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On Two Classes of Extended 3-Lie Algebras

Yu Cheng[1], **Yansha Gao**[2]

[1]School of Data Science and Software Engineering, Baoding University, Baoding, China
[2]College of Mathematics and Information Science, Hebei University, Baoding, China
Email: chengyu72@sina.com

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Abstract

In this paper, based on the existing research results, we obtain the unary extension 3-Lie algebras by one-dimensional extension of the known Lie algebra L. For two known 3-Lie algebras H, M, the (μ,ρ,β)-extension of H through M is given, and the necessary and sufficient conditions for the (μ,ρ,β)-extension algebra of H through M being 3-Lie algebra are obtained, and the structural characteristics and properties of these two kinds of extended 3-Lie algebras are given.

Keywords

The Unary Extension 3-Lie Algebras, Lie Algebra, (μ,ρ,β)-Extension

1. Introduction

In recent years, the study of 3-Lie algebra has been paid much attention because of its wide application in mathematics and physics. 3-Lie algebra is a special form of n-Lie algebra, which is an algebraic system with ternary linearly oblique symmetric multiplication table satisfying the generalized Jacobi equation [1]. 3-Lie algebra has extremely profound and rich algebraic and analytical structure. In this paper, the extension problem of 3-Lie algebra is studied on the basis of the existing research. Firstly, we define the unary extended 3-Lie algebra for a known Lie algebra L by one-dimensional extension, and study its properties. Secondly, for two known 3-Lie algebras H, M, the (μ,ρ,β)-extension of H through M is defined, and the (μ,ρ,β)-extension of H through M is given as a necessary and sufficient condition for the 3-Lie algebra. Finally, the structure and properties of this extended 3-Lie algebra are discussed. Thus, it lays a foundation for the further study of the properties of the derivatives of two kinds of 3-Lie algebras.

2. Fundamental Notions

Firstly, the basic knowledge [1]-[9] to be used in this paper is given.

Definition 2.1 Let A be a vector space over a domain F and have a 3-element linear operation $[\ ,\ ,\ ]: A \wedge A \wedge A \to A$, satisfied for arbitrary, $x_1, x_2, x_3, y_2, y_3 \in A$

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3} [x_1, [x_i, y_2, y_3], x_3], \tag{1}$$

$(A, [\ ,\ ,\ ])$ is called 3-Lie algebra. Without confusion, A is called 3-Lie algebra for short.

Definition 2.2 Let A be a 3-Lie algebra, and D be a linear transformation of A, if this equation is satisfied

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D([x, y, z]), \quad x, y, z \in A \tag{2}$$

Then D is the derivative of A, and the set of derivatives is denoted by $Der(A)$. It is easy to prove that $Der(A)$ is a subalgebra of the general linear Lie algebra $gl(A)$.

The map

$$ad(x_1, x_2): A \to A, \quad ad(x_1, x_2)(x) = [x_1, x_2, x]$$

for $x \in A$ is called the left multiplication defined by elements $x_1, x_2 \in A$. Obviously the left multiplication is the derivative. The linear combination of the left multiplication is called the inner derivative, denoted by $ad(A)$.

Let B be a subspace of A, and if $[B, B, B] \subseteq B([B, A, A] \subseteq B)$, then B be a subalgebra (ideal) of A. And if $[B, B, B] = 0([B, B, A] = 0)$, then B is called a *Abel* subalgebra

(*Abel* ideal). In particular, the subalgebra spanned by $[x_1, x_2, x_3](\forall x_1, x_2, x_3 \in A)$ is called the derivative algebra of A, denoted by A^1. If $A^1 = 0$, then A is called *Abel* algebra. If an ideal I of A is a *Abel* subalgebra but not an *Abel* ideal, that is $[I, I, I] = 0$, but $[I, I, A] \neq 0$, then I is called an *hypo-abelian* ideal.

The ideal I of a 3-Lie algebra A is called s-solvable, $2 \le s \le 3$, if $I^{(k,s)} = 0$ for some $k \ge 0$, where $I^{(0,s)} = I$, $I^{(k+1,s)}$ is defined as

$I^{(k+1,s)} = \Big[\underbrace{I^{(k,s)}, \cdots, I^{(k,s)}}_{s}, \underbrace{A, \cdots, A}_{3-s}\Big]$. Where 2-solvable is also called solvable, and $I^{(k,s)}$ is abbreviated as $I^{(k)}$.

An ideal I of a 3-Lie algebra A is called nilpotent if $I^s = 0$ for some $s \ge 0$, where $I^0 = I$ and $I^s = [I^{s-1}, I, A]$.

The center of A is denoted by $Z(A) = \{x \in A | [x, A, A] = 0\}$. Obviously $Z(A)$ is the *Abel* ideal of A.

Let A is a 3-Lie algebra over the field F, V is a vector space, $\rho: A \wedge A \to End(V)$ is a linear mapping, if ρ satisfies for any $x_1, x_2, x_3, x_4 \in A$

$$[\rho(x_1, x_2), \rho(x_3, x_4)] = \rho([x_1, x_2, x_3]x_4) - \rho([x_1, x_2, x_4]x_3), \tag{3}$$

$$\begin{aligned}\rho([x_1, x_2, x_3], x_4) = {} & \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) \\ & + \rho(x_3, x_1)\rho(x_2, x_4)\end{aligned} \tag{4}$$

Then (V,ρ) is called the representation of A (or (V,ρ) is A-module).

Lemma 2.1 Let A is a 3-Lie algebra over the field F, V is a vector space, $\rho: A\wedge A \to End(V)$ is a linear mapping. If (V,ρ) is an A-module, then for any $x,y,z,u\in A$, the following equation is true:

$$\rho([x,y,z],u)-\rho([x,y,u],z)+\rho([x,z,u],y)-\rho([y,z,u],x)=0, \quad (5)$$

$$\begin{aligned}&\rho(x,u)\rho(y,z)+\rho(y,z)\rho(x,u)+\rho(x,y)\rho(z,u)+\rho(z,u)\rho(x,y)\\&-\rho(x,z)\rho(y,u)-\rho(y,u)\rho(x,z)=0.\end{aligned} \quad (6)$$

3. The Unary Extension 3-Lie Algebra of Lie Algebras

Definition 3.1 Let $(L,[\ ,\ ])$ be a Lie algebra over a field F, let $A=L\oplus Fx_0$ $x_0\in F$, and $x_0\notin L$. Linear operation $[\ ,\ ,\ ]: A\wedge A\wedge A\to A$ for all $x,y,z\in L$ that satisfy the following multiplication table:

$$[x,y,x_0]=[x,y],[x,y,z]=0. \quad (7)$$

Then A is called the unary extension of Lie algebra L. If $(A,[\ ,\ ,\ ])$ is a 3-Lie algebra, then $(A,[\ ,\ ,\ ])$ is a unary extension 3-Lie algebra of the Lie algebra L.

Lemma 3.1 let L be a Lie algebra over a field F. If let $A=L\oplus Fx_0$ $x_0\in F$, $x_0\notin L$ and the multiplication of is defined by (7), then A is a 3-Lie algebra, and for positive integers m, the following equation holds

$$A^{(m)}=L^{(m)}, A^{(m,2)}=L^{(m,2)}=L^{(m)}, A^{(2,3)}=0.$$

Proof: By multiplication (7), direct calculation A is 3-Lie algebra. Due to the

$$A^1=[A,A,A]=[L,L,L]+[L,L,Fx_0]=L^1,$$

$$A^2=\left[A^1,A,A\right]=\left[L^1,L,Fx_0\right]=L^2,$$

Assume $A^{m-1}=L^{m-1}$, then

$$A^m=\left[A^{m-1},A,A\right]=\left[L^{m-1},L+Fx_0,L+Fx_0\right]=\left[L^{m-1},L\right]=L^m.$$

similarly, $A^{(m,2)}=L^{(m,2)}=L^{(m)}$ and $A^{(2,3)}=0$. The conclusion is proved.

Theorem 3.1 Let L be a Lie algebra on the field F and $A=L\oplus Fx_0$ be a unary extension 3-Lie algebra, where $x_0\in F$ and $x_0\notin L$, then

1) A is 2-solvable if and only if L is a solvable Lie algebra.

2) A is nilpotent if and only if L is a nilpotent Lie algebra.

3) A is 3-solvable.

4) $Z(A)=Z(L)$.

Proof: According to lemma 3.1, (1), (2) and (3) can be obtained directly. It is proved below that (4) is true. If $L^1=0$, then $A^1=L^1=0$ and $Z(A)=Z(L)$. If $L^1\neq 0$, then exists $y,z\in L$ such that $[y,z]\neq 0$. For any $x\in L$, $\lambda\in F$, $x+\lambda x_0\in Z(A)$, because of $[x+\lambda x_0,y,z]=\lambda[y,z]=0$, therefore $\lambda=0$ .And because $[x+\lambda x_0,A,x_0]=[x,L]=0$, so $x\in Z(L)$. Therefore $Z(A)\subseteq Z(L)$. Obviously, the conclusion of $Z(L)\subseteq Z(A)$ is true.

Theorem 3.2 Let L be a Lie algebra on the field F and I be a subspace of L:

1) I is an ideal of A if and only if I is an ideal of L.

2) Let $J = I \oplus Fx_0$, then J is ideal of A if and only if $L^1 \subseteq I$.

3) If $L^1 \subseteq I$, then for positive integers m, $J^{(m,2)} \subseteq I^{(m-1)}$. If I is a solvable ideal of L, then J is a 2-solvable ideal of A.

4) If L is a simple Lie algebra, then L is *hypo-abelian* ideal of A.

Proof: From $[I, A, A] = [I, L, x_0] = [I, L]$, we can get (1). From Equation (7),

$$[J, A, A] = [I, L, x_0] + [x_0, L, L] = [I, L] + [L, L],$$

So $[J, A, A] \subseteq J$ if and only if $[L, L] \subseteq I$. That means (2) is true.

If I is the ideal of L and $L^1 \subseteq I$, then

$$J^{(1,2)} = [J, J, A] = [I, I] + [I, L] = I^{(1)} + [I, L] \subseteq I^{(1)} + I \subseteq I = I^{(0)},$$

$$J^{(2,2)} = \left[J^{(1,2)}, J^{(1,2)}, A\right] = [I, I, L + Fx_0] \subseteq I^{(1)},$$

Assuming $J^{(m-1,2)} \subseteq I^{(m-2)}$ is true, then

$$J^{(m,2)} = \left[J^{(m-1,2)}, J^{(m-1,2)}, A\right] \subseteq \left[I^{(m-2)}, I^{(m-2)}, L + Fx_0\right] \subseteq I^{(m-1)}.$$

Therefore (3) holds. If L is a simple Lie algebra, then L is ideal of A, and $[L, L, L] = 0$, $[L, L, A] = [L, L, x_0] = L^1 \neq 0$. Therefore, L is *hypo-abelian* ideal of A. That's the end of the argument.

4. (μ, ρ, β)-Extension of 3-Lie Algebras

Definition 4.1 Let $(H, [\ ,\ ,\ ]_H)$ and $(M, [\ ,\ ,\ ]_M)$ be two 3-Lie algebras over the field F, $A = M \oplus H$, and

$$\rho : M \wedge M \to Der(H), \quad \beta : M \wedge H \to Der(H), \quad \mu : M \wedge M \wedge M \to H$$

is linear mappings. Define a linear operation $[\ ,\ ,\ ]_{\mu\rho\beta} : A \wedge A \wedge A \to A$, for any $x, y, z \in M$, $h, h_1, h_2 \in H$ that satisfies the multiplication table:

$$[x, y, z]_{\mu\rho\beta} = [x, y, z]_M + \mu(x, y, z), \quad [x, y, h]_{\mu\rho\beta} = \rho(x, y)h \tag{8}$$

$$[h_1, h_2, h_3]_{\mu\rho\beta} = [h_1, h_2, h_3]_H, \quad [x, h_1, h_2]_{\mu\rho\beta} = \beta(x, h_1)h_2.$$

Then $(A, [\ ,\ ,\ ]_{\mu\rho\beta})$ is called the (μ, ρ, β)-extension of H through M. If $(A, [\ ,\ ,\ ]_{\mu\rho\beta})$ is a 3-Lie algebra, then $(A, [\ ,\ ,\ ]_{\mu\rho\beta})$ is (μ, ρ, β)-extension algebra of 3-Lie algebra. If $\beta = 0$, then A is called (μ, ρ)-extension of H through M, and $[\ ,\ ,\ ]_{\mu\rho\beta}$ denoted as $[\ ,\ ,\ ]_{\mu\rho}$. For convenience, we will abbreviate $[\ ,\ ,\ ]_M$ and $[\ ,\ ,\ ]_H$ as $[\ ,\ ,\ ]$ and $[\ ,\ ,\ ]_{\mu\rho\beta}$ as $[\ ,\ ,\ ]_A$.

Lemma 4.1 Let $(H, [\ ,\ ,\ ]_H)$ and $(M, [\ ,\ ,\ ]_M)$ be two 3-Lie algebras over the field F, and A be the (μ, ρ, β)-extension of H through M, and for all $x_1, x_2, x_3, x_4 \in M$ satisfy

$$\begin{aligned}\rho(x_4, [x_1, x_2, x_3]) = {} & \rho(x_3, x_1)\rho(x_4, x_2) - \rho(x_2, x_1)\rho(x_4, x_3) \\ & + \rho(x_2, x_3)\rho(x_4, x_1) - \beta(x_4, \mu(x_1, x_2, x_3)).\end{aligned} \tag{9}$$

Then Equation (6) is true if and only if the following equation

$$\begin{aligned}&\rho\left(x_4,\left[x_1,x_2,x_3\right]\right)\\&=\rho\left(x_3,\left[x_1,x_2,x_4\right]\right)-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)\\&\quad-\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)+\rho\left(x_3,x_4\right)\rho\left(x_1,x_2\right).\end{aligned}\tag{10}$$

Proof: From Equation (9), we can get

$$\begin{aligned}\rho\left(x_3,\left[x_1,x_2,x_4\right]\right)=&\rho\left(x_2,x_4\right)\rho\left(x_3,x_1\right)-\rho\left(x_1,x_4\right)\rho\left(x_3,x_2\right)\\&+\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)-\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right),\end{aligned}$$

$$\begin{aligned}&\rho\left(x_4,\left[x_1,x_2,x_3\right]\right)-\rho\left(x_3,\left[x_1,x_2,x_4\right]\right)\\&=\rho\left(x_1,x_3\right)\rho\left(x_2,x_4\right)-\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)-\rho\left(x_2,x_3\right)\rho\left(x_1,x_4\right)\\&\quad-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)+\rho\left(x_2,x_4\right)\rho\left(x_1,x_3\right)-\rho\left(x_1,x_4\right)\rho\left(x_2,x_3\right)\\&\quad-\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)\\&=\rho\left(x_1,x_3\right)\rho\left(x_2,x_4\right)+\rho\left(x_2,x_4\right)\rho\left(x_1,x_3\right)-\rho\left(x_2,x_3\right)\rho\left(x_1,x_4\right)\\&\quad-\rho\left(x_1,x_4\right)\rho\left(x_2,x_3\right)-2\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)\\&\quad+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right).\end{aligned}$$

So Equation (10) holds. On the other hand, if

$$\begin{aligned}\rho\left(x_4,\left[x_1,x_2,x_3\right]\right)=&\rho\left(x_2,x_4\right)\rho\left(x_3,x_1\right)-\rho\left(x_1,x_4\right)\rho\left(x_3,x_2\right)\\&+\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)-\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)\\&-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)\\&-\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)+\rho\left(x_3,x_4\right)\rho\left(x_1,x_2\right).\end{aligned}$$

Through Equation (9), it can be concluded that Equation (6) holds.

Lemma 4.2. Let A be the (μ,ρ,β)-extension of H through M, for all $x_1,x_2\in M$, $h_1,h_2,h\in H$ satisfies

$$\begin{aligned}&\beta\left(y,h_2\right)\beta\left(x,h_1\right)h-\beta\left(y,h\right)\beta\left(x,h_1\right)h_2-\beta\left(x,h_1\right)\beta\left(y,h_2\right)h\\&=\left[\rho(x,y)h_1,h_2,h\right]\end{aligned}\tag{11}$$

There are

$$\begin{aligned}&\rho(x,y)\left[h_1,h_2,h\right]+\beta\left(y,h_1\right)\beta\left(x,h_2\right)h-\beta\left(x,h_1\right)\beta\left(y,h_2\right)h\\&=\left[\rho(x,y)h_1,h_2,h\right]\end{aligned}\tag{12}$$

Proof: From Equation (11) and the $\rho(x,y)$ is derivative of H, we can get

$$\begin{aligned}&\left[h_1,\rho(x,y)h_2,h\right]\\&=\beta\left(y,h_1\right)\beta(x,h)h_2+\beta(y,h)\beta\left(x,h_2\right)h_1+\beta\left(x,h_2\right)\beta\left(y,h_1\right)h\\&\left[h_1,h_2,\rho(x,y)h\right]\\&=\beta\left(y,h_2\right)\beta\left(x,h_1\right)h+\beta\left(y,h_1\right)\beta(x,h)h_2+\beta(x,h)\beta\left(y,h_2\right)h_1\end{aligned}$$

$$\begin{aligned}&\rho(x,y)\left[h_1,h_2,h\right]\\&=2\left(\beta\left(y,h_1\right)\beta(x,h)h_2+\beta\left(y,h_2\right)\beta\left(x,h_1\right)h+\beta(y,h)\beta\left(x,h_2\right)h_1\right)\\&\quad+\beta\left(x,h_1\right)\beta(y,h)h_2+\beta\left(x,h_2\right)\beta\left(y,h_1\right)h+\beta(x,h)\beta\left(y,h_2\right)h_1,\end{aligned}$$

so

$$\beta(y,h_1)\beta(y,h)h_2+\beta(y,h_2)\beta(x,h_1)h+\beta(y,h)\beta(x,h_2)h_1$$
$$+\beta(x,h_1)\beta(y,h)h_2+\beta(x,h_2)\beta(y,h_1)h+\beta(y,h)\beta(y,h_2)h_1=0,$$
$$\rho(x,y)[h_1,h_2,h]$$
$$=\beta(y,h_1)\beta(x,h)h_2+\beta(y,h_2)\beta(x,h_1)h+\beta(y,h)\beta(x,h_2)h_1$$

Namely

$$\beta(y,h_2)\beta(x,h_1)h-\beta(y,h)\beta(x,h_1)h_2$$
$$=\rho(x,y)[h_1,h_2,h]+\beta(y,h_1)\beta(x,h_2)h.$$

Using Equation (11) again, Equation (12) can be obtained.

Lemma 4.3. Let A be the (μ,ρ,β)-extension of H through M. If for all $x\in M$, $h_1,h_2,h_3,h_4\in H$ satisfies

$$ad(\beta(x,h_1)h_3,h_2)+ad(h_3,\beta(x,h_1)h_2)+ad(\beta(x,h_3)h_2,h_1)$$
$$=\beta(x,[h_1,h_2,h_3]), \tag{13}$$

Then

$$[h_1,h_2,\beta(x,h_3)h_4]-\beta(x,[h_1,h_2,h_3])h_4-[h_3,h_4,\beta(x,h_1)h_2]$$
$$=\beta(x,h_3)[h_1,h_2,h_4]. \tag{14}$$

Proof: According to Equation (13),

$$\beta(x,[h_1,h_2,h_3])h_4$$
$$=[\beta(x,h_1)h_3,h_2,h_4]+[h_3,\beta(x,h_1)h_2,h_4]+[\beta(x,h_3)h_2,h_1,h_4].$$

Because of $\beta(x,h_3)\in Der(H)$, therefore

$$\beta(x,[h_1,h_2,h_3])h_4$$
$$=[\beta(x,h_1)h_3,h_2,h_4]+[h_3,\beta(x,h_1)h_2,h_4]+[\beta(x,h_3)h_2,h_1,h_4]$$
$$=-[\beta(x,h_3)h_1,h_2,h_4]+[h_3,\beta(x,h_1)h_2,h_4]+[\beta(x,h_3)h_2,h_1,h_4]$$
$$=-\beta(x,h_3)[h_1,h_2,h_4]+[h_1,h_2,\beta(x,h_3)h_4]+[h_3,\beta(x,h_1)h_2,h_4].$$

Hence, Equation (14) holds.

Theorem 4.1. Let A be the (μ,ρ,β)-extension of H through M, then A is a 3-Lie algebra if and only if for any $x_1,x_2,x_3,x_4,x_5\in M$, $h_1,h_2\in H$, Equations (6), (9), (11), (13) and the following are true,

$$[\mu(x_1,x_2,x_3),h_1,h_2]$$
$$=\rho(x_2,x_3)\beta(x_1,h_1)h_2-\rho(x_1,x_3)\beta(x_2,h_1)h_2$$
$$+\rho(x_1,x_2)\beta(x_3,h_1)h_2-\beta([x_1,x_2,x_3],h_1)h_2, \tag{15}$$

$$\beta(x_1,h_1)\rho(x_2,x_3)h_2+\beta(x_3,h_2)\rho(x_1,x_2)h_1$$
$$=\rho(x_2,x_3)\beta(x_1,h_1)h_2+\beta(x_2,h_2)\rho(x_1,x_3)h_1, \tag{16}$$

$$\mu(x_1,x_2,[x_3,x_4,x_5])-\mu([x_1,x_2,x_3],x_4,x_5)$$
$$-\mu(x_3,[x_1,x_2,x_4],x_5)-\mu(x_3,x_4,[x_1,x_2,x_5])$$
$$=\rho(x_3,x_4)\mu(x_1,x_2,x_5)-\rho(x_3,x_5)\mu(x_1,x_2,x_4)$$
$$-\rho(x_1,x_2)\mu(x_3,x_4,x_5)+\rho(x_4,x_5)\mu(x_1,x_2,x_3). \tag{17}$$

Proof: If A is a 3-Lie algebra, the Equations (11), (15), (16) and (17) are obtained from the Equations (1). The following proves that Equations (6), (9) and (13) are true.

For $x_i \in M, i = 1,2,3,4$, $h \in H$, according to (8),

$$\begin{aligned}&\left[h, x_1, [x_2, x_3, x_4]_A\right]_A \\ &= \left[h, x_1, [x_2, x_3, x_4]\right]_A + \left[h, x_1, \mu(x_2, x_3, x_4)\right]_A \\ &= \rho\left(x_1, [x_2, x_3, x_4]\right)h + \beta\left(x_1, \mu(x_2, x_3, x_4)\right)h\end{aligned}$$

$$\begin{aligned}&\left[[h, x_1, x_2]_A, x_3, x_4\right]_A + \left[x_2, [h, x_1, x_3]_A, x_4\right]_A + \left[x_2, x_3, [h, x_1, x_4]_A\right]_A \\ &= \rho(x_3, x_4)\rho(x_1, x_2)h - \rho(x_2, x_4)\rho(x_1, x_3)h + \rho(x_2, x_3)\rho(x_1, x_4)h\end{aligned}$$

As a result,

$$\begin{aligned}&\rho\left(x_1, [x_2, x_3, x_4]\right) + \beta\left(x_1, \mu(x_2, x_3, x_4)\right) \\ &= \rho(x_3, x_4)\rho(x_1, x_2) - \rho(x_2, x_4)\rho(x_1, x_3) + \rho(x_2, x_3)\rho(x_1, x_4).\end{aligned}$$

In the above formula, x_1, x_2, x_3, x_4 is replaced by x_4, x_1, x_2, x_3, and Equation (9) can be obtained.

Because, $\left[x_1, x_2, [x_3, x_4, h]_A\right]_A = \rho(x_1, x_2)\rho(x_3, x_4)h$

$$\begin{aligned}&\left[[x_1, x_2, x_3]_A, x_4, h\right]_A + \left[x_3, [x_1, x_3, x_4]_A, h\right]_A + \left[x_3, x_4, [x_1, x_2, h]_A\right]_A \\ &= \rho\left([x_1, x_2, x_3], x_4\right)h - \beta\left(x_4, \mu(x_1, x_2, x_3)\right)h + \rho\left(x_3, [x_1, x_2, x_4]\right)h \\ &\quad + \beta\left(x_3, \mu(x_1, x_2, x_4)\right)h + \rho(x_3, x_4)\rho(x_1, x_2)h.\end{aligned}$$

So Equation (10) holds. Equation (6) is obtained from lemma 4.1.

For arbitrary $h_i \in H, i = 1,2,3,4$, $x \in M$, it can be known from (8) that,

$$\left[h_1, h_2, [h_3, h_4, x]_A\right]_A = \left[h_1, h_2, \beta(x, h_3)h_4\right],$$

$$\begin{aligned}&\left[[h_1, h_2, h_3]_A, h_4, x\right]_A + \left[h_3, [h_1, h_2, h_4]_A, x\right]_A + \left[h_3, h_4 [h_1, h_2, x]_A\right]_A \\ &= \beta\left(x, [h_1, h_2, h_3]\right)h_4 + \beta(x, h_3)[h_1, h_2, h_4] + \left[h_3, h_4, \beta(x, h_1)h_2\right],\end{aligned}$$

As a result,

$$\begin{aligned}&\left[h_1, h_2, \beta(x, h_3)h_4\right] \\ &= \beta\left(x, [h_1, h_2, h_3]\right)h_4 + \beta(x, h_3)[h_1, h_2, h_4] + \left[h_3, h_4, \beta(x, h_1)h_2\right].\end{aligned}$$

Because of $\beta(x, h_3) \in Der(H)$, therefore

$$\begin{aligned}&\beta\left(x, [h_1, h_2, h_3]\right)h_4 \\ &= \left[h_1, h_2, \beta(x, h_3)h_4\right] - \beta(x, h_3)[h_1, h_2, h_4] - \left[h_3, h_4, \beta(x, h_1)h_2\right] \\ &= -\left[\beta(x, h_3)h_1, h_2, h_4\right] - \left[h_1, \beta(x, h_3)h_2, h_4\right] + \left[h_3, \beta(x, h_1)h_2, h_4\right] \\ &= ad\left(\beta(x, h_1)h_3, h_2\right)h_4 + ad\left(h_3, \beta(x, h_1)h_2\right)h_4 + ad\left(\beta(x, h_3)h_2, h_1\right)h_4.\end{aligned}$$

Equation (13) holds.

Conversely, to prove that A is a 3-Lie algebra, it is only necessary to prove that (8) satisfies Equation (1).

Case 1. For all $x_i \in M, i = 1,2,3,4,5$, known by (8)

$$\begin{aligned}&\left[x_1,x_2,\left[x_3,x_4,x_5\right]_A\right]_A\\&=\left[x_1,x_2,\left[x_3,x_4,x_5\right]\right]+\mu\left(x_1,x_2,\left[x_3,x_4,x_5\right]\right)+\rho\left(x_1,x_2\right)\mu\left(x_3,x_4,x_5\right),\end{aligned}$$

$$\begin{aligned}&\left[\left[x_1,x_2,x_3\right]_A,x_4,x_5\right]_A+\left[x_3,\left[x_1,x_2,x_4\right]_A,x_5\right]_A+\left[x_3,x_4,\left[x_1,x_2,x_5\right]_A\right]_A\\&=\left[\left[x_1,x_2,x_3\right],x_4,x_5\right]+\mu\left(\left[x_1,x_2,x_3\right],x_4,x_5\right)+\rho\left(x_4,x_5\right)\mu\left(x_1,x_2,x_3\right)\\&\quad+\left[x_3,\left[x_1,x_2,x_4\right],x_5\right]+\mu\left(x_3,\left[x_1,x_2,x_4\right],x_5\right)+\rho\left(x_5,x_3\right)\mu\left(x_1,x_2,x_4\right)\\&\quad+\left[x_3,x_4,\left[x_1,x_2,x_5\right]\right]+\mu\left(x_3,x_4,\left[x_1,x_2,x_5\right]\right)+\rho\left(x_3,x_4\right)\mu\left(x_1,x_2,x_5\right).\end{aligned}$$

From Equation (17), we can get

$$\begin{aligned}&\left[x_1,x_2,\left[x_3,x_4,x_5\right]_A\right]_A\\&=\left[\left[x_1,x_2,x_3\right]_A,x_4,x_5\right]_A+\left[x_3,\left[x_1,x_2,x_4\right]_A,x_5\right]_A+\left[x_3,x_4,\left[x_1,x_2,x_5\right]_A\right]_A.\end{aligned}$$

Case 2. For all $x_i\in M, i=1,2,3,4$, $h\in H$, know from (8)

$$\begin{aligned}\left[h,x_1,\left[x_2,x_3,x_4\right]_A\right]_A&=\left[h,x_1,\left[x_2,x_3,x_4\right]\right]_A+\left[h,x_1,\mu\left(x_2,x_3,x_4\right)\right]_A\\&=\rho\left(x_1,\left[x_2,x_3,x_4\right]\right)h+\beta\left(x_1,\mu\left(x_2,x_3,x_4\right)\right)h\end{aligned}$$

$$\begin{aligned}&\left[\left[h,x_1,x_2\right]_A,x_3,x_4\right]_A+\left[x_2,\left[h,x_1,x_3\right]_A,x_4\right]_A+\left[x_2,x_3\left[h,x_1,x_4\right]_A\right]_A\\&=\rho\left(x_3,x_4\right)\rho\left(x_1,x_2\right)h-\rho\left(x_2,x_4\right)\rho\left(x_1,x_3\right)h+\rho\left(x_2,x_3\right)\rho\left(x_1,x_4\right)h.\end{aligned}$$

In Equation (9), by x_1,x_2,x_3,x_4 substitution for x_4,x_1,x_2,x_3, we can get

$$\begin{aligned}\rho\left(x_1,\left[x_2,x_3,x_4\right]\right)&=\rho\left(x_3,x_4\right)\rho\left(x_1,x_2\right)-\rho\left(x_2,x_4\right)\rho\left(x_1,x_3\right)\\&\quad+\rho\left(x_2,x_3\right)\rho\left(x_1,x_4\right)-\beta\left(x_1,\mu\left(x_2,x_3,x_4\right)\right).\end{aligned}$$

As a result,

$$\begin{aligned}&\left[h,x_1,\left[x_2,x_3,x_4\right]_A\right]_A\\&=\left[\left[h,x_1,x_2\right]_A,x_3,x_4\right]_A+\left[x_2,\left[h,x_1,x_3\right]_A,x_4\right]_A+\left[x_2,x_3,\left[h,x_1,x_4\right]_A\right]_A.\end{aligned}$$

Due to the $\left[x_1,x_2,\left[x_3,x_4,h\right]_A\right]_A=\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)h$,

$$\begin{aligned}&\left[\left[x_1,x_2,x_3\right]_A,x_4,h\right]_A+\left[x_3,\left[x_1,x_2,x_4\right]_A,h\right]_A+\left[x_3,x_4,\left[x_1,x_2,h\right]_A\right]_A\\&=\rho\left(\left[x_1,x_2,x_3\right]_A,x_4\right)h-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)h+\rho\left(x_3,\left[x_1,x_2,x_4\right]\right)h\\&\quad+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)h+\rho\left(x_3,x_4\right)\rho\left(x_1,x_2\right)h.\end{aligned}$$

Through lemma 4.1 and Equation (9), we can get

$$\begin{aligned}&\rho\left(x_4,\left[x_1,x_2,x_3\right]\right)-\rho\left(x_3,\left[x_1,x_2,x_4\right]\right)\\&=\rho\left(x_2,x_3\right)\rho\left(x_4,x_1\right)-\rho\left(x_1,x_3\right)\rho\left(x_4,x_2\right)+\rho\left(x_1,x_2\right)\rho\left(x_4,x_3\right)\\&\quad-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)-\rho\left(x_2,x_4\right)\rho\left(x_3,x_1\right)+\rho\left(x_1,x_4\right)\rho\left(x_3,x_2\right)\\&\quad-\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right)+\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)\\&=\beta\left(x_3,\mu\left(x_1,x_2,x_4\right)\right)-\beta\left(x_4,\mu\left(x_1,x_2,x_3\right)\right)-\rho\left(x_2,x_3\right)\rho\left(x_1,x_4\right)\\&\quad-\rho\left(x_1,x_4\right)\rho\left(x_2,x_3\right)+\rho\left(x_1,x_3\right)\rho\left(x_2,x_4\right)\\&\quad+\rho\left(x_2,x_4\right)\rho\left(x_1,x_3\right)-2\rho\left(x_1,x_2\right)\rho\left(x_3,x_4\right).\end{aligned}$$

According to Equations (6) and (10),

$$\left[x_1,x_2,\left[x_3,x_4,h\right]_A\right]_A$$
$$=\left[\left[x_1,x_2,x_3\right]_A,x_4,h\right]_A+\left[x_3,\left[x_1,x_2,x_4\right]_A,h\right]_A+\left[x_3,x_4,\left[x_1,x_2,h\right]_A\right]_A.$$

Case 3. For all $x_i \in M, i=1,2,3$, $h_1,h_2 \in H$, it is obtained from Equations (15), (16)

$$\left[x_1,h_1,\left[x_2,x_3,h_2\right]_A\right]_A$$
$$=\left[\left[x_1,h_1,x_2\right]_A,x_3,h_2\right]_A+\left[x_2,\left[x_1,h_1,x_3\right]_A,h_2\right]_A+\left[x_2,x_3,\left[x_1,h_1,h_2\right]_A\right]_A,$$

$$\left[h_1,h_2,\left[x_1,x_2,x_3\right]_A\right]_A$$
$$=\left[\left[h_1,h_2,x_1\right]_A,x_2,x_3\right]_A+\left[x_1,\left[h_1,h_2,x_2\right]_A,x_3\right]_A+\left[x_1,x_2,\left[h_1,h_2,x_3\right]_A\right]_A.$$

Because $\left[x_1,x_2,\left[x_3,h_1,h_2\right]_A\right]_A=\rho(x_1,x_2)\rho(x_3,h_1)h_2$,

$$\left[\left[x_1,x_2,x_3\right]_A,h_1,h_2\right]_A+\left[x_3,\left[x_1,x_2,h_1\right]_A,h_2\right]_A+\left[x_3,h_1,\left[x_1,x_2,h_2\right]_A\right]_A$$
$$=\beta(\left[x_1,x_2,x_3\right],h_1)h_2+\beta(x_3,\rho(x_1,x_2)h_1)h_2$$
$$+\beta(x_3,h_1)\rho(x_1,x_2)h_2+\left[\mu(x_1,x_2,x_3),h_1,h_2\right].$$

Through the direct calculation of Equations (15) and (16),

$$\left[\mu(x_1,x_2,x_3),h_1,h_2\right]$$
$$=\beta(x_1,h_1)\rho(x_2,x_3)h_2+\beta(x_3,h_2)\rho(x_1,x_2)h_1-\beta(x_2,h_2)\rho(x_1,x_3)h_1$$
$$+\rho(x_1,x_3)\beta(x_2,h_2)h_1-\rho(x_1,x_2)\beta(x_3,h_2)h_1-\beta(\left[x_1,x_2,x_3\right],h_1)h_2$$

$$\left[\mu(x_1,x_2,x_3),h_1,h_2\right]$$
$$=\beta(x_2,h_2)\rho(x_1,x_3)h_1+\beta(x_3,h_1)\rho(x_2,x_1)h_2-\beta(x_1,h_1)\rho(x_2,x_3)h_2$$
$$+\rho(x_2,x_3)\beta(x_1,h_1)h_2-\rho(x_2,x_1)\beta(x_3,h_1)h_2-\beta(\left[x_1,x_2,x_3\right],h_1)h_2$$

$$\left[\mu(x_1,x_2,x_3),h_1,h_2\right]=\rho(x_1,x_2)\beta(x_3,h_1)h_2-\beta(x_3,h_1)\rho(x_1,x_2)h_2$$
$$+\beta(x_3,h_2)\rho(x_1,x_2)h_1-\beta(\left[x_1,x_2,x_3\right],h_1)h_2.$$

As a result,

$$\left[x_1,x_2,\left[x_3,h_1,h_2\right]_A\right]_A$$
$$=\left[\left[x_1,x_2,x_3\right]_A,h_1,h_2\right]_A+\left[x_3,\left[x_1,x_2,h_1\right]_A,h_2\right]_A+\left[x_3,h_1,\left[x_1,x_2,h_2\right]_A\right]_A.$$

Case 4. For all $x_1,x_2 \in M$, $h_i \in H, i=1,2,3$, due to the $\beta(x_1,x_2)\in Der(H)$, it can be concluded from Equation (11) that,

$$\left[x_1,x_2,\left[h_1,h_2,h_3\right]_A\right]_A$$
$$=\left[\left[x_1,x_2,h_1\right]_A,h_2,h_3\right]_A+\left[h_1,\left[x_1,x_2,h_2\right]_A,h_3\right]_A+\left[h_1,h_2,\left[x_1,x_2,h_3\right]_A\right]_A$$

$$\left[x_1,h_1,\left[x_2,h_2,h_3\right]_A\right]_A$$
$$=\left[\left[x_1,h_1,x_2\right]_A,h_2,h_3\right]_A+\left[x_2,\left[x_1,h_1,h_2\right]_A,h_3\right]_A+\left[x_2,h_2,\left[x_1,h_1,h_3\right]_A\right]_A$$

Because of $\left[h_1,h_2,\left[h_3,x_1,x_2\right]_A\right]_A=\left[h_1,h_2,\rho(x_1,x_2)h_3\right]$,

Then

$$\left[\left[h_1,h_2,h_3\right]_A,x_1,x_2\right]_A+\left[h_3,\left[h_1,h_2,x_1\right]_A,x_2\right]_A+\left[h_3,x_1,\left[h_1,h_2,x_2\right]_A\right]_A$$
$$=\rho(x_1,x_2)\left[h_1,h_2,h_3\right]+\beta(x_2,h_3)\beta(x_1,h_1)h_2-\beta(x_1,h_3)\beta(x_2,h_1)h_2$$

According to lemma 4.2,

$$\begin{aligned}&\rho(x_1,x_2)[h_1,h_2,h_3]\\&=\beta(x_2,h_1)\beta(x_1,h_3)h_2+\beta(x_2,h_2)\beta(x_1,h_1)h_3+\beta(x_2,h_3)\beta(x_1,h_2)h_1\end{aligned}$$

$$\begin{aligned}&\beta(x_2,h_2)\beta(x_1,h_1)h_3-\beta(x_2,h_3)\beta(x_1,h_1)h_2\\&=\rho(x_1,x_2)[h_1,h_2,h_3]+\beta(x_2,h_1)\beta(x_1,h_2)h_3\end{aligned}$$

Using Equation (11) again, we can get

$$\begin{aligned}&[\rho(x_1,x_2)h_1,h_2,h_3]\\&=\rho(x_1,x_2)[h_1,h_2,h_3]+\beta(x_2,h_1)\beta(x_1,h_2)h_3-\beta(x_1,h_1)\beta(x_2,h_2)h_3\end{aligned}$$

namely

$$\begin{aligned}&[h_1,h_2,[h_3,x_1,x_2]_A]_A\\&=[[h_1,h_2,h_3]_A,x_1,x_2]_A+[h_3,[h_1,h_2,x_1]_A,x_2]_A+[h_3,x_1,[h_1,h_2,x_2]_A]_A.\end{aligned}$$

Case 5. For all $x\in M$, $h_i\in H, i=1,2,3,4$, because $\beta(x,h_1)\in Der(H)$, through Equation (13),

$$\begin{aligned}&[x,h_1,[h_2,h_3,h_4]_A]_A\\&=[[x,h_1,h_2]_A,h_3,h_4]_A+[h_2,[x,h_1,h_3]_A,h_4]_A+[h_2,h_3,[x,h_1,h_4]_A]_A\end{aligned}$$

$$\begin{aligned}&[h_1,h_2,[h_3,h_4,x]_A]_A\\&=[[h_2,h_3,h_4]_A,h_4,x]_A+[h_3,[h_1,h_3,h_4]_A,x]_A+[h_3,h_4,[h_1,h_2,x]_A]_A\end{aligned}$$

To sum up, (8) satisfies Equation (1). The conclusion is proved.

Theorem 4.2 Let $A=M\oplus H$ be the (μ,ρ,β)-extension of 3-Lie algebra H through *M*. So (H,ρ) is *M*-module if and only if $\beta(M,\mu(M,M,M))=0$.

Proof: If $\beta(M,\mu(M,M,M))=0$, obviously (H,ρ) is an *M*-module.

On the other hand, to any $x_j\in M, j=1,2,3,4$, by theorem 4.1 and Equation (9), (10),

$$\begin{aligned}&\rho([x_1,x_2,x_4],x_3)-\rho([x_1,x_2,x_3],x_4)\\&=-\beta(x_4,\mu(x_1,x_2,x_3))+\beta(x_3,\mu(x_1,x_2,x_4))\\&\quad-\rho(x_1,x_2)\rho(x_3,x_4)-\rho(x_3,x_4)\rho(x_1,x_2)\end{aligned}$$

$$\begin{aligned}\rho(x_2,[x_1,x_3,x_4])&=\rho(x_3,x_4)\rho(x_2,x_1)-\rho(x_1,x_4)\rho(x_2,x_3)\\&\quad+\rho(x_1,x_3)\rho(x_2,x_4)-\beta(x_2,\mu(x_1,x_3,x_4))\end{aligned}$$

As a result,

$$\begin{aligned}&-\rho([x_1,x_2,x_3],x_4)+\rho([x_1,x_2,x_4],x_3)+\rho(x_2,[x_1,x_3,x_4])-\rho(x_1,[x_2,x_3,x_4])\\&=-\beta(x_4,\mu(x_1,x_2,x_3))+\beta(x_3,\mu(x_1,x_2,x_4))-\rho(x_1,x_2)\rho(x_3,x_4)\\&\quad+\rho(x_3,x_4)\rho(x_1,x_2)+\rho(x_3,x_4)\rho(x_2,x_1)-\rho(x_1,x_4)\rho(x_2,x_3)\\&\quad+\rho(x_1,x_3)\rho(x_2,x_4)-\beta(x_2,\mu(x_1,x_3,x_4))-\rho(x_3,x_4)\rho(x_1,x_2)\\&\quad+\rho(x_2,x_4)\rho(x_1,x_3)-\rho(x_2,x_3)\rho(x_1,x_4)+\beta(x_1,\mu(x_2,x_3,x_4))\end{aligned}$$

$$
\begin{aligned}
&= \beta\left(x_3, \mu\left(x_1, x_2, x_4\right)\right) - \beta\left(x_4, \mu\left(x_1, x_2, x_3\right)\right) + \beta\left(x_1, \mu\left(x_2, x_3, x_4\right)\right) \\
&\quad - \beta\left(x_2, \mu\left(x_1, x_3, x_4\right)\right) - \rho\left(x_1, x_2\right)\rho\left(x_3, x_4\right) - \rho\left(x_3, x_4\right)\rho\left(x_1, x_2\right) \\
&\quad + \rho\left(x_1, x_3\right)\rho\left(x_2, x_4\right) + \rho\left(x_2, x_4\right)\rho\left(x_1, x_3\right) - \rho\left(x_1, x_4\right)\rho\left(x_2, x_3\right) \\
&\quad - \rho\left(x_2, x_3\right)\rho\left(x_1, x_4\right) \\
&= \beta\left(x_3, \mu\left(x_1, x_2, x_4\right)\right) - \beta\left(x_4, \mu\left(x_1, x_2, x_3\right)\right) \\
&\quad + \beta\left(x_1, \mu\left(x_2, x_3, x_4\right)\right) - \beta\left(x_2, \mu\left(x_1, x_3, x_4\right)\right) \\
&= 0
\end{aligned}
$$

According to Equation (9), $\beta\left(x_4, \mu\left(x_1, x_2, x_3\right)\right) = 0$. And the theorem is proved.

Theorem 4.3 Let $A = M \oplus H$ be the $\left(H, \rho\right)$-extension of 3-Lie algebra H through M and $\left(H, \rho\right)$ be an M-module. So A is a 3-Lie algebra if and only if $\mu\left(M, M, M\right) \subseteq Z\left(H\right)$, $\rho\left(M, M\right) \subseteq Z\left(DerH\right)$ and Equation (17) is true.

Proof: If A is a 3-Lie algebra, Equation (17) holds by theorem 4.1. Since $\left(H, \rho\right)$ is M-module, then $\beta = 0$. And $\mu\left(M, M, M\right) \subseteq Z\left(H\right)$, $\rho\left(M, M\right) \subseteq Z\left(DerH\right)$ can be obtained by Equations (9) and (13). Conversely, from theorems 4.1 and 4.2, A is a 3-Lie algebra.

The above conclusions about 3-Lie algebras will be helpful for further study of their derivation algebras.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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