frac{\psi^{(6)}(3/4)}{5852160}=1.00834\cdots,$$
$$\zeta(9)=\frac{\pi^9E_8}{41207040}-\frac{\psi^{(8)}(3/4)}{5274501120}=1.00200$$

where the ellipsis $\cdots$ denotes the decimal representations are unending. In general, it will be demonstrated

$$\zeta(2n+1)=\frac{(-4)^n\pi^{2n+1}E_{2n}-2\psi^{(2n)}(3/4)}{2^{2n+1}\left(2^{2n+1}-1\right)(2n)!},\quad n=1,2,3,\cdots \tag{10}$$

where E_n are the Euler numbers and $\psi^{(n)}(z)$ are the polygamma functions.

Observe the $\zeta(2n+1)$ is related to π^{2n+1}. Apéry's constant is

$$\zeta(3)=\frac{-4\pi^3 E_2-2\psi^{(2)}(3/4)}{112}=1.20205690316\cdots$$

5. Polygamma Functions

The digamma function $\psi(s)$ is defined

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln\Gamma(s)=\frac{\Gamma'(s)}{\Gamma(s)}=\psi(s) \tag{11}$$

where $\Gamma(s)$ is the gamma function.

$$\Gamma(s)=\int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x,\quad Re(s)>0 \tag{12}$$

The gamma function satisfies the functional equation $\Gamma(x+1)=x\Gamma(x)$ so one can write

$$\begin{aligned}\Gamma(x+1)&=x\Gamma(x)\\ \Gamma(x+2)&=(x+1)\Gamma(x+1)\\ &\vdots\\ \Gamma(x+n+1)&=(x+n)\Gamma(x+n)\end{aligned} \tag{13}$$

and consequently

$$\Gamma(x)=\frac{\Gamma(x+n+1)}{x(x+1)(x+2)\cdots(x+n)} \tag{14}$$

Take logarithms on both sides of equation (14) and then differentiate to show

$$\psi(x)=-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\cdots-\frac{1}{x+n}+\psi(x+n+1)$$

Differentiate again and show

$$\psi'(x)=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\cdots+\frac{1}{(x+n)^2}+\psi'(x+n+1) \tag{15}$$

From the reference [2] or reference [4] (5.15), one can show that in the limit as n increases without bound the derivative term $\psi'(n)$ behaves like $1/n$ and approaches zero. By repeated differentiation of equation (15) one can obtain the polygamma functions $\psi^{(n)}(x)$ defined by

$$\begin{aligned}\psi^{(n)}(x)&=\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\ln\Gamma(x)=(-1)^{n+1}n!\left[\frac{1}{x^{n+1}}+\frac{1}{(x+1)^{n+1}}+\cdots\right]\\ \psi^{(n)}(x)&=(-1)^{n+1}n!\sum_{\ell=0}^{\infty}\frac{1}{(x+\ell)^{n+1}}\end{aligned} \tag{16}$$

for $n=1,2,3,\cdots$.

6. Additional Functions

Related to the study of the zeta function are the Dirichlet[1] eta, lambda and beta

[1]Peter Gustav Lejeune Dirichlet (1805-1859).

series defined

$$\eta(s)=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots=\sum_{k=1}^{\infty}(-1)^{k-1}k^{-s},\quad s>0,$$
$$\lambda(s)=\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots=\sum_{k=0}^{\infty}(2k+1)^{-s},\quad s>1$$
$$\beta(s)=\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\cdots=\sum_{k=0}^{\infty}(-1)^{k}(2k+1)^{-s},\quad s>0$$

The first two Dirichlet series are related to the zeta function by the identities

$$\frac{\zeta(s)}{2^s}=\frac{\lambda(s)}{2^s-1}=\frac{\eta(s)}{2^s-2}\quad\text{and}\quad\zeta(s)+\eta(s)=2\lambda(s)\tag{17}$$

Make note of the fact that knowing the equations (7) and (10) one can construct closed form expressions for the Dirichlet eta and lambda functions evaluated at odd and even integers greater than one.

7. Preliminary Observations

Define the function

$$r(s)=\frac{2}{3^s}+\frac{2}{7^s}+\frac{2}{11^s}+\cdots=2\sum_{k=0}^{\infty}\frac{1}{(3+4k)^s}\tag{18}$$

where s is a positive integer greater than 1. One can then verify that

$$\beta(s)+r(s)=\lambda(s)=\left(1-2^{-s}\right)\zeta(s)\tag{19}$$

We examine the special cases

$$\beta(2k+1)+r(2k+1)=\lambda(2k+1)=\left(1-2^{-(2k+1)}\right)\zeta(2k+1)\tag{20}$$

from which $\zeta(2k+1)$ can be obtained and

$$\beta(2k)+r(2k)=\lambda(2k)=\left(1-2^{-(2k)}\right)\zeta(2k)\tag{21}$$

from which an expression for $\beta(2k)$, k an integer, can be obtained. From these two equations one can develop closed form expressions for $\zeta(2k+1)$ and $\beta(2k)$.

8. Calculation of $r(2k+1)$ and $r(2k)$

Observe that by using equation (16) with $n=2k$, and again with $n=2k-1$, one can obtain the series representations

$$\begin{aligned} r(2k+1)&=\frac{2}{4^{2k+1}}\sum_{\ell=0}^{\infty}\frac{1}{\left(\frac{3}{4}+\ell\right)^{2k+1}}=\frac{-2}{4^{2k+1}}\frac{\psi^{(2k)}(3/4)}{(2k)!}\\ r(2k)&=\frac{2}{4^{2k}}\sum_{\ell=0}^{\infty}\frac{1}{\left(\frac{3}{4}+\ell\right)^{2k}}=\frac{2\psi^{(2k-1)}(3/4)}{4^{2k}(2k-1)!}\end{aligned}\tag{22}$$

These results will be used shortly.

9. Calculation of $\beta(2k+1)$ and $\beta(2k)$

We begin by examining the trigonometric function $f(x)=\sec(x)+\tan(x)$ which can be expressed in many different forms. One form is $f(x)=\dfrac{\cos(x)}{1-\sin(x)}$ where one can examine the zeros of the denominator and write

$$f(x)=\sum_{n=0}^{\infty}\left[\frac{a_n}{1-\dfrac{x}{(1+4n)\dfrac{\pi}{2}}}+\frac{b_n}{1+\dfrac{x}{(3+4n)\dfrac{\pi}{2}}}\right]$$

where a_n, b_n are constants which can be determined from the limits

$$\lim_{x\to(1+4m)\pi/2}\left(x-(1+4m)\pi/2\right)f(x)=a_m=\frac{4}{(1+4m)\pi}$$

$$\lim_{x\to(3+4m)\pi/2}\left(x-(3+4m)\pi/2\right)f(x)=b_m=\frac{-4}{(3+4m)\pi}$$

This produces the expression

$$f(x)=\frac{4}{\pi}\sum_{n=0}^{\infty}\left[\frac{1}{1+4n}\frac{1}{1-\dfrac{x}{(1+4n)\pi/2}}-\frac{1}{3+4n}\frac{1}{1+\dfrac{x}{(3+4n)\pi/2}}\right]$$

which can now be expanded into the series

$$\begin{aligned}f(x)=&\frac{4}{\pi}\left[1+\frac{2x}{\pi}+\frac{2^2x^2}{\pi^2}+\cdots+\frac{2^nx^n}{\pi^n}+\cdots\right]\\&+\frac{4}{5\pi}\left[1+\frac{2x}{5\pi}+\frac{2^2x^2}{5^2\pi^2}+\cdots+\frac{2^nx^n}{5^n\pi^n}+\cdots\right]+\cdots\\&-\frac{4}{3\pi}\left[1-\frac{2x}{3\pi}+\frac{2^2x^2}{3^2\pi^2}+\cdots+\frac{(-1)^n2^nx^n}{3^n\pi^n}+\cdots\right]\\&-\frac{4}{7\pi}\left[1-\frac{2x}{7\pi}+\frac{2^2x^2}{7^2\pi^2}+\cdots+\frac{(-1)^n2^nx^n}{7^n\pi^n}+\cdots\right]-\cdots\end{aligned}\tag{23}$$

Another form for $f(x)$ is

$$f(x)=\sum_{k=0}^{\infty}K_n\frac{x^n}{n!}=K_0+K_1x+K_2\frac{x^2}{2!}+\cdots+K_n\frac{x^n}{n!}+\cdots\tag{24}$$

where the coefficients K_n are known as the Euler zigzag numbers. Still another form for $f(x)$ is

$$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}+\sum_{n=1}^{\infty}(-1)^{n-1}2^{2n}\left(2^{2n}-1\right)\frac{B_{2n}}{(2n)!}x^{2n-1}\tag{25}$$

where

$$\sec(x)=\sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}$$

and

$$\tan(x)=\sum_{n=1}^{\infty}(-1)^{n-1}2^{2n}\left(2^{2n}-1\right)\frac{B_{2n}}{(2n)!}x^{2n-1}$$

with E_n and B_n denoting the Euler and Bernoulli numbers.

Comparing like powers of x from equations (23) and (24) one can establish the relation

$$\frac{K_n}{n!}=\frac{2^{n+2}}{\pi^{n+1}}\left[1+\left(-\frac{1}{3}\right)^{n+1}+\left(\frac{1}{5}\right)^{n+1}+\left(-\frac{1}{7}\right)^{n+1}+\left(\frac{1}{9}\right)^{n+1}+\cdots\right] \tag{26}$$

where the right-hand side of the equation is recognized as the β series or λ series, depending upon the value of n. Replace n by $2n$ in equation (26) to obtain

$$\beta(2n+1)=\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}K_{2n} \tag{27}$$

Comparing like powers of x using the equations (24) and (25) one can show

$$K_{2n}=(-1)^n E_{2n} \quad \text{and} \quad K_{2n-1}=\frac{(-1)^{n-1}2^{2n}\left(2^{2n}-1\right)}{2n}B_{2n} \tag{28}$$

which expresses the zigzag numbers in terms of the Euler and Bernoulli numbers. Therefore, the equation (27) can be expressed in the alternative form

$$\beta(2n+1)=\frac{(-1)^n\pi^{2n+1}}{2^{2n+2}(2n)!}E_{2n}, \quad n=1,2,3,\cdots \tag{29}$$

a result also found in references [5] [6].

In equation (26) let $n=2m-1$ and show

$$\lambda(2m)=\frac{K_{2m-1}\pi^{2m}}{(2m-1)!2^{2m+1}}=\frac{(-1)^{m-1}\left(2^{2m}-1\right)(2\pi)^{2m}}{2^{2m+1}(2m)!}B_{2m}$$

and consequently the equation (21) can be written

$$\beta(2m)=\frac{(-1)^{m-1}\left(2^{2m}-1\right)(2\pi)^{2m}}{2^{2m+1}(2m)!}B_{2m}-\frac{2\psi^{(2m-1)}(3/4)}{4^{2m}(2m-1)!} \tag{30}$$

giving a closed form expression for $\beta(2m)$ where $m=1,2,3,\cdots$. Note Catalan's constant is given by

$$\beta(2)=\frac{3(2\pi)^2B_2}{16}-\frac{2\psi^{(1)}(3/4)}{16}=0.915965559\cdots$$

10. Calculation of $\zeta(2k+1)$

Use the results from equations (29), (20) and (22) one can demonstrate the equation

$$\beta(2k+1)+r(2k+1)=\lambda(2k+1)=\left(1-2^{-(2k+1)}\right)\zeta(2k+1)$$

can be expressed in the form

$$\frac{(-1)^k \pi^{2k+1}}{2^{2k+2}(2k)!}E_{2k} + \frac{-2}{4^{2k+1}}\frac{\psi^{(2k)}(3/4)}{(2k)!} = \left(1-2^{-(2k+1)}\right)\zeta(2k+1)$$

for $k=1,2,3,\cdots$. Solving for $\zeta(2k+1)$ one obtains the closed form expression given by equation (10) for the zeta function evaluated at odd positive integers greater than or equal to three.

11. Riemann Zeta Functional Equation

Several derivations of the Riemann zeta functional equation can be found in the reference [3]. One derivation is as follows. Using the definition of the gamma function

$$\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr, \quad \alpha > 0 \tag{31}$$

make the substitutions $\alpha = \frac{s}{2}$ and $r = \pi n^2 x$ to obtain after simplification

$$\frac{\pi^{-s/2}}{n^s}\Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{s/2-1} e^{-\pi n^2 x} dx$$

A summation of both sides of this equation over the index n and interchanging summation and integration one can show the above equation reduces to

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty \frac{1}{n^s} = \int_0^\infty x^{s/2-1}\sum_{n=1}^\infty e^{-\pi n^2 x} dx \tag{32}$$

Here $\sum_{n=1}^\infty \frac{1}{n^s} = \zeta(s)$ is the Riemann zeta function and $\sum_{n=1}^\infty e^{-\pi n^2 x}$ is related to the Jacobi theta function

$$\theta(x) = \sum_{n=-\infty}^\infty e^{-\pi n^2 x} = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 x} \tag{33}$$

Define $\phi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ and express equation (33) in the form

$$\theta(x) = 1 + 2\phi(x)$$

The Jacobi theta function satisfies the property $\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right)$ which can be written in terms of the function ϕ as

$$1 + 2\phi(x) = \frac{1}{\sqrt{x}}\left(1 + 2\phi\left(\frac{1}{x}\right)\right) \quad \text{or} \quad \phi(x) = \frac{1}{\sqrt{x}}\phi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \tag{34}$$

The equation (32) can now be expressed

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{s/2-1}\phi(x) dx = \int_0^1 x^{s/2-1}\phi(x) dx + \int_1^\infty x^{s/2-1}\phi(x) dx$$

The first integral on the right-hand side can be written in a different form as follows.

$$\begin{aligned}\int_0^1 x^{s/2-1}\phi(x) dx &= \int_0^1 x^{s/2-1}\left(\frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right) dx \\ &= \int_0^1 x^{s/2-1}\frac{1}{\sqrt{x}}\phi\left(\frac{1}{x}\right) dx + \int_0^1 \left(\frac{1}{2}x^{s/2-3/2} - \frac{1}{2}x^{s/2-1}\right) dx\end{aligned}$$

which simplifies to

$$\int_0^1 x^{s/2-1}\phi(x)\,\mathrm{d}x = \int_0^1 x^{s/2-3/2}\phi\left(\frac{1}{x}\right)\mathrm{d}x + \frac{1}{s(s-1)}$$

This integral is further simplified by making the substitution $x = \frac{1}{u}$ to obtain

$$\int_0^1 x^{s/2-1}\phi(x)\,\mathrm{d}x = \int_1^\infty u^{-s/2-1/2}\phi(u)\,\mathrm{d}u + \frac{1}{s(s-1)}$$

This last integral allows one to express the equation (32) in the form

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \left(x^{s/2-1} + x^{-(s+1)/2}\right)\phi(x)\,\mathrm{d}x + \frac{1}{s(s-1)} \tag{35}$$

Observe that the right-hand side of equation (35) remains unchanged when s is replaced by $1-s$. This implies

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{36}$$

which is the Riemann zeta functional equation. Multiplication of equation on both sides by $\Gamma\left(\frac{s+1}{2}\right)$ and using the Euler reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

and the Legendre duplication formula

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x)$$

the functional equation can be expressed in the alternative form

$$\pi^{-\left(\frac{1-s}{2}\right)}\frac{\pi}{\sin\left(\pi\left(\frac{s+1}{2}\right)\right)}\zeta(1-s) = \pi^{-s/2}2^{1-s}\sqrt{\pi}\,\Gamma(s)\zeta(s)$$

which simplifies to

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s), \quad Re s > 0 \tag{37}$$

Replacing s by $1-s$ the Riemann zeta functional equation can also be expressed in the form

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \quad Re s < 1 \tag{38}$$

12. Zeta Function for 0 and Negative Integers

The Riemann zeta functional equation is used to demonstrate

$$\zeta(-2n) = 0, \quad n = 1,2,3,\cdots \tag{39}$$

since $\sin(n\pi) = 0$ for all values of the integer n. These values for the zeta function are known as the trivial zeros. The nontrivial zeros lie in the complex plane. Also the Riemann zeta functional equation gives

$$\zeta(-2n+1) = 2^{-2n+1}\pi^{-2n}\Gamma(2n)\zeta(2n)$$

Using the results from equation (7), this simplifies to

$$\zeta(-2n+1) = \frac{-B_{2n}}{2n} \tag{40}$$

where B_n are the Bernoulli numbers.

Using the fact that $B_n = 0$ for odd integers greater than one the equations (39) and (40) can be combined into the form

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \tag{41}$$

for n a positive integer or zero.

This last equation also gives the integer values

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12} \tag{42}$$

Recall the value $\zeta(1)$ does not exist as the series is the harmonic series which diverges for $\sigma = 1$. These values added to the values presented earlier will give the value of the zeta function at integer values, different from 1, along the real line.

For additional representations involving the zeta function in various forms and evaluated at other values the reader is referred to the references [2] [3] [6] [7] [8].

13. Zeros of the Zeta Function

The Euler product formula is used to demonstrate $\zeta(s) \neq 0$ whenever $Re(s) > 1$. The Dirichlet eta function $\eta(s)$ is used to study the zeros of the zeta function for $Re(s) > 0$, $s \neq 1$, since it is related to the zeta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1-2^{1-s}\right)\zeta(s), \quad Re(s) > 0, \quad s \neq 1 \tag{43}$$

The eta function is a converging alternating series for $Re(s) > 0, s \neq 1$ and is sometimes referred to as the alternating zeta function. The equation (43) shows $\eta(s) = 0$ whenever $\zeta(s) = 0$. The factor ($1-2^{1-s}$) is zero at the points $s = 1 + \frac{i2\pi n}{\ln 2}$, for all nonzero integer values for n. These are additional zeros of the eta function.

Writing $\eta(s) = \eta(\sigma + it) = u(\sigma,t) + iv(\sigma,t)$ where for $0 < \sigma < 1$ one can show

$$u(\sigma,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}\cos(t\ln n), \quad v(\sigma,t) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}\sin(t\ln n) \tag{44}$$

and verify that $\frac{\partial u}{\partial \sigma} = \frac{\partial v}{\partial t}, \frac{\partial v}{\partial \sigma} = -\frac{\partial u}{\partial t}$ so the Cauchy-Riemann equations are satisfied. This show $\eta(s)$ is an holomorphic function which satisfies $\eta(\bar{s})=\overline{\eta(s)}$. This implies that if $\eta(s)=0$ for some value of *s*, then its conjugate $\bar{s}$ satisfies $\eta(\bar{s})=0$. This demonstrates that the zeros of the zeta function are symmetric about the σ-axis. The equation $\eta(s)=0$ is satisfied if both the real part u and imaginary part v of η are zero simultaneously. The condition $u=0$ and $v=0$ simultaneously is illustrated in **Figure 1** by plotting $u[\sigma,t]^2+v[\sigma,t]^2$ vs t in the special case where $\sigma=1/2$. The special case $\sigma=1/2$ was selected for **Figure 1** because of the Riemann hypothesis which is a conjecture that the nontrivial zeros of the zeta function have a real part equal to one-half. The values $t_1,t_2,\cdots$ are the values of t where $u=0$ and $v=0$ simultaneously for $\sigma=1/2$. Here $\sigma=1/2$ is called the critical line and the region $0<\sigma<1$ is called the critical strip. To see the first one hundred imaginary parts of the complex zeros one can visit the web site https://wow.Imfdb.org/zeros/zeta/. A huge number of these complex zeros have been calculated and all lie on the critical line where $\sigma=1/2$. Currently there is no proof that all of the nontrivial zeros of the zeta function must lie on the critical line.

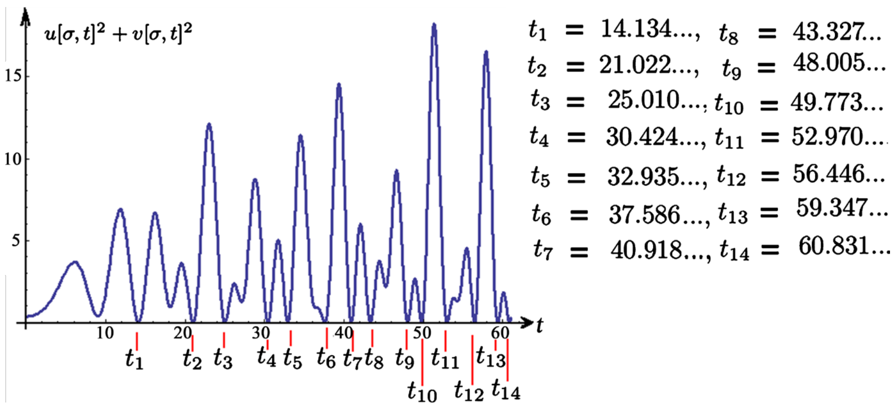

Figure 1. Plot of $u[\sigma,t]^2+v[\sigma,t]^2$ vs *t*, for $0\le t<61$, with $\sigma=1/2$.

14. Conclusion

A closed form expression for the Riemann zeta function evaluated at odd positive integers greater than three has been presented having the form

$$\zeta(2n+1)=\frac{(-4)^n\pi^{2n+1}E_{2n}-2\psi^{(2n)}(3/4)}{2^{2n+1}(2^{2n+1}-1)(2n)!},\quad n=1,2,3,\cdots$$

where E_{2n} are the Euler numbers and $\psi^{(2n)}(3/4)$ are the polygamma functions. It has been demonstrated that knowing closed form expressions for $\zeta(2n)$ and $\zeta(2n+1)$ for $n=1,2,3,\cdots$ one can construct closed form expressions for the Dirichlet eta, lambda and beta series at the even and odd integers different from unity. Closed form representations of the Apéry's constant $\zeta(3)=\dfrac{-4\pi^3E_2-2\psi^{(2)}(3/4)}{112}=1.20205690316\cdots$ and the Catalan's constant

$\beta(2) = \frac{3(2\pi)^2 B_2}{16} - \frac{2\psi^{(1)}(3/4)}{16} = 0.915965559\cdots$ are obtained.

15. The Riemann Hypothesis

The Riemann hypothesis is a conjecture that all nontrivial zeros of the zeta function have a real part equal to one-half. If this is true, all nontrivial zeros are complex numbers of the form $\frac{1}{2} + it$, called the critical line. Whether this is true or not is still an open question.

The Clay Mathematics Institute in Petersborough, New Hampshire is offering a one million dollar prize to anyone who can prove this conjecture and show how to calculate all the zeros of the zeta function. For additional information and conditions to be met in order to win the prize, the reader can consult reference [2] under the search name *Riemann zeta function prize*.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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