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Singular Hammerstein-Volterra Integral Equation and Its Numerical Processing

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Abstract

In this paper, the existence and uniqueness of solution of singular Hammerstein-Volterra integral equation (**H-VIE**) are considered. Toeplitz matrix (**TMM**) and product Nystrom method (**PNM**) to solve the **H-VIE** with singular logarithmic kernel are used. The absolute error is calculated.

Keywords

Integral Equation, Hammerstein, Logarithmic Kernel

1. Introduction

The singular integral equations are considered to be of more interest than the others and a close form of solution is generally not available. Therefore, great attention must be considered for the numerical solution of these equations. Abdou in [1], studied Fredholm-Volterra integral equation with singular kernel. Al-Bugami, in [2], studied some numerical methods for solving singular and nonsingular integral equations. Abdou, El-Sayed and Deebs, in [3], obtained a solution of nonlinear integral equation. Also in [4], Abdou and Hendi used numerical solution for solving Fredholm integral equation with Hilbert kernel. In [5], Al-Bugami used **TMM** and Volterra-Hammerstein integral equation with a generalized singular kernel. In [6], Abdou, Borai, and El-Kojok used **TMM** and nonlinear integral equation of Hammerstein type. Al-Bugami, in [7], studied the error analysis for numerical solution of **HIE** with a generalized singular kernel. A. Shahsavaran in [8], studied Lagrange functions method for solving nonlinear **F-VIE**. In [9], Darwish, studied the nonlinear Fredholm-Volterra integral equations with hysteresis. In [10], Mirzaee used numerical solution of nonlinear **F-VIEs** via Bell polynomials. In [11], Raad studied linear **F-VIE** with logarithmic kernel and solved the linear system of Fredholm integral equations numerical with logarithmic form.

2. Existence and Uniqueness of the Solution of H-VIE

Consider:

$$\mu\phi(x,t)=f(x,t)+\lambda\int_{-a}^{a}K(x,y)\gamma\big(y,t,\phi(y,t)\big)\mathrm{d}y+\lambda\int_{0}^{t}F(t,\tau)\phi(x,\tau)\mathrm{d}\tau \quad (1)$$

This formula is measured in $L_2[-a,a]\times C[0,T], T<\infty$, where the **FI** term is measured with respect to position. While the **VI** term is considered in time, and $f(x,t)$ is known function. λ is the parameter, while μ defines the kind of the integral Equation (1).

We assume:

1) $K(x,y)\in C([-a,a]\times[-a,a])$, and satisfies:

$$\left[\int_{-a}^{a}\int_{-a}^{a}|K(x,y)|^2\,\mathrm{d}y\mathrm{d}x\right]^{\frac{1}{2}}=A_1<\infty,\ (A_1 \text{ is a constant})$$

2) $F(t,\tau)\in C([0,T]\times[0,T]), 0\le\tau\le t\le T\le\infty$, satisfies:

$$|F(t,\tau)|\le A_2$$

3) $f(x,t)$ is continuous in $L_2[-a,a]\times C[0,T]$ where:

$$\|f(x,t)\|=\max_{0\le t\le T}\int_0^t\left[\int_a^b|f(x,\tau)|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\mathrm{d}\tau=A_3$$

4) $\gamma\big(x,t,\phi(x,t)\big)$, satisfies for the constant $B>B_1, B>p$, the following conditions:

a) $\int_0^t\int_a^b\left(\left|\gamma\big(x,t,\phi(x,t)\big)\right|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\le B_1\|\phi(x,t)\|_{L_2[a,b]\times C[0,T]}$

b) $\left\|\gamma\big(x,t,\phi_1(x,t)\big)-\gamma\big(x,t,\phi_2(x,t)\big)\right\|\le N(x,t)\left|\phi_1(x,t)-\phi_2(x,t)\right|$

where $\|N(x,t)\|_{L_2[a,b]\times C[0,T]}=p$

In other words, we prove that the solution exists using the successive approximation method, also called the Picard method, that we pick up any real continuous function $\phi_0(x,t)$ in $L_2[-a,a]\times C[0,T]$, we assume $\phi_0(x,t)=f(x,t)$, then construct a sequence ϕ_n defined by

$$\phi_n(x,t)=f(x,t)+\lambda\int_{-a}^{a}K(x,y)\gamma\big(y,t,\phi_{n-1}(y,t)\big)\mathrm{d}y$$
$$+\lambda\int_0^t F(t,\tau)\phi_{n-1}(x,\tau)\mathrm{d}\tau,\ (\mu=1)$$

$$\phi_{n-1}(x,t)=f(x,t)+\lambda\int_{-a}^{a}K(x,y)\gamma\big(y,t,\phi_{n-2}(y,t)\big)\mathrm{d}y$$
$$+\lambda\int_0^t F(t,\tau)\phi_{n-2}(x,\tau)\mathrm{d}\tau,\ (\mu=1)$$

$$\begin{aligned}\psi_n(x,t) &= \phi_n(x,t)-\phi_{n-1}(x,t)\\ &= \lambda\int_{-a}^{a}K(x,y)\left[\gamma\left(y,t,\phi_{n-1}(y,t)\right)-\gamma\left(y,t,\phi_{n-2}(y,t)\right)\right]\mathrm{d}y\\ &\quad+\lambda\int_0^t F(t,\tau)\left[\phi_{n-1}(x,\tau)-\phi_{n-2}(x,\tau)\right]\mathrm{d}\tau,\ \ n=1,2,\cdots\end{aligned}$$

Then:

$$\phi_n(x,t)=\sum_{i=0}^{n}\psi_i(x,t) \tag{2}$$

Hence

$$\psi_n(x,t)=f(x,t)+\lambda\int_{-a}^{a}K(x,y)\gamma\left(y,t,\psi_{n-1}(y,t)\right)\mathrm{d}y+\lambda\int_0^t F(t,\tau)\psi_{n-1}(x,\tau)\mathrm{d}\tau$$

Using the properties of the norm, we obtain:

$$\left\|\psi_n(x,t)\right\|\le|\lambda|\left\|\int_{-a}^{a}K(x,y)\gamma\left(y,t,\psi_{n-1}(y,t)\right)\mathrm{d}y\right\|+|\lambda|\left\|\int_0^t F(t,\tau)\psi_{n-1}(x,\tau)\mathrm{d}\tau\right\|$$

For $n=1$, we get

$$\begin{aligned}\left\|\psi_1(x,t)\right\| &\le|\lambda|\left\|\int_{-a}^{a}K(x,y)\gamma\left(y,t,\psi_0(y,t)\right)\mathrm{d}y\right\|+|\lambda|\left\|\int_0^t F(t,\tau)\psi_0(x,\tau)\mathrm{d}\tau\right\|\\ &\le|\lambda|\left\|\left(\int_{-a}^{a}\left|K(x,y)\right|^2\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{-a}^{a}\left|\gamma\left(y,t,\psi_0(y,t)\right)\right|^2\mathrm{d}y\right)^{\frac{1}{2}}\right\|\\ &\quad+|\lambda|\left\|\int_0^t\left|F(t,\tau)\right|\left|\psi_0(x,\tau)\right|\mathrm{d}\tau\right\|\end{aligned}$$

Using Cauchy Schwarz inequality and from conditions (i)-(iv-a) with $\psi_0=f(x,t)$ and $\|f\|=A_3$, we get

$$\begin{aligned}\left\|\psi_1(x,t)\right\| &\le|\lambda|\max_{0\le t\le T}\int_0^t\left[\int_{-a}^{a}\left(\int_{-a}^{a}\left|K(x,y)\right|^2\mathrm{d}y\int_{-a}^{a}\left|\gamma\left(y,t,\psi_0(y,t)\right)\right|^2\mathrm{d}y\right)\mathrm{d}x\right]^{\frac{1}{2}}\mathrm{d}\tau\\ &\quad+|\lambda|A_2\int_0^t\left\|\psi_0(x,\tau)\right\|\mathrm{d}\tau\\ &\le|\lambda|A_1A_3B_1+|\lambda|A_2A_3\|t\|\end{aligned}$$

We have $0\le\tau\le t\le T\le\infty$, then $\max|t|=T=L$, and then we have:

$$\left\|\psi_1(x,t)\right\|\le|\lambda|A_3\left(A_1B_1+A_2L\right)$$

In general, we get:

$$\left\|\psi_1(x,t)\right\|\le|\lambda|^n A_3\left(A_1B_1+A_2L\right)^n=A_3\alpha^n,\ \alpha=|\lambda|\left(A_1B_1+A_2L\right) \tag{3}$$

This bound makes the sequence $\psi_n(x,t)$ converges if

$$\alpha<1\Rightarrow|\lambda|<\frac{1}{A_1B_1+A_2L} \tag{4}$$

The result (4), leads us to say that the formula (2) has a convergent solution. So let $n \to \infty$, we have:

$$\phi(x,t)=\sum_{i=0}^{\infty}\psi_i(x,t)=\frac{A_3}{1-\alpha},\ (\alpha<1) \tag{5}$$

The infinite series of (5) is convergent, and $\phi(x,t)$ represents the convergent solution of Equation (1). Also each of ψ_i is continuous, therefore $\phi(x,t)$ is also continuous.

To show that $\phi(x,t)$ is unique, we assume that $\bar{\phi}(x,t)$ is also a continuous solution of (1) then, we write

$$\phi(x,t)-\bar{\phi}(x,t)=\lambda\int_{-a}^{a}K(x,y)\left[\gamma(y,t,\phi(y,t))-\gamma(y,t,\bar{\phi}(y,t))\right]\mathrm{d}y$$
$$+\lambda\int_{0}^{t}F(t,\tau)\left[\phi(x,\tau)-\bar{\phi}(x,\tau)\right]\mathrm{d}\tau,\ (\mu=1)$$

which leads us to the following:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|\le|\lambda|\left\|\int_{-a}^{a}|K(x,y)|\left|\gamma(y,t,\phi(y,t))-\gamma(y,t,\bar{\phi}(y,t))\right|\mathrm{d}y\right\|$$
$$+|\lambda|\left\|\int_{0}^{t}|F(t,\tau)|\left|\phi(x,\tau)-\bar{\phi}(x,\tau)\right|\mathrm{d}\tau\right\|$$

Using conditions (iv-b), then we have:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|$$
$$\le|\lambda|\max_{0\le t\le T}\int_{0}^{t}\left[\left(\int_{-a}^{a}\int_{-a}^{a}|K(x,y)|\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{-a}^{a}N^2(x,t)\left|\phi(x,t)-\bar{\phi}(x,t)\right|^2\mathrm{d}y\right)^{\frac{1}{2}}\right]\mathrm{d}\tau$$
$$+|\lambda|\left\|\int_{0}^{t}|F(t,\tau)|\left|\phi(x,t)-\bar{\phi}(x,t)\right|\mathrm{d}\tau\right\|$$

Finally, with the aid of conditions (i) and (ii):

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|\le\alpha\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|$$

Then:

$$(1-\alpha)\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|\le 0$$

Since $\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|$ is necessarily non-negative, and $\alpha<1$:

$$\left\|\phi(x,t)-\bar{\phi}(x,t)\right\|=0\Rightarrow\phi(x,t)=\bar{\phi}(x,t)$$

It follows that if (1) has a solution it must be unique.

3. SHIEs

Consider:

$$\phi(x,t)=f(x,t)+\lambda\int_{-a}^{a}K(x,y)\gamma(y,t,\phi(y,t))\mathrm{d}y+\lambda\int_{0}^{t}F(t,\tau)\phi(x,\tau)\mathrm{d}\tau \tag{6}$$

when $t=0$ Equation (13) becomes:

$$\phi_0(x) = f_0(x) + \lambda\int_{-a}^{a} K(x,y)\gamma(y,\phi_0(y))\mathrm{d}y \tag{7}$$

where $\phi_0(x) = \phi(x,0)$, $f_0(x) = f(x,0)$.

The formula (7) represents **HIE** of the second kind at $t = 0$. Divide the interval $[0,T]$, $0 \le t \le T < \infty$ as $0 = t_0 \le t_1 < \cdots < t_k < \cdots < t_N = T$, then using the quadrature formula, the Volterra integral term in (6) becomes:

$$\int_0^{t_k} F(t,\tau)\phi(x,\tau)\mathrm{d}\tau = \sum_{j=0}^{k} u_j F(t_k,t_j)\phi(x,t_j) + o(\hbar_i^{\tilde{p}+1}), (\hbar_k \to 0, \tilde{p} > 0) \tag{8}$$

where $\hbar_k = \max_{0\le j\le k} h_j$, $h_j = t_{j+1} - t_j$

Using (8) in (6), we have:

$$\phi_k(x) = f_k(x) + \lambda\int_{-a}^{a} K(x,y)\gamma(y,t_k,\phi_k(y))\mathrm{d}y + \lambda\sum_{j=0}^{k} u_j F_{kj}\phi_j(x) \tag{9}$$

where $\phi_k(x) = \phi(x,t_k)$, $f_k(x) = f(x,t_k)$, $F_{kj} = F(t_k,t_j)$.

$$\mu_n\phi_n(x) = G_n(x) + \lambda\int_{-a}^{a} K(x,y)\phi_n(y)\mathrm{d}y \tag{10}$$

where $\mu_n = 1 - \lambda F_{nn}u_n, G_n(x) = f_n(x) + \lambda\sum_{j=0}^{n-1} u_j F_{nj}\gamma(x,t_j,\phi_j(x)), n = 0,1,\cdots,N$.

The formula (10) represents **SHIEs** of the second kind, and we have N unknown $\phi_n(x)$.

4. Some Numerical Techniques for Solving SHIEs

4.1. The TMM

In this section, we present the **TMM** to obtain numerical solution for **HIE** of the second kind with singular kernel. Consider:

$$\phi(x) = f(x) + \lambda\int_{-a}^{a} K(|x-y|)\gamma(y,\phi(y))\mathrm{d}y \tag{11}$$

Write the integral term in the form:

$$\int_{-a}^{a} K(|x-y|)\gamma(y,\phi(y))\mathrm{d}y = \sum_{n=-N}^{N-1}\int_{nh}^{nh+h} K(|x-y|)\gamma(y,\phi(y))\mathrm{d}y, \left(h = \frac{2a}{N}\right) \tag{12}$$

Approximate the integral in the right hand side of Equation (12) by:

$$\begin{aligned}&\int_{nh}^{nh+h} K(|x-y|)\gamma(y,\phi(y))\mathrm{d}y \\ &= A_n(x)\gamma(nh,\phi(nh)) + B_n(x)\gamma(nh+h,\phi(nh+h)) + R\end{aligned} \tag{13}$$

where $A_n(x)$ and $B_n(x)$ are two arbitrary functions. Putting $\phi(x) = 1, x$ in Equation (13), where in this case we choose $R = 0$. By solving the result, then we take:

$$A_n(x) = \frac{1}{h}\left[\gamma(nh+h,nh+h)I(x) - \gamma(nh+h,1)J(x)\right] \tag{14}$$

And

$$B_n(x) = \frac{1}{h}\left[\gamma(nh+h,1)J(x) - \gamma(nh,nh)I(x)\right] \tag{15}$$

where:

$$I(x)=\int_{nh}^{nh+h} K(|x-y|)\gamma(y,1)\,\mathrm{d}y \tag{16}$$

$$J(x)=\int_{nh}^{nh+h} K(|x-y|)\gamma(y,y)\,\mathrm{d}y \tag{17}$$

The relation (12), becomes:

$$\int_{-a}^{a} K(|x-y|)\gamma(y,\phi(y))\,\mathrm{d}y=\sum_{n=-N}^{N} D_n(x)\gamma(nh,\phi(nh))$$

where

$$D_n(x)=\begin{cases} A_{-N}(x), & n=-N \\ A_n(x)+B_n(x), & -N<n<N \\ B_{N-1}(x), & n=N \end{cases} \tag{18}$$

The **IE** (11) becomes:

$$\phi(x)-\lambda\sum_{n=-N}^{N} D_n(x)\gamma(nh,\phi(nh))=f(x) \tag{19}$$

Putting $x=mh$, we have:

$$\phi_m-\lambda\sum_{n=-N}^{N} D_{n,m}\gamma_n(\phi_n)=f_m,\ -N\le m\le N \tag{20}$$

where $\phi_m=\phi(mh), D_{n,m}=D_n(mh), f_m=f(mh)$.

The matrix $D_{n,m}$ may be written as $D_{n,m}=G_{n,m}+E_{n,m}$, where:

$$G_{n,m}=A_n(mh)+B_{n-1}(mh),\ -N\le n,m\le N \tag{21}$$

Is a Toeplitz matrix of order $2N+1$ and:

$$E_{n,m}(x)=\begin{cases} B_{-N-1}(x), & n=-N, m=-N+i \\ 0, & -N<n<N \\ A_N(x), & n=N, m=-N+i \end{cases} \tag{22}$$

where $0\le i\le 2n$. The solution of the formula (20):

$$\phi_m=\left[I-\lambda\left(G_{n,m}+E_{n,m}\right)\right]^{-1} f_m,\ \left|I-\lambda\left(G_{n,m}+E_{n,m}\right)\right|\ne 0 \tag{23}$$

Also

$$R=\left|\int_{-a}^{a} K(|x-y|)\gamma(y,\phi(y))\,\mathrm{d}y-\sum_{n=-N}^{N} D_{nm}\gamma(nh,\phi(nh))\right| \tag{24}$$

4.2. The PNM

Consider:

$$\phi(x)-\lambda\int_{-a}^{a} p(x,y)\bar{K}(x,y)\gamma(y,\phi(y))\,\mathrm{d}y=f(x) \tag{25}$$

where p and $\bar{K}$ are badly behaved and well-behaved functions of their arguments, respectively. Then, we get:

$$\phi(x_i)-\lambda\sum_{j=0}^{N} w_{ij}\bar{K}(x_i,y_j)\gamma\left(y_j,\phi(y_j)\right)=f(x_i) \tag{26}$$

where $x_i = y_i = a+ih, i=0,1,\cdots,N$ with $h=\frac{2a}{N}$, N even and w_{ij} are the weights. When $x=x_i$, we write:

$$\int_{-a}^{a} p(x_i,y)\bar{K}(x_i,y)\gamma\left(y,\phi(y)\right)\mathrm{d}y=\sum_{j=0}^{\frac{N-2}{2}}\int_{y_{2j}}^{y_{2j+2}} p(x_i,y)\bar{K}(x_i,y)\gamma\left(y,\phi(y)\right)\mathrm{d}y \tag{27}$$

Form relation (25) through (27) we find:

$$\sum_{j=0}^{N} w_{ij}\bar{K}(x_i,y_j)\gamma\left(y_j,\phi(y_j)\right)=\sum_{j=0}^{\frac{N-2}{2}}\int_{y_{2j}}^{y_{2j+2}} p(x_i,y)\bar{K}(x_i,y)\gamma\left(y,\phi(y)\right)\mathrm{d}y \tag{28}$$

Then, we obtain:

$$\begin{aligned}&\int_{-a}^{a} p(x_i,y)\bar{K}(x_i,y)\gamma\left(y,\phi(y)\right)\mathrm{d}y\\&=\sum_{j=0}^{\frac{N-2}{2}}\int_{y_{2j}}^{y_{2j+2}} p(x_i,y)\left\{\frac{(y_{2j+1}-y)(y_{2j+2}-y)}{2h^2}\gamma\left(y_{2j},\phi(y_{2j})\right)\right.\\&\quad+\frac{(y-y_{2j})(y_{2j+2}-y)}{h^2}\gamma\left(y_{2j+1},\phi(y_{2j+1})\right)\\&\quad\left.+\frac{(y_{2j+1}-y)(y_{2j}-y)}{2h^2}\gamma\left(y_{2j+2},\phi(y_{2j+2})\right)\right\}\mathrm{d}y\end{aligned}$$

Therefore:

$$\begin{aligned}&w_{i,0}=\beta_1(y_i) && w_{i,2j+1}=2\gamma_{j+1}(y_i)\\&w_{i,2j}=\alpha_i(y_i)+\beta_{j+1}(y_i) && w_{i,N}(y_i)=\alpha_{\frac{N}{2}}(y_i)\end{aligned} \tag{29}$$

where:

$$\begin{aligned}\alpha_j(y_i)&=\frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y-y_{2j-2})(y-y_{2j-1})\mathrm{d}y\\\beta_j(y_i)&=\frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y-y_{2j-1})(y-y_{2j})\mathrm{d}y\\\gamma_j(y_i)&=\frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y-y_{2j-2})(y_{2j}-y)\mathrm{d}y\end{aligned} \tag{30}$$

We now introduce the change of variable $y=y_{2j-2}+\zeta h, 0\le\zeta\le 2$ thus the system (30) becomes:

$$\alpha_j(y_i)=\frac{h}{2}\int_0^2\zeta(\zeta-1)\,p\left(y_{2j-2}+\zeta h,y_i\right)\mathrm{d}\zeta$$

$$\beta_j(y_i)=\frac{h}{2}\int_0^2(\zeta-1)(\zeta-2)\,p\left(y_{2j-2}+\zeta h,y_i\right)\mathrm{d}\zeta$$

$$\gamma_j(y_i)=\frac{h}{2}\int_0^2\zeta(2-\zeta)\,p\left(y_{2j-2}+\zeta h,y_i\right)\mathrm{d}\zeta$$

If we define:

$$\psi_i=\int_0^2\zeta^i\,p\left(y_{2j-2}+\zeta h,y_i\right)\mathrm{d}\zeta$$

For $p(x,y)=p(x-y)$, we have:

$$\psi_i=\int_0^2\zeta^i\,p\left(y_i,y_{2j-2}+\zeta h\right)\mathrm{d}\zeta,\ \ i=0,1,2 \tag{31}$$

When $y_i-y_{2j-2}=(i-2j+2)h$. If we assume $z=i-2j+2$, then:

$$\begin{aligned}\alpha_j(y_i)&=\frac{h}{2}\int_0^2\zeta(\zeta-1)\,p(z-\zeta)\mathrm{d}\zeta\\ \beta_j(y_i)&=\frac{h}{2}\int_0^2(\zeta-1)(\zeta-2)\,p(z-\zeta)\mathrm{d}\zeta\\ \gamma_j(y_i)&=\frac{h}{2}\int_0^2\zeta(2-\zeta)\,p(z-\zeta)\mathrm{d}\zeta\end{aligned} \tag{32}$$

Hence, the system (29) becomes:

$$\begin{aligned}w_{i,0}&=\frac{h}{2}\left[2\psi_0(z)-3\psi_1(z)+\psi_2(z)\right], && z=i\\ w_{i,2j+1}&=h\left[2\psi_1(z)-\psi_2(z)\right], && z=i-2j\\ w_{i,2j}&=\frac{h}{2}\left[\psi_2(z)-\psi_1(z)+2\psi_0(z-2)-3\psi_1(z-2)+\psi_2(z-2)\right], && z=i-2j+2\\ w_{i,N}&=\frac{h}{2}\left[\psi_2(z)-\psi_1(z)\right], && z=i-N+2\end{aligned} \tag{33}$$

Therefore, the integral Equation (25) is reduced to **SLAEs** as in (26) or: $(I-\lambda W)\phi=F$

Which has the solution:

$$\phi=(I-\lambda W)^{-1}F,\ \ |I-\lambda W|\neq 0 \tag{34}$$

The **PNM** is said to be convergent of order r in $[-a,a]$. If for N sufficiently large, there exists a constant $C>0$ independent of N such that:

$$\left\|\phi(x)-\phi_N(x)\right\|\le CN^{-r}$$

5. Numerical Applications

We using **TMM** and **PNM** at $N=20,40$, $T=0.03,0.7$, $\lambda=1$, and $\mu=1$. **In** Tables 1-4:

$\phi_{Exact}\to$ Exact solution, $\phi_T\to$ appro. sol. of **TMM**, $E_T\to$ the absolute error of **TMM**, $\phi_N\to$ appro. sol. of **PNM**, $E_N\to$ the absolute error of **PNM**.

Example 1

Consider:

$$\phi(x,t)=f(x,t)+\lambda\int_{-1}^{1}\ln|x-y|(yt)^2\,\mathrm{d}y+\lambda\int_0^t\tau^2\phi(x,\tau)\mathrm{d}\tau$$

Table 1. The values of exact, approximate solutions, and errors by using **TMM**, **PNM** at $N = 20$.

T	x	ϕ_{Exact}	ϕ_T	E_T	ϕ_N	E_N
	−1.0	−0.03000000	−0.030701591	7.0159E−4	−0.03002738	2.7386E−5
	−0.8	−0.02400000	−0.023838516	1.6148E−4	−0.02403073	3.0733E−5
	−0.6	−0.01800000	−0.017855743	1.4425E−4	−0.01803683	3.6835E−5
	−0.4	−0.01200000	−0.011903314	9.6685E−5	−0.01203176	3.1761E−5
	−0.2	−0.00600000	−0.005955438	4.4561E−5	−0.00632818	3.2818E−5
0.03	0	0	0.000000452	4.5279E−7	−0.00002599	2.5995E−5
	0.2	0.006000000	0.0059710686	2.8931E−5	0.005976067	2.3932E−5
	0.4	0.012000000	0.0011959300	4.6995E−5	0.011983505	1.6494E−5
	0.6	0.018000000	0.0179646910	3.5308E−5	0.017987883	1.2116E−5
	0.8	0.024000000	0.0239827313	1.7268E−5	0.023995269	4.7302E−6
	1.0	0.030000000	0.029999603	3.9617E−7	0.030003419	3.4192E−6
	−1.0	−0.70000000	−0.716358582	1.6358E−2	−0.70031893	3.1893E−4
	−0.8	−0.56000000	−0.556088118	3.9118E−3	−0.56033293	3.3293E-4
	−0.6	−0.42000000	−0.416502750	3.4972E−3	−0.42054763	5.4763E−4
	−0.4	−0.28000000	−0.277643540	2.3564E−3	−0.28051997	5.1997E−4
	−0.2	−0.14000000	−0.138874201	1.1257E−3	−0.14061900	6.1900E−4
0.7	0	0	0.0001142305	1.1423E−4	−0.00050225	5.0225E−4
	0.2	0.140000000	0.1394831440	5.1685E−4	0.139540294	4.5979E−4
	0.4	0.280000000	0.2792953877	7.0461E−4	0.279740200	2.5979E−4
	0.6	0.420000000	0.4195246008	4.7539E−4	0.419885056	1.1494E−4
	0.8	0.560000000	0.5600309299	3.0929E−5	0.560081782	8.1782E−4
	1.0	0.700000000	0.7003734045	3.7340E−4	0.700159944	1.5994E−4

Table 2. The values of exact, approximate solutions, and errors by using **TMM**, **PNM** at $N = 40$.

T	x	ϕ_{Exact}	ϕ_T	E_T	ϕ_N	E_N
	−1.0	−0.03000000	−0.031354257	1.3542E−3	−0.03002817	2.8170E−5
	−0.8	−0.02400000	−0.023848333	1.5156E−4	−0.02403524	3.5249E−5
	−0.6	−0.01800000	−0.017860806	1.3919E−4	−0.01803612	3.6128E−5
	−0.4	−0.01200000	−0.011906018	9.3981E−5	−0.01203478	3.4780E−5
	−0.2	−0.00600000	−0.005956673	4.3326 E−5	−0.00603193	3.1939E−5
0.03	0	0	0.0000002134	2.1349E−7	−0.00002798	2.7985E−5
	0.2	0.006000000	0.005971517	2.8482E−5	0.059768287	2.3171E−5
	0.4	0.012000000	0.0119602026	3.9797E−5	0.011982306	1.7693E−5
	0.6	0.018000000	0.0179658449	3.4155E−5	0.017988279	1.1720E−5
	0.8	0.024000000	0.0239839397	1.6060E−5	0.0239946215	5.3784E−6
	1.0	0.030000000	0.0300003812	3.8126E−7	0.030002272	2.2728E-6
	−1.0	−0.70000000	−0.731583858	3.1585E−2	−0.700337200	3.3720E−4
	−0.8	−0.56000000	−0.556312874	3.6871E−3	−0.560438272	4.3827E−4
	−0.6	−0.42000000	−0.416615893	3.3841E−3	−0.420531161	5.3116E−4
	−0.4	−0.28000000	−0.277703301	2.2966E−3	−0.280590379	5.9037E−4
	−0.2	−0.14000000	−0.138901367	1.0986E−3	−0.140598518	5.9851E−4
0.7	0	0	0.0001086431	1.0864E−4	−0.000548653	5.4865E−4
	0.2	0.140000000	0.1394919213	5.0807E−4	0.1395580310	4.4196E−4
	0.4	0.280000000	0.2793130480	6.8695E−4	0.2791722263	2.8777E−4
	0.6	0.420000000	0.4195464227	4.5357E−4	0.4198942815	1.0571E−4
	0.8	0.560000000	0.5600523036	5.2303E−5	0.560066655	6.6665E−5
	1.0	0.700000000	0.7003829896	3.8298E−4	0.7001331775	1.3317E−4

Table 3. The values of exact, approximate solutions, and errors by using **TMM**, **PNM** at $N = 20$.

T	x	ϕ_{Exact}	ϕ_T	E_T	ϕ_N	E_N
	−1.0	−0.015000000	−0.015350899	3.5089E−4	−0.0150137693	1.3789E−5
	−0.8	−0.012000000	−0.119193964	8.0603E−5	−0.0120155049	1.5504E−5
	−0.6	−0.009000000	−0.008927989	7.2013E−5	−0.0090185327	1.8532E−5
	−0.4	−0.006000000	−0.005951798	4.8260E−5	−0.0060159635	1.5963E−5
	−0.2	−0.003000000	−0.002977777	2.2222E−5	−0.0030164679	1.6467E−5
0.03	0	0	−0.176673371	1.7667E−7	−0.0001304779	1.3047E−5
	0.2	0.0030000000	0.0029854757	1.4524E−5	0.00298797512	1.2024E−5
	0.4	0.0060000000	0.0059795673	2.0432E−5	0.00599166981	8.3301E-6
	0.6	0.0090000000	0.0089822306	1.7769E−5	0.00899382675	6.1732E−6
	0.8	0.0120000000	0.0119912274	8.7725E−6	0.01199749658	2.5034E−6
	1.0	0.0150000000	0.0149997058	2.9414E−7	0.01500161353	1.6135E−6
	−1.0	−0.350000000	−0.358222414	8.2224E−3	−0.3502016929	2.0169E−4
	−0.8	−0.280000000	−0.278111233	1.8887E−3	−0.2802337016	2.3370E−4
	−0.6	−0.210000000	−0.208307768	1.6922E−3	−0.2103304274	3.3042E−4
	−0.4	−0.140000000	−0.138862748	1.1372E−3	−0.1403011727	3.0117E−4
	−0.2	−0.700000000	−0.069466949	5.3305E−4	−0.0703395348	3.3953E−4
0.7	0	0	0.0000300394	3.0039E−5	−0.000278348	2.7834E−4
	0.2	0.7000000000	0.069707680	2.9231E−4	0.0697361145	2.6388E−4
	0.4	0.1400000000	0.1395986121	4.0138E−4	0.1398209052	1.7909E−4
	0.6	0.2100000000	0.209637503	3.0262E−4	0.209873904	1.2609E−4
	0.8	0.2800000000	0.279932202	6.7797E−5	0.2799576210	4.2379E−4
	1.0	0.3500000000	0.3501244044	1.2440E−4	0.3500176757	1.7675E−5

Table 4. The values of exact, approximate solutions, and errors by using **TMM**, **PNM** at $N = 40$.

T	x	ϕ_{Exact}	ϕ_T	E_T	ϕ_N	E_N
	−1.0	−0.015000000	−0.015677228	6.7722E−4	−0.015014180	1.4180E−5
	−0.8	−0.012000000	−0.011924354	7.5645E−5	−0.120177632	1.7763E−5
	−0.6	−0.009000000	−0.008930517	6.9482E−5	−0.009018179	1.8179E−5
	−0.4	−0.006000000	−0.005953091	4.6908E−5	−0.006017473	1.7473E−5
	−0.2	−0.003000000	−0.002978395	2.1604E−5	−0.003016028	1.6028E−5
0.03	0	0	0.0000000570	5.7023E−8	−0.000014042	1.4042E−5
	0.2	0.0030000000	0.0029857000	1.4299E−5	0.002988355	1.1644E−5
	0.4	0.0060000000	0.0059800185	1.9981E−5	0.0059910700	8.9299E−6
	0.6	0.0090000000	0.0089828075	1.7192E−5	0.008994024	5.9752E−6
	0.8	0.0120000000	0.0119918316	8.1683E−6	0.0119971724	2.8275E−6
	1.0	0.0150000000	0.0150009457	9.457E−8	0.015001040	1.0403E−6
	−1.0	−0.350000000	−0.36583575	1.5835E−2	−0.350210888	2.1082E−4
	−0.8	−0.280000000	−0.27822360	1.7763E−3	−0.280286378	2.8637E−4
	−0.6	−0.210000000	−0.20836433	1.6356E−3	−0.210322188	3.2218E−4
	−0.4	−0.140000000	−0.13889262	1.1073E−3	−0.14033638	3.3638E−4
	−0.2	−0.700000000	−0.06948053	5.1946E−4	−0.070329283	3.2928E−4
0.7	0	0	0.000027245	2.7245E−5	−0.000301555	3.0155E−4
	0.2	0.7000000000	0.697120697	2.8793E−4	0.6974498644	2.5501E−4
	0.4	0.1400000000	0.139607442	3.9255E−4	0.139806915	1.9308E−4
	0.6	0.2100000000	0.209704661	2.9533E−4	0.209878520	1.2147E−4
	0.8	0.2800000000	0.279942888	5.7111E−4	0.279950061	4.9938E−5
	1.0	0.3500000000	0.350129197	1.2919E−4	0.3500042976	4.2976E−6

Exact solution: $\phi(x,t) = xt$

Example 2

Consider:

$$\phi(x,t) = f(x,t) + \lambda \int_{-1}^{1} \ln|x-y| \left(\frac{yt}{2}\right)^2 \mathrm{d}y + \lambda \int_{0}^{t} t\tau\phi(x,\tau)\,\mathrm{d}\tau$$

Exact solution: $\phi(x,t) = \frac{xt}{2}$

6. Conclusion

The goal of this work is to study the **H-VIE** with singular kernel of the second kind. **TMM** and **PNM** are successive to solve this equation numerically. As N is increasing, the errors are decreasing. As t is increasing, the errors are increasing.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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