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Wave Interactions of the Aw-Rascle Model for Generalized Chaplygin Gas*

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Abstract

In this paper, we investigate the elementary wave interactions of the Aw-Rascle model for the generalized Chaplygin gas. We construct the unique solution by the characteristic analysis method and obtain the stability of the corresponding Riemann solutions under such small perturbations on the initial values. We find that the elementary wave interactions have a much more simple structure for Temple class than general systems of conservation laws. It is important to study the elementary waves interactions of the traffic flow system for the generalized Chaplygin gas not only because of their significance in practical applications in the traffic flow system, but also because of their basic role for the general mathematical theory.

Keywords

Wave Interaction, Aw-Rascle Model, Generalized Chaplygin Gas, Riemann Problem, Delta Shock, Hyperbolic Conservation Laws

1. Introduction

In the present paper, we study the Aw-Rascle (AR) macroscopic model of traffic flow

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho(u+P))_t + (\rho u(u+P))_x = 0, \end{cases} \tag{1}$$

where $\rho \geq 0$ is the density, $u \geq 0$ is the velocity, P is the velocity offset which is called as the "pressure" inspired from gas dynamics. The derivation process of the above AR model and the application can be discovered in [1]-[7].

In [8], Aw and Rascle studied the limit behavior and found that the pressure term is active. In [9], Shen and Sun investigated the limit behavior without the constraint of the maximal density.

In [10] [11], M. N. Sun studied the following model

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u + \rho^\gamma)_t + (\rho u^2 + \rho^\gamma u)_x = 0, \end{cases} \tag{2}$$

In [10] [11], they studied the elementary wave interactions and obtained the stability of the Riemann solutions under such a perturbation on the initial data.

In [12], G. D. Wang investigated the Riemann problem of (1) and

$$p = -\frac{B}{\rho^\alpha}, \tag{3}$$

where $0 < \alpha \le 1$, $B > 0$. This is the so-called generalized Chaplygin gas for (1).

In the present paper, we investigate the elementary wave interactions for (1) and (3). In our paper [13], we study the wave interactions containing no delta shock, so we just consider the wave interactions for (1) containing delta shock wave with the three piecewise constants

$$(u,\rho)(x,0) = \begin{cases} (u_-, \rho_-), & -\infty < x < -\varepsilon, \\ (u_m, \rho_m), & -\varepsilon < x < \varepsilon, \\ (u_+, \rho_+), & \varepsilon < x < +\infty, \end{cases} \tag{4}$$

where the perturbation parameter ε is sufficiently small. (4) can be regarded as a local perturbation on the initial values

$$(u,\rho)(x,0) = (u_\pm, \rho_\pm), \quad \pm x > 0, \tag{5}$$

where $u_\pm, \rho_\pm > 0$.

This paper is arranged as follows. In Section 2, we give curtly the Riemann problem for the model (1) (3) and (5) for the convenience of the readers. In Section 3, we investigate the elementary wave interactions by the characteristic analysis method. In Section 4, we summarize our main conclusion.

2. Preliminaries

We give briefly the Riemann problem for (1) (3) and (5) [12].

The characteristic roots of (1) are $\mu_1 = u - \frac{B\alpha}{\rho^\alpha}$, $\mu_2 = u$ which shows that (1) is strictly hyperbolic. The corresponding right characteristic vector of μ_1 and μ_2 is respectively given by

$$\boldsymbol{\nu}_1 = \left(-\frac{B\alpha}{\rho^{1+\alpha}}, 1\right)^{\mathrm{T}}, \quad \boldsymbol{\nu}_2 = (0,1)^{\mathrm{T}}, \tag{6}$$

If $0 < \gamma < 1$, we get

$$\nabla\mu_1 \cdot \boldsymbol{\nu}_1 = -\frac{(1-\alpha)B\alpha}{\rho^{1+\alpha}} < 0, \quad \nabla\mu_2 \cdot \boldsymbol{\nu}_2 = 0, \tag{7}$$

which indicates that μ_1 is genuinely nonlinear and the associated wave is either shock wave or rarefaction wave, μ_2 is always linearly degenerate and the associated wave is the contact discontinuity, where ∇ denotes the gradient.

We construct the self-similar solution $(u,\rho)(x,t)=(u,\rho)(\zeta)$, $\zeta=\frac{x}{t}$. The Riemann problem (1) (5) becomes the following boundary value problem of the ordinary differential equations

$$\begin{cases} -\zeta\rho_\zeta+(\rho u)_\zeta=0, \\ -\zeta\left(\rho u-\dfrac{B}{\rho^{\alpha-1}}\right)_\zeta+\left(\rho u^2-\dfrac{uB}{\rho^{\alpha-1}}\right)_\zeta=0, \end{cases} \tag{8}$$

and $(u,\rho)(\pm\infty)=(u_\pm,\rho_\pm)$. For smooth solutions, let $V=(u,\rho)^{\mathrm{T}}$, (8) becomes

$$A(V)V_\zeta=0, \tag{9}$$

where

$$A(V)=\begin{pmatrix} \rho & u-\zeta \\ u-\dfrac{B\alpha}{\rho^\alpha}-\zeta & 0 \end{pmatrix}.$$

Besides the constant state solution $(u,\rho)=\text{constant}$, (9) has a rarefaction wave solution. For the given left state (u_-,ρ_-), the rarefaction wave curve is given by

$$R(u_-,\rho_-):\begin{cases} \zeta=\lambda_1=u-\dfrac{\alpha B}{\rho^\alpha}, \\ u=u_-+\dfrac{B}{\rho^\alpha}-\dfrac{B}{\rho_-^\alpha}, \\ \rho<\rho_-,\ u>u_-. \end{cases} \tag{10}$$

For a bounded discontinuity at $\zeta=\tau$, it holds the Rankine-Hugoniot conditions

$$\begin{cases} -\tau[\rho]+[\rho u]=0, \\ -\tau\left[\rho u-\dfrac{B}{\rho^{\alpha-1}}\right]+\left[\rho u^2-\dfrac{uB}{\rho^{\alpha-1}}\right]=0, \end{cases} \tag{11}$$

where $[\rho]=\rho_r-\rho_l$, $\rho_l=\rho(\tau-0)$, $\rho_r=\rho(\tau+0)$, etc.

For the given left state (u_-,ρ_-), the shock wave is given by

$$S(u_-,\rho_-):\begin{cases} \tau=u-\dfrac{B}{\rho^\alpha}-\dfrac{\left(\rho_-^{1-\alpha}-\rho^{1-\alpha}\right)B}{\rho-\rho_-}, \\ u=u_-+\dfrac{B}{\rho^\alpha}-\dfrac{B}{\rho_-^\alpha}, \\ \rho>\rho_-,\ u<u_-. \end{cases} \tag{12}$$

Since λ_2 is linearly degenerate, from (9) or (11) we know the contact discontinuity

$$J:\zeta=u=u_-. \tag{13}$$

All the above rarefaction waves *R*, shock waves *S*, and contact discontinuities *J* are the elementary waves for (1). Notice the shock curves coincide with the rarefaction curves in the phase plane (u,ρ) [14]. It is very important because it can simplify the process of the elementary wave interactions.

According to the right state (u_+,ρ_+) in the different region (**Figure 1**), we obtain the unique Riemann solution. When $(u_+,\rho_+)\in \mathrm{I}$ or II, the unique Riemann solution is $R+J$, when $(u_+,\rho_+)\in \mathrm{III}$ or IV, the unique Riemann solution is $S+J$, when $(u_+,\rho_+)\in \mathrm{V}$, *i.e.*, $u_+ < u_- - \frac{B}{\rho_-^\alpha} < u_-$, we should construct the delta shock wave solution as follows.

Consider a piecewise smooth solution of (1) with the form

$$(u,\rho)(x,t)\begin{cases}(u_-,\rho_-), & x<\tau t,\\ (u_\delta,\beta(t)\delta(x-\sigma t)), & x=\tau t\\ (u_+,\rho_+), & x>\tau t,\end{cases} \tag{14}$$

where

$$\tau = u_\delta = \frac{[2\rho u+\rho p]+\sqrt{[\rho p]^2+4\rho_+\rho_-[u][u+p]}}{2[\rho]}, \tag{15}$$
$$\beta(t)=([\rho]u_\delta-[\rho u])t.$$

δ-measure solutions (u,ρ) of (1) (3) and (5) is given

$$u(x,t)=u_-+[u]H(x-\sigma t),\quad \rho(x,t)=\rho_-+[\rho]H(x-\sigma t)+\beta(t)\delta_L, \tag{16}$$

here $L=\{(\sigma t,t):0\le t<+\infty\}$, $H(x)$ is the Heaviside function.

The measure solution (14) and (15) satisfies the generalized Rankine-Hugoniot condition

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=u_\delta(t),\\ \dfrac{\mathrm{d}\beta(t)}{\mathrm{d}t}=[\rho]u_\delta(t)-[\rho u],\\ \dfrac{\mathrm{d}(\beta(t)u_\delta(t))}{\mathrm{d}t}=[\rho(u+P)]u_\delta(t)-[\rho u(u+P)],\end{cases} \tag{17}$$

where $[u]=u(x(t)+0)-u(x(t)-0)$ is the jump of *u* across the discontinuity $x=x(t)$, etc.

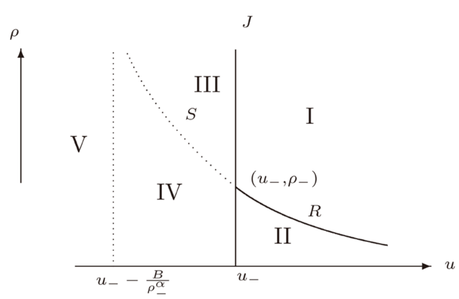

Figure 1. Wave curves in (u,ρ).

The δ-entropy condition is

$$\lambda_1(u_+,\rho_+) \le \lambda_2(u_+,\rho_+) \le u_\delta \le \lambda_1(u_-,\rho_-) \le \lambda_1(u_-,\rho_-), \tag{18}$$

which is

$$u_+ - \frac{B\alpha}{\rho_+^\alpha} \le u_+ \le u_\delta \le u_- - \frac{B\alpha}{\rho_-^\alpha} \le u_- . \tag{19}$$

We know that all the characteristics on both sides of the δ-shock wave curve are incoming.

In order to consider the interaction of elementary waves containing delta shock wave, we briefly recall the concept of left- and right-hand side delta functions as follows.

Let $\overline{R_+^2}$ be divided into two disjoint open sets Ω_1 and Ω_2 with a piecewise smooth boundary curve *L*, which satisfies $\Omega_1 \cap \Omega_2 = \varnothing$ and $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{R_+^2}$. Let $\mathcal{C}(\Omega_i)$ and $\mathcal{M}(\Omega_i)$ be the space of bounded and continuous real-valued functions equipped with the L^∞-norm and the space of measures on $\Omega_i\,(i=1,2)$, respectively. Let us assume that $\mathcal{C}_L = (\mathcal{C}(\Omega_1),\mathcal{C}(\Omega_2))$ and $\mathcal{M}_L = (\mathcal{M}(\Omega_1),\mathcal{M}(\Omega_2))$, then the product of $G=(G_1,G_2)\in\mathcal{C}_L$ and $D=(D_1,D_2)\in\mathcal{M}_L$ is defined as an element $GD=(G_1D_1,G_2D_2)\in\mathcal{M}_L$, where $G_iD_i\,(i=1,2)$ can be defined as the usual product of a continuous function and a measure. Thus, it is obvious that the above-defined product makes sense.

We view the measure on $\overline{\Omega_i}$ as a measure on $\overline{R_+^2}$ with support in $\overline{\Omega_i}\,(i=1,2)$. Then the mapping $m: M_L \to M\left(\overline{R_+^2}\right)$ can be obtained by taking $m(D)=D_1+D_2$. In a similar way, we have $m(GD)=G_1D_1+G_2D_2$. The solution concept used in our paper when we consider the delta shock can be described as follows: carry out the multiplication and composition in the space M_L and take the mapping $m: M_L \to M\left(\overline{R_+^2}\right)$ before differentiation in the space of distributions.

Based on the above analysis, we have the following conclusion.

Theorem 2.1 *The Riemann solution of the initial value problem* (1) (3) *and* (5) *is unique.*

3. Interactions of Elementary Waves Containing Delta Shock Wave

Now we study the elementary wave interactions for (1) (3) with (4). (4) is regarded as the perturbation on the Riemann initial values (5). In order to cover all the cases containing delta shock completely, we have three possibilities according to the different combinations from $(-\varepsilon,0)$ and $(\varepsilon,0)$ as follows: δS and δS, $S+J$ and δS, $R+J$ and δS.

Case 1: δS_1 **and** δS_2**.**

We consider the interaction of a delta shock wave δS_1 emanating from $(-\varepsilon,0)$ and a delta shock wave δS_2 emanating from $(\varepsilon,0)$. When *t* is small enough, the solution of the initial value problem (1) (3) and (4) is expressed as (**Figure 2**).

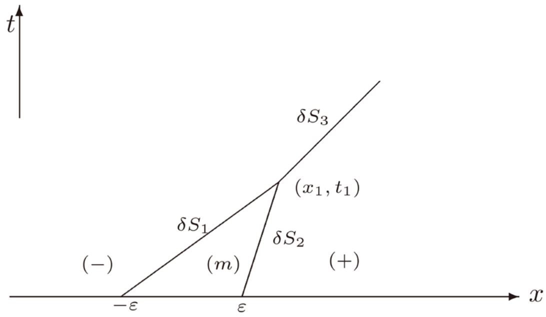

Figure 2. Interaction of δS_1 and δS_2.

$$\left(u_-,\rho_-\right)+\delta S_1+\left(u_m,\rho_m\right)+\delta S_2+\left(u_+,\rho_+\right),$$

The propagation speed of δS_1 and δS_2 satisfy respectively the δ-entropy conditions

$$u_m-\frac{B\alpha}{\rho_m^\alpha}<u_m<\tau_{\delta_1}<u_--\frac{B\alpha}{\rho_-^\alpha}<u_-,$$

$$u_+-\frac{B\alpha}{\rho_+^\alpha}<u_+<\tau_{\delta_2}<u_m-\frac{B\alpha}{\rho_m^\alpha}<u_m,$$

and we know that

$$\tau_{\delta_1}=\frac{\left(2\rho_m u_m+\rho_m P_m-2\rho_-u_--\rho_-P_-\right)+\sqrt{\left(\rho_m P_m-\rho_-P_-\right)^2+4\rho_-\rho_m\left(u_m-u_-\right)\left(u_m+P_m-u_-P_-\right)}}{2\left(\rho_m-\rho_-\right)},$$

$$\tau_{\delta_2}=\frac{\left(2\rho_+u_++\rho_+P_+-2\rho_m u_m-\rho_m P_m\right)+\sqrt{\left(\rho_+P_+-\rho_m P_m\right)^2+4\rho_m\rho_+\left(u_+-u_m\right)\left(u_++P_+-u_m P_m\right)}}{2\left(\rho_+-\rho_m\right)},$$

$$\beta_1(t)=\left(\left(\rho_m-\rho_-\right)\tau_{\delta_1}-\left(\rho_m u_m-\rho_-u_-\right)\right)t,$$

$$\beta_2(t)=\left(\left(\rho_+-\rho_m\right)\tau_{\delta_2}-\left(\rho_+u_+-\rho_m-u_m\right)\right)t,$$

where τ_{δ_1} and τ_{δ_2} are respectively the propagating speed of δS_1 and δS_2, $\beta_1(t)$ and $\beta_2(t)$ are respectively the strength of δS_1 and δS_2.

It easy to see that $\tau_{\delta_1}>\tau_{\delta_2}$ which shows that δS_1 will overtake δS_2 at a point $\left(x_1,t_1\right)$ which is determined by

$$\begin{cases}x_1+\varepsilon=\tau_{\delta_1}t_1,\\ x_1-\varepsilon=\tau_{\delta_1}t_1,\end{cases}\tag{20}$$

which yields

$$\left(x_1,t_1\right)=\left(\frac{\left(u_{\delta_1}+u_{\delta_2}\right)\varepsilon}{u_{\delta_1}-u_{\delta_2}},\frac{2\varepsilon}{u_{\delta_1}-u_{\delta_2}}\right).\tag{21}$$

At the intersection point $\left(x_1,t_1\right)$, a new initial data is formed as follows

$$u\big|_{t_1}=\begin{cases}u_-, & x<x_1,\\ u_+, & x>x_1,\end{cases}\quad \rho\big|_{t_1}=\begin{Bmatrix}\rho_-, & x<x_1,\\ \rho_+, & x>x_1,\end{Bmatrix}+\beta\left(t_1\right)\delta_{\left(x_1,t_1\right)},\tag{22}$$

where $\beta(t_1)=\beta_1(t_1)+\beta_1(t_1)$ is the sum of the strengths of the incoming delta shock wave δS_1 and δS_2. A new delta shock wave will generate after interaction and we denote it by δS_3, which is given by

$$\begin{cases} u(x,t)=u_-+(u_+-u_-)H, \\ \rho(x,t)=\rho_-+(\rho_+-\rho_-)H+\beta_-(t)D^-+\beta_+(t)D^+, \end{cases} \tag{23}$$

where H is the Heaviside function and $\beta(t)=\beta_-(t)D^-+\beta_+(t)D^+$ is a split delta function. All of them are supported by the line $x=x_1+(t-t_1)\tau_{\delta_3}$, τ_{δ_3} is the propagating speed of δS_3. Although they are supported by the same line, D^- is the delta measure on the set $\overline{R_+^2}\cap\{(x,t)\,|\,x\le x_1+(t-t_1)\tau_{\delta_3}\}$ and D^+ is the delta measure on the set $\overline{R_+^2}\cap\{(x,t)\,|\,x\ge x_1+(t-t_1)\tau_{\delta_3}\}$ respectively.

From (23), we obtain

$$\rho_t=(\rho_+-\rho_-)(-\tau_{\delta_3})\delta+\beta_-'(t)\delta+\beta_+'(t)\delta-\tau_{\delta_3}(\beta_-(t)+\beta_+(t))\delta', \tag{24}$$

$$(\rho u)_x=(\rho_+u_+-\rho_-u_-)\delta+(u_-\beta_-(t)+u_+\beta_+(t))\delta'. \tag{25}$$

Substituting (24) and (25) into the first equation of (1), we obtain

$$\beta(t)=\beta(t_1)+\left(\tau_{\delta_3}(\rho_+-\rho_-)-(\rho_+u_+-\rho_-u_-)\right)(t-t_1).$$

From (15), we get

$$\tau_{\delta_3}=\frac{(2\rho_+u_++\rho_+P_+-2\rho_-u_--\rho_-P_-)+\sqrt{(\rho_+P_+-\rho_-P_-)^2+4\rho_-\rho_+(u_+-u_-)(u_++P_+-u_-P_-)}}{2(\rho_+-\rho_-)}.$$

Case 2: $S+J$ **and** δS.

In this case, a shock wave followed by a contact discontinuity emits from $(-\varepsilon,0)$ and a delta shock wave emits from $(\varepsilon,0)$ (**Figure 3**). The propagating speed of the contact discontinuity is $\tau_1=u_m=u_*$, and the propagating speed of the delta shock wave satisfies the δ-entropy condition

$u_+-\frac{B\alpha}{\rho_+^\alpha}<u_+<\tau_{\delta_1}<u_m-\frac{B\alpha}{\rho_m^\alpha}<u_m$. It easy to see that J will overtake δS_1 at (x_1,t_1) which given by

$$\begin{cases} x_1+\varepsilon=u_mt_1, \\ x_1-\varepsilon=\tau_{\delta_1}t_1. \end{cases} \tag{26}$$

From (26), we get

$$(x_1,t_1)=\left(\frac{(u_{\delta_1}+u_m)\varepsilon}{u_m-u_{\delta_1}},\frac{2\varepsilon}{u_m-u_{\delta_1}}\right). \tag{27}$$

The delta shock wave δS_1 will pass through J with the same speed as before but the strength changes due to the difference between ρ_* and ρ_m. We still denote it with δS_1 after the time t_1, and

$\beta(t_1)=(\rho_+-\rho_m)\tau_{\delta_1}-(\rho_+u_+-\rho_mu_m)$.

From the δ-entropy condition

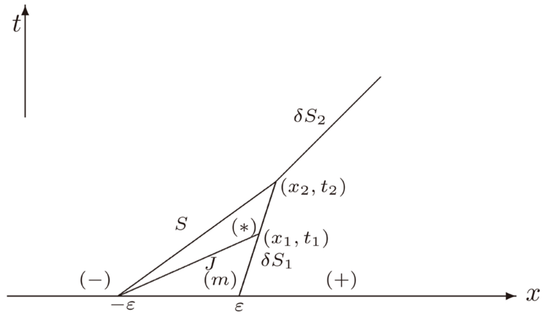

Figure 3. Interaction of $S+J$ and δS.

$$u_+ - \frac{B\alpha}{\rho_+^\alpha} < u_+ < \tau_{\delta_1} < u_* - \frac{B\alpha}{\rho_*^\alpha} < u_*,$$

and the shock entropy condition

$$u_* - \frac{B\alpha}{\rho_*^\alpha} < \tau_1 < u_- - \frac{B\alpha}{\rho_-^\alpha},$$

we know that S will overtake δS_1 at (x_2,t_2) which satisfies

$$\begin{cases} x_2 + \varepsilon = \tau_1 t_2, \\ (x_2 - x_1) = \tau_{\delta_1}(t_2 - t_1), \end{cases} \tag{28}$$

thus

$$(x_2,t_2) = \left(2\varepsilon, \frac{3\varepsilon}{\tau_1}\right). \tag{29}$$

The new initial data will be formulated at (x_2,t_2) as follows

$$u\big|_{t_2} = \begin{cases} u_-, & x < x_2, \\ u_+, & x > x_2, \end{cases} \quad \rho\big|_{t_1} = \begin{cases} \rho_-, & x < x_2, \\ \rho_+, & x > x_2, \end{cases} + \beta(t_2)\delta_{(x_2,t_2)}, \tag{30}$$

$\beta(t_2) = (\rho_+ - \rho_*)\tau_{\delta_1} - (\rho_+ u_+ - \rho_* u_*)$ denotes the strength of δS_1 at the time t_2 and from $u_* = u_m$ and $u_* = u_- + \frac{B}{\rho_*^\alpha} - \frac{B}{\rho_-^\alpha}$ we can determine the value of (u_*, ρ_*).

A new delta shock wave will be generated after the interaction of S and δS_1, denoted by δS_2 here. It satisfies that

$$\begin{cases} u(x,t) = u_- + (u_+ - u_-)H, \\ \rho(x,t) = \rho_- + (\rho_+ - \rho_-)H + \beta_-(t)D^- + \beta_+(t)D^+. \end{cases} \tag{31}$$

The Heaviside function H and the split delta function $\beta(t)D = \beta_-(t)D^- + \beta_+(t)D^+$ are supported by the line $x = x_2 + (t - t_2)\tau_{\delta_2}$, τ_{δ_2} is the propagating speed of δS_2. From (31), similarly with the above case we obtain that the strength of δS_2 after the interaction of S and δS_1 is

$$\beta(t) = \beta(t_2) + \left(\tau_{\delta_2}(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-)\right)(t - t_2),$$

$$\beta_-(t) = \frac{\tau_{\delta_2} - u_+}{u_- - u_+}\beta(t),$$

$$\beta_+(t) = \frac{u_- - \tau_{\delta_2}}{u_- - u_+}\beta(t).$$

As $t > t_2$, the delta shock wave δS_2 will propagate with the invariant speed τ_{δ_2} which is given by (15) with (u_+, ρ_+) and (u_-, ρ_-) as its right and left state. Furthermore, from the condition $u_- < u_- - \frac{B}{\rho_-^\alpha} < u_m < u_-$ we know the δ-entropy condition for the new delta shock wave τ_{δ_2} holds which shows τ_{δ_2} is an overcompresive wave.

Case 3: $R + J$ **and** δS.

When t is small enough, the solution of the initial value problem (1) (3) and (5) can be described by (**Figure 4**)

$$(u_-, \rho_-) + R + (u_*, \rho_*) + J + (u_m, \rho_m) + \delta S_1 + (u_+, \rho_+).$$

Similar with the above case, the contact discontinuity J will overtake the delta shock wave δS_1 at the point (x_1, t_1) given by (27). The delta shock wave δS_1 will pass through J with the same speed as before but the strength changes due to the difference between ρ_* and ρ_m. We still denote it with δS_1 after the time t_1, and $\beta(t_1) = (\rho_+ - \rho_m)\tau_{\delta_1} - (\rho_+ u_+ - \rho_m u_m)$. Since the propagation speed of wave front in the rarefaction wave is $\lambda_1 = u_* - \frac{B\alpha}{\rho_*^\alpha}$ and that of the delta shock wave δS_1 satisfies the δ-entropy condition

$$u_+ - \frac{B\alpha}{\rho_+^\alpha} < u_+ < \tau_{\delta_1} < u_* - \frac{B\alpha}{\rho_*^\alpha} < u_*,$$

it is shown that R will interact with δS_1 at (x_2, t_2) which is determined by

$$\begin{cases} x_2 + \varepsilon = \lambda_1 t_2, \\ (x_2 - x_1) = \tau_{\delta_1}(t_2 - t_1), \end{cases} \tag{32}$$

it follows that $(x_2, t_2) = \left(\frac{(\lambda_1 + \tau_{\delta_1})\varepsilon}{\lambda_1 - \tau_{\delta_1}}, \frac{2\varepsilon}{\lambda_1 - \tau_{\delta_1}}\right)$. The strength of δS_1 at (x_2, t_2) can be calculated by $\beta(t_2) = \left((\rho_+ - \rho_*)\tau_{\delta_1} - (\rho_+ u_+ - \rho_* u_*)\right)t_2$. At the same time, a new delta shock wave δS_2 with varying speed is generated. Here we use $\Gamma : x = x(t), t \ge t_2$ to express the curve of δS_2 and it is given in the following form

$$u(x,t) = \begin{cases} \frac{1}{1-\alpha}\frac{x-\varepsilon}{t} + \frac{\alpha}{1-\alpha}\left(-u_* + \frac{B}{\rho_*^\alpha}\right), & x < x(t), \\ u_+, & x > x(t), \end{cases} \tag{33}$$

$$\rho(x,t) = \begin{Bmatrix} \left(B(1-\alpha)\right)^{\frac{1}{\alpha}}\left(\frac{x-\varepsilon}{t} - u_* + \frac{B}{\rho_*^\alpha}\right)^{-\frac{1}{\alpha}}, & x < x(t), \\ \rho_+ & x > x(t), \end{Bmatrix} \tag{34}$$
$$+ \beta_-(t)D_\Gamma^- + \beta_+(t)D_\Gamma^+,$$

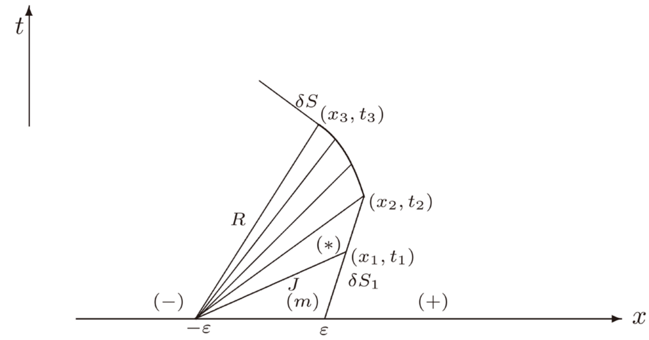

Figure 4. Interaction of $R + J$ and δS_1.

where $\beta(t) D_\Gamma = \beta_-(t) D_\Gamma^- + \beta_+(t) D_\Gamma^+$ is a split delta function on the new delta shock, and $\beta(t) = \beta_-(t) + \beta_+(t)$ is the strength of the new delta shock at the time t.

When $u_+ > u_-$, the delta shock wave cannot penetrate the rarefaction wave; when $u_+ \le u_-$, the delta shock wave can penetrate the rarefaction wave completely.

4. Conclusions

Now we construct the unique solution of the elementary wave interactions and get the following main conclusion. Using the characteristic analysis method, *i.e.*, by analyzing the elementary wave curves in the phrase plane, we get the unique solution of the initial problem (1) with the state equation (3) and the initial values (4). We observe that the elementary wave interactions have a much simpler structure for Temple class than general systems of conservation laws since the wave interaction of the same family does not generate wave of other families for Temple systems. It is important to study the elementary waves interactions for (1) not only because of their significance in practical applications in the traffic flow system for the generalized Chaplygin gas, but also because of their basic role as building blocks for the general mathematical theory of the traffic flow system.

Theorem 4.1 *The Riemann solutions of the initial value problem* (1) (3) *with the initial data* (4) *are constructed which are stable under such small perturbation on the initial data*.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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