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Standing Waves for Quasilinear Schrödinger Equations with Indefinite Nonlinearity

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Abstract

In this article, we consider quasilinear Schrödinger equations of the form

$$-\Delta u+V(x)u-u\Delta(u^2)=f(x,u) \text{ in } \mathbb{R}^N.$$

Such equations have been derived as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics. Unlike all known results in the literature, the nonlinearity is allowed to be indefinite. It is very interesting from physical and mathematical viewpoint. By mountain pass theorem and some special techniques, we prove the existence of solutions for the quasilinear Schrödinger equations with indefinite nonlinearity. This indefinite problem had never been considered so far. So our main results can be regarded as complementary work in the literature.

Keywords

Quasilinear Schrödinger Equations, Indefinite Nonlinearity, Standing Waves

1. Introduction

Solutions of semilinear elliptic equations

$$-\Delta u+V(x)u=a(x)|u|^{p-2}u \text{ in } \mathbb{R}^N \tag{1.1}$$

are standing waves of the corresponding time-dependent Schrödinger. For the existence of solutions of Equation (1.1), one of the important role is the sign of $V(x)$ and $a(x)$. We say Equation (1.1) is linearly indefinite if $V(x)$ changes sign, and superlinearly indefinite if $a(x)$ changes sign. There are many results of Equation (1.1) for the superlinearly indefinite problem, linearly indefinite or not, we refer to [1] [2] [3]. In this paper, we consider the following modified Schrödinger equations

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$$-\Delta u+V(x)u-u\Delta(u^2)=a(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{1.2}$$

This kind of equations arise when we are looking for standing waves $\Psi(t,x)=\mathrm{e}^{-iwt}u(t)$ for the time-dependent quasilinear Schrödinger equation

$$-i\partial_t\Psi=-\Delta\Psi+V(x)\Psi-\left(\Delta|\Psi|^2\right)\Psi-a(x)|\Psi|^{p-2}\Psi$$

which was used for the superfluid film equation in plasma physics by Kurihar [4]. This model also appears in plasma physics and fluid mechanics, dissipative quantum mechanics and condensed matter theory. For more information on the relevance of these models and their deduction, we refer to [5].

To the best of our knowledge, the first mathematical studies of the Equation (1.2) seem to be Poppenberg *et al.* [6] for the one dimensional case and Liu-Wang [7] for higher dimensional case. The proofs in these papers are based on constrained minimization argument. Formally, Equation (1.2) associates with the Euler functional

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(1+2u^2\right)|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^N}a(x)|u|^p\,\mathrm{d}x.$$

Unfortunately, the functional J is not defined for all $u\in X$, unless $N=1$. Therefore, it is difficult to use the standard variational methods to study the functional J. To overcome this difficulty, Jeanjean [8] introduced a transformation f so that if v is a critical point of

$$\Phi(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)f(v)^2\,\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^N}a(x)|f(v)|^p\,\mathrm{d}x, \tag{1.3}$$

where f is defined by

$$f'(t)=\frac{1}{\left(1+2f^2(t)\right)^{\frac{1}{2}}} \text{ on } [0,\infty) \text{ and } f(t)=-f(t) \text{ on } (-\infty,0].$$

Then $u=f(v)$ is a solution of (1.2).

Since the publication of [8], Problem (1.2) has been studied extensively. For example, the case that the potential V is $\mathbb{Z}^N$ is studied in Silva-Vieira [9]. By Nehari manifold method, Fang-Szulkin [10] studied the case that the nonlinearity is 4-superlinear and the potential has a positive lower bound. For problems with critical nonlinearities, see Silva-Vieira [9].

In all these papers, it is required that the potential V and nonlinearity satisfy the positive condition. With this condition and suitable conditions on the nonlinearity, the mountain pass theorem can be applied to produce a solution of (1.2).

In the literature, there are some existence results which allow the potential V to be negative somewhere. The strategy is to write $V=V^+-V^-$ with $V^\pm=\max\{0,\pm V\}$. Then if V^- is in some sense small, it can be absorbed and the functional still verifies the mountain pass geometry. We refer the reader to [11]. Recently, by a local linking argument and Morse theory, Liu-Zhou [12] obtains a nontrivial solution for the problem (1.2) with indefinite potential. For linearly indefinite case, we also refer to [13].

However, this is a gap in the high dimensional quasilinear Schrödinger equations with indefinite nonlinearity. The one dimensional case has been partially studied in [14] by critical point theory. The purpose of this paper is to present some results about indefinite quasilinear Schrödinger equations in higher dimensional. More precisely, we present our assumptions on the potential $V(x)$ and $a(x)$

(V1) $\alpha = \inf_{x\in\mathbb{R}^N} V(x) > 0$;

(V2) $V(x) \in C(\mathbb{R}^N)$ and for each $M > 0$, $\left|\{x \in \mathbb{R}^N \mid V(x) \le M\}\right| < \infty$, where α is a constant and $|A|$ denotes the Lebesgue measure of a measurable set $A \in \mathbb{R}^N$;

(A1) $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\left|\{\Omega^+\}\right| \ne 0$, where $\Omega^+ = \{x \in \mathbb{R}^N \mid a(x) > 0\}$.

(P1) $4 < p < 2\cdot 2^*$ where the critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \ge 3$ and $2^* = \infty$ for $N = 2$.

Then we have

Theorem 1. *Suppose that* $(V1)$, $(V2)$, $(A1)$ *and* $(P1)$ *hold. Then Equation* (1.2) *has at least one nontrivial solutions.*

Notation. $C, C_1, C_2, \cdots$ will denote different positive constants whose exact value is inessential.

2. Preliminaries

Before prove our results, we shall introduce the appropriate space to find critical points of the Euler functional. Let

$$X = \left\{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < \infty\right\}.$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv\mathrm{d}x$$

and the norm

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Then X is a Hilbert space. By Bartsch and Wang [15], we know that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ for is compact for $s \in [2, 2^*)$.

Below we summarize the properties of f in (1.3). Proofs may be found in [8].

Lemma 2.1. *The function f has the following properties*:

(f1) f is uniquely defined, C^∞ and invertible.

(f2) $|f(t)| \le |t|$ and $|f'(t)| \le 1$ for all $t \in \mathbb{R}$. Moreover, $f'(0) = 1$.

(f3) For all $t > 0$ we have $\frac{1}{2} f(t) \le f'(t)t \le f(t)$.

(f4) For all $t \in \mathbb{R}$ we have $f^2(t) \ge f(t)f'(t)t$ and $|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$.

(f5) There exists a positive constant κ such that $|f(t)| > \kappa|t|$ for $|t| \le 1$,

$|f(t)| > \kappa |t|^{\frac{1}{2}}$ for $|t| \ge 1$.

By Lemma 2.1, it is easy to see that $\Phi(v) \in C^1(X)$, moreover

$$\begin{aligned}\langle \Phi'(v), w\rangle &= \int_{\mathbb{R}^N} \nabla u \nabla w + \int_{\mathbb{R}^N} V(x) f(v) f'(v) w \\ &\quad - \int_{\mathbb{R}^N} a(x) |f(v)|^{p-2} f(v) f'(v) w\end{aligned} \tag{2.1}$$

for all $v, w \in X$.

3. Proof of the Theorem 1

Because the principle part of Φ, denoted by

$$Q(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v) \mathrm{d}x$$

is not a quadratic form on *v*, it's not so obvious to verify that Φ satisfies the mountain pass geometry. Similar to [12], by taking into account the Taylor expansion of Q at the origin, it is easy to deduce that $u = 0$ is a strict local minimizer of Φ.

Lemma 3.1. *Under the assumptions of Theorem* 1, *then*

(i) $u = 0$ is a strict local minimizer of Φ.

(ii) There is $\omega \in X, \rho \in \mathbb{R}^+$ with $\|\omega\| > \rho$ such that $\Phi(\omega) < 0$.

Proof. By the properties of the transformation *f*, it is easy to see that Q is a C^2-functional on *X*. Since $f(0) = 0, f'(0) = 1$, we get $Q(0) = Q'(0) = 0$. According to the Taylor formula, as $\|v\|^2 \to 0$, we have

$$\begin{aligned}Q(v) &= \frac{1}{2}\langle Q''(0)v, v\rangle + o\left(\|v\|^2\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)\left(f'(0)^2 + f(0) f''(0)\right) v^2 \mathrm{d}x + o\left(\|v\|^2\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x) v^2 \mathrm{d}x + o\left(\|v\|^2\right).\end{aligned}$$

Therefore, combining this with Lemma 2.1 (f2), there exists $c > 0$ such that

$$\begin{aligned}\Phi(v) &= Q(v) - \frac{1}{p}\int_{\mathbb{R}^N} a(x) |f(v)|^p \,\mathrm{d}x \\ &\ge \frac{1}{2}\|v\|^2 - c\|v\|^p + o\left(\|v\|^2\right).\end{aligned}$$

this implies that the zero function 0 is a strict local minimizer of Φ.

On the other hand, since $|\Omega^+| \ne 0$ and $a(x)$ is continuous in $\mathbb{R}^N$, we may choose $\varphi \in X$ such that $supp\varphi \subset \Omega^+$ and $\varphi(x) \ge 0$ for all $x \in \Omega^+$. Then for any $s > 0$, using Lemma 2.1 (f2),(f5), we deduce

$$\begin{aligned}\Phi(s\varphi) &\le \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(s\varphi) \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} a(x) |f(s\varphi)|^p \,\mathrm{d}x \\ &\le \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x + \frac{s^2}{2}\int_{\mathbb{R}^N} V(x) \varphi^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} a(x) |f(s\varphi)|^p \,\mathrm{d}x \\ &= \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x + \frac{s^2}{2}\int_{\mathbb{R}^N} V(x) \varphi^2 \mathrm{d}x - \frac{1}{p}\int_{\Omega^+ \cap \{x | s\varphi(x) < 1\}} a(x) |f(s\varphi)|^p \,\mathrm{d}x\end{aligned}$$

$$-\frac{1}{p}\int_{\Omega^+\cap\{x|s\varphi(x)\geq 1\}}a(x)|f(s\varphi)|^p\,\mathrm{d}x$$
$$\leq \frac{s^2}{2}\int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}x+\frac{s^2}{2}\int_{\mathbb{R}^N}V(x)\varphi^2\mathrm{d}x-\frac{1}{p}\int_{\Omega^+\cap\{x|s\varphi(x)\geq 1\}}a(x)|f(s\varphi)|^p\,\mathrm{d}x$$
$$\leq \frac{s^2}{2}\int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}x+\frac{s^2}{2}\int_{\mathbb{R}^N}V(x)\varphi^2\mathrm{d}x-\frac{1}{p}s^{\frac{p}{2}}\kappa\int_{\Omega^+\cap\{x|s\varphi(x)\geq 1\}}a(x)|\varphi|^{\frac{p}{2}}\,\mathrm{d}x$$

Since $p>4$, we know that $\Phi(s\varphi)<0$ for s sufficiently large. Thus the conclusion(ii) follows from choosing $\omega=\overline{s}\varphi$ with $\overline{s}$ large. □

Lemma 3.2. *Under the assumptions of Theorem* 1. *Then the functional* Φ *satisfies Cerami condition.*

Proof. Let $\{v_n\}$ be a Cerami sequence of Φ, that is $\Phi(v_n)\to d$, $(1+\|v_n\|)\Phi'(v_n)\to 0$ for some $d\in\mathbb{R}$.

First we claim that there exists $C>0$ such that

$$Q(v_n)\leq C. \tag{3.1}$$

Let $\phi_n=\sqrt{1+2f^2(v_n)}f(v_n)$. By direct computation, we get $|\nabla\phi_n|=\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|$. By (1.3) and (2.1), there exists $C_1>0$ such that

$$\begin{aligned}
d+o(1)&=\Phi(v_n)-\frac{1}{p}\Phi'(v_n)\phi_n\\
&=\int_{\mathbb{R}^N}\left(\frac{1}{2}-\frac{1}{p}\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)\right)|\nabla v_n|^2\,\mathrm{d}x\\
&\quad+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x\\
&\geq\left(\frac{1}{2}-\frac{2}{p}\right)\int_{\mathbb{R}^N}|\nabla v_n|^2+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x\\
&\geq C_1Q(v_n).
\end{aligned}$$

Therefore, our claim is true.

Next, we claim that there exists $C_2>0$ such that

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)f'(v_n)f(v_n)v_n\mathrm{d}x\geq C_2\|v_n\|^2. \tag{3.2}$$

Indeed, we may assume that $v_n\neq 0$ (otherwise the conclusion is trivial). We argue by contradiction and assume that

$$\int_{\mathbb{R}^N}|\nabla w_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)g_n(x)w_n^2\mathrm{d}x\to 0, \tag{3.3}$$

where $w_n=\frac{v_n}{\|v_n\|}$ and $g_n=\frac{f(v_n)f'(v_n)}{v_n}$. By direct computation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f'(t)f(t)\right)=\frac{1}{\left(1+2f^2(t)\right)^{\frac{1}{2}}}>0, \tag{3.4}$$

This implies $f'(t)f(t)$ is strictly increasing. So we get $g_n(x)$ is positive if

$w_n(x) \neq 0$. Combining this with (3.3), we obtain

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 \mathrm{d}x \to 0, \int_{\mathbb{R}^N} V(x) g_n(x) w_n^2 \mathrm{d}x \to 0, \int_{\mathbb{R}^N} V(x) w_n^2 \mathrm{d}x \to 1. \quad (3.5)$$

We claim that for each $\varepsilon > 0$, there exists a constant $C_3 > 0$ independent of n such that $|A_n| < \varepsilon$, where $A_n := \{x \in \mathbb{R}^N : |v_n| \geq C_3\}$. Otherwise, there is an $\varepsilon_0 > 0$ and a subsequence $\{v_{nk}\}$ of $\{v_n\}$ such that for any positive integer *k*, $|A_{nk}| \geq \varepsilon_0 > 0$, where $A_{nk} := \{x : |v_{nk}| \geq k\}$. By the properties of *f* described in Lemma 2.1 and (V1), there exists a constant $C_4 > 0$ such that

$$Q(v_{nk}) \geq \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v_{nk}) \mathrm{d}x \geq \frac{1}{2}\int_{A_{nk}} V(x) f^2(v_{nk}) \mathrm{d}x \geq C_4 k \varepsilon_0 \to \infty \text{ as } k \to \infty,$$

a contradiction. Hence the assertion is true. Then for each $\varepsilon > 0$, C_3 may be chosen so that $|A_n| \leq \varepsilon$. Next, keeping $|v_n| \leq C_3$ in mind. Let $B_n = \mathbb{R}^N / A_n$. By (3.4), as in the proof of the Lemma 3.10 in [10], it is easy to see that as $|t| \leq C_3$, there exists $\delta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f'(t) f(t)\right) \geq \delta > 0.$$

Combining this with (3.3) and the Mean Value Theorem, we have

$$\delta \int_{B_n} V(x) w_n^2 \mathrm{d}x \leq \int_{B_n} V(x) g_n(x) w_n^2 \mathrm{d}x \to 0. \quad (3.6)$$

Since $\int_{\mathbb{R}^N} V(x) w_n^2 \mathrm{d}x$ is uniformly bounded, by the integral absolutely continuity there exists $\varepsilon > 0$ such that whenever $|A_n| \leq \varepsilon$, $\int_{A_n} V(x) w_n^2 \mathrm{d}x < \frac{1}{2}$. For this ε, we have

$$\int_{\mathbb{R}^N} V(x) w_n^2 \mathrm{d}x = \int_{B_n} V(x) w_n^2 \mathrm{d}x + \int_{A_n} V(x) w_n^2 \mathrm{d}x \leq \frac{1}{2} + \int_{B_n} V(x) w_n^2 \mathrm{d}x.$$

This and (3.6) contradict with (3.5). Therefore, this claim is true.

Lastly, together (3.2) and Lemma 2.1(f4) give us

$$Q(v_n) \geq C_2 \|v_n\|.$$

Combining this with (3.1) implies $\|v_n\|$ is bounded in *X*. Up to a subsequence we may assume $v_n \rightharpoonup v$ in *X*. Since embedding $X \hookrightarrow L^p$ is compact for $p \in [2, 2^*)$, by a standard argument, we can show that v_n has a convergent subsequence, see [16] (Theorem 2.1, Step 3). We omit it here. This completes the proof.

To prove Theorem 1, we will apply the following Mountain Pass Theorem [17].

Theorem 2. *Let X be a Banach space and* $\Phi \in C^1(X)$ *be a functional satisfying the Cerami condition. If* $e \in X$ *and* $0 < r < \|e\|$ *are such that*

$$a = \max\{\Phi(0), \Phi(e)\} < \inf_{\|u\|=r} \Phi(u) = b,$$

then

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t))$$

is a critical value of Φ with $c \geq b$, where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Proof of the Theorem 1

Proof. From Lemma 3.1 and Lemma 3.2, we know Φ satisfies the conditions of Theorem 2. Hence Equation (1.2) has at least one nontrivial solution under assumptions (V1), (V2), (A1) and (P1).

4. Conclusion

By mountain pass theorem and Taylor expansion, we prove the existence of solutions for the quasilinear Schrödinger equations with indefinite nonlinearity. This indefinite problem had never been considered so far. So our main results can be regarded as complementary work in the literature. On the other hand, our approach seems much simpler than those presented in [9] [16].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Costa, D.G., Guo, Y. and Ramos, M. (2002) Existence and Multiplicity Results for Nonlinear Elliptic Problems in RN with an Indefinite Functional. *Electronic Journal of Differential Equations*, **2002**, 1-15.

[2] Du, Y. (2004) Multiplicity of Positive Solutions for an Indefinite Superlinear Elliptic Problem on $\mathbb{R}^N$. *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire*, **21**, 657-672. https://doi.org/10.1016/j.anihpc.2003.09.002

[3] Szulkin, A. and Weth, T. (2009) Ground State Solutions for Some Indefinite Variational Problems. *Journal of Functional Analysis*, **257**, 3802-3822. https://doi.org/10.1016/j.jfa.2009.09.013

[4] Kurihura, S. (1981) Large-Amplitude Quasi-Solitons in Superfluid Films. *Journal of the Physical Society of Japan*, **50**, 3262-3267. https://doi.org/10.1143/JPSJ.50.3262

[5] Benguria, R., Brzis, H. and Lieb, E. (1981) The Thomas-Fermi-Von Weizsäcker Theory of Atoms and Molecules. *Communications in Mathematical Physics*, **79**, 167-180. https://doi.org/10.1007/BF01942059

[6] Poppenberg, M., Schmitt, K. and Wang, Z.Q. (2002) On the Existence of Soliton Solutions to Quasilinear Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **14**, 329-344. https://doi.org/10.1007/s005260100105

[7] Liu, J.Q. and Wang, Z.Q. (2003) Soliton Solutions for Quasilinear Schrödinger Equations. *Proceedings of the American Mathematical Society*, **131**, 486-490. https://doi.org/10.1090/S0002-9939-02-06783-7

[8] Colin, M. and Jeanjean, L. (2004) Solutions for a Quasilinear Schrödinger Equation: A Dual Approach. *Nonlinear Analysis*, **56**, 213-226.

https://doi.org/10.1016/j.na.2003.09.008

[9] Silva, E.A.B. and Vieira, G.F. (2010) Quasilinear Asymptotically Periodic Schrödinger Equations with Subcritical Growth. *Nonlinear Analysis*, **72**, 2935-2949. https://doi.org/10.1016/j.na.2009.11.037

[10] Fang, X.D. and Szulkin, A. (2013) Multiple Solutions for a Quasilinear Schrödinger Equation. *Journal of Differential Equations*, **254**, 2015-2032. https://doi.org/10.1016/j.jde.2012.11.017

[11] Zhang, J., Tang, X. and Zhang, W. (2014) Infinitely Many Solutions of Quasilinear Schrödinger Equation with Sign-Changing Potential. *Journal of Mathematical Analysis and Applications*, **420**, 1762-1775. https://doi.org/10.1016/j.jmaa.2014.06.055

[12] Liu, S. and Zhou, J. (2018) Standing Waves for Quasilinear Schrödinger Equations with Indefinite Potentials. *Journal of Differential Equations*, **265**, 3970-3987. https://doi.org/10.1016/j.jde.2018.05.024

[13] Shen, Z. and Han, Z.-Q. (2015) Existence of Solutions to Quasilinear Schrödinger Equations with Indefinite Potential. *Electronic Journal of Differential Equations*, **2015**, 1-9.

[14] Chen, J.Q. (2014) Multiple Positive Solutions to a Class of Modified Nonlinear Schrödinger Equations. *Journal of Mathematical Analysis and Applications*, **415**, 525-542. https://doi.org/10.1016/j.jmaa.2014.01.067

[15] Bartsch, T. and Wang, Z.Q. (1995) Existence and Multiplicity Results for Some Superlinear Elliptic Problems on $\mathbb{R}^N$. *Communications in Partial Differential Equations*, **20**, 1725-1741. https://doi.org/10.1080/03605309508821149

[16] Wu, X. (2014) Multiple Solutions for Quasilinear Schrödinger Equations with a Parameter. *Journal of Differential Equations*, **256**, 2619-2632. https://doi.org/10.1016/j.jde.2014.01.026

[17] Bartsch, T. and Li, S. (1997) Critical Point Theory for Asymptotically Quadratic Functionals and Applications to Problems with Resonance. *Nonlinear Analysis*, **28**, 419-441. https://doi.org/10.1016/0362-546X(95)00167-T