Journal of Applied Mathematics and Physics, 2021, 9, 211-219
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On the (r,ϕ,ψ)-Suzuki Contraction for the Multi-Valued Mappings

Zamir Selko

Department of Mathematics, Faculty of Natural Sciences, University of Elbasan, "Aleksandër XHUVANI", Elbasan, Albania
Email: zamir.selko@uniel.edu.al

How to cite this paper: Selko, Z. (2021) On the (r,ϕ,ψ)-Suzuki Contraction for the Multi-Valued Mappings. *Journal of Applied Mathematics and Physics*, **9**, 211-219.
https://doi.org/10.4236/jamp.2021.92015

Received: December 16, 2020
Accepted: January 31, 2021
Published: February 3, 2021

Abstract

In this work we try to give a new contraction type in multi-valued mapping on complete metric spaces. We prove the existence of fixed point for (r,ϕ,ψ)-Suzuki contraction in such spaces. Around our paper, the function ψ is absolutely continuous, and in this case, the contraction proposed by as has a fixed point.

Keywords

Suzuki-Contraction, Multi-Valued Mappings, Fixed Point Theorem, Absolutely Continuous Functions

1. Introduction

Banach's contraction principle is a magnific tool in many fields of nonlinear analysis and in mathematical analysis. Applications in these fields are very interests and promise to other new applications. Banach's contraction principle has been generalized and extended in many directions. The authors [1] have given an important result about contractions in multi-valued complete metric spaces. We have given the generalization of this result which is a particular case of author's [2] paper. In this paper, we prove a new fixed point theorem for multi-valued mapping defined on complete metric spaces. To realize this result we give the proof of an intuitive lemma which is used to complete the proof of the main result.

Theorem 1.1 (Đorić and Lazović) *Define a nonincreasing function* ϕ *from* $[0,1)$ *into* $(0,1]$ *by*

$$\phi(r)=\begin{cases}1 & 0\le r<\dfrac{1}{2}\\ 1-r & \dfrac{1}{2}\le r<1\end{cases} \tag{1}$$

Let (X,d) be a complete metric space and T be a mapping from X into $CB(X)$. Assuming that there exists $r\in[0,1)$ such that $\phi(r)d(x,Tx)\le d(x,y)$ implies

$$H(Tx,Ty)\le r\cdot\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \tag{2}$$

for all $x,y\in X$. Then, there exists $z\in X$, such that $z\in Tz$.

This theorem is a particular case of our main result when $\phi=id$.

Theorem 1.2 (*Kaliaj*) *Let (X,d) be a complete metric space and let $T:X\to CB(X)$ be a multi-valued mapping, assuming that T is (r,ϕ,ψ)-Suzuki integral contraction. Then, for any $x\in X$ we have*

$$\psi(H(Tx,Ty))\le r\cdot\int_0^{d(x,y)}\psi'(t)\mathrm{d}t \tag{3}$$

Our main result is a particular case of this theorem when ψ is absolutely continuous.

Definition 1.3 *Let (X,d) be a complete metric space and let $CB(X)$ be the family of all nonempty closed bounded subsets of X. Define the Hausdorff metric*

$$H:CB(X)\times CB(X)\to[0,+\infty)$$

$$H(A,B)=\max\left\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\right\} \tag{4}$$

for all $A,B\in CB(X)$.

It is well-known that, if (X,d) is a complete metric space, then $(CB(X),H)$ is also a complete metric space.

Definition 1.4 *Let $T\to CB(X)$ be a multi-valued mapping. We say that T is a (r,ϕ)-Suzuki contraction with*

$$\phi(r)=\begin{cases}1 & 0\le r<\dfrac{1}{2}\\ 1-r & \dfrac{1}{2}\le r<1\end{cases} \tag{5}$$

if there exists $r\in[0,1)$ such that, the implication

$$\phi(r)d(x,Tx)\le d(x,y)\Rightarrow H(Tx,Ty)\le T_M(x,y) \tag{6}$$

holds whenever $x,y\in X$, where

$$T_M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \tag{7}$$

Definition 1.5 *The multi-valued mapping T is an (r,ϕ,ψ)-Suzuki integral contraction if ϕ is defined in conditions of Theorem* 1.1 *and there exist $r\in[0,1)$ and $\psi\in\Psi$, such that the implication*

$$\phi(r)\cdot\int_0^{d(x,Tx)}\psi'(t)\mathrm{d}t\le\psi(d(x,y))\Rightarrow\psi(H(Tx,Ty))\le r\cdot T_\int(x,y) \tag{8}$$

holds whenever $x,y\in X$, where

$$T_{\mathrm{j}}(x,y)=\max\left\{\int_0^{d}(x,y)\psi'(t)\mathrm{d}t,\int_0^{d}(x,Tx)\psi'(t)\mathrm{d}t,\right.$$
$$\left.\int_0^{d}(y,Ty)\psi'(t)\mathrm{d}t,\int_0^{\frac{d(x,Ty)+d(y,Tx)}{2}}\psi'(t)\mathrm{d}t\right\} \tag{9}$$

Definition 1.6 *The multi-valued mapping T is said to be a* (r,ϕ,ψ) *-Suzuki contraction with*

$$\phi(r)=\begin{cases}1 & 0\le r<\frac{1}{2}\\ 1-r & \frac{1}{2}\le r<1\end{cases} \tag{10}$$

if there exists $r\in[0,1)$ and $\psi\in\Psi$ such that, the implication

$$\phi(r)\psi(d(x,Tx))\le\psi(d(x,y))\Rightarrow\psi(H(Tx,Ty))\le T_{\psi}(x,y) \tag{11}$$

holds whenever $x,y\in X$, where

$$T_{\psi}(x,y)=\max\left\{\psi(d(x,y)),\psi(d(x,Tx)),\psi(d(y,Ty)),\right.$$
$$\left.\psi\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right)\right\} \tag{12}$$

Definition 1.7 *A function* $f:[a,b]\to\mathbb{R}$ *is said to be absolutely continuous in* $[a,b]$ *if, given* $\varepsilon>0$*, there exists some* $\delta>0$ *such that*

$$\sum_{i=1}^{n}|f(y_i)-f(x_i)|<\varepsilon, \tag{13}$$

whenever $\{[x_i,y_i]:i=1,\cdots,n\}$ is a finite collection of mutually disjoint sub-intervals of $[a,b]$ with $\sum_{i=1}^{n}|y_i-x_i|<\delta$.

Lemma 1.8 (*Kaliaj*) *Let* (X,d) *be a complete metric space and let* $T:X\to CB(X)$ *be a* (r,ϕ,ψ) *-Suzuki integral contraction and* $z_n\in X$ *with*

$$\lim_{n\to\infty}z_n=z\in X \tag{14}$$

and $z_{n+1}\in Tz_n$ for all $n\in\mathbb{N}$.

Then, exists a $r\in[0,1)$ such that

$$\psi(d(z,Tx))\le r\cdot\max\left\{\int_0^{d(z,x)}\psi'(t)\mathrm{d}t,\int_0^{d(x,Tx)}\psi'(t)\mathrm{d}t\right\} \tag{15}$$

for all $x\in X\setminus\{z\}$.

When ψ is absolutely continuous, we have this

Corollary 1.9 *Let* (X,d) *be a complete metric space and let* $T:X\to CB(X)$ *be a* (r,ϕ,ψ) *-Suzuki contraction and* $z_n\in X$ *with*

$$\lim_{n\to\infty}z_n=z\in X \tag{16}$$

and $z_{n+1}\in Tz_n$ for all $n\in\mathbb{N}$.

Then, exists a $r\in[0,1)$ such that

$$\psi(d(z,Tx))\le r\cdot\max\{\psi(d(z,x)),\psi(d(x,Tx))\} \tag{17}$$

for all $x \in X \setminus \{z\}$.

The main result is Theorem 2.1. First, we give the proof of this lemma:

Lemma 1.10 *Let* (X,d) *be a complete metric space and* $T: X \to CB(X)$ *be a* (r,ϕ,ψ) *-Suzuki contraction. Then, the implication*

$$\phi(r)\psi(d(x,Tx)) \le \psi(d(x,y)) \Rightarrow \psi(H(Tx,Ty)) \le r \cdot \psi(d(x,y))$$

holds for all $y \in Tx$.

Proof: First, since $y \in Tx$, it follows that $d(y,Tx) = 0$. By the definition of distance of a point to the set we can obtain this inequality:

$$d(x,Tx) = \inf\{d(x,y) : y \in Tx\} \le d(x,y)$$

Because the function $\phi(r)$ takes values in $(0,1]$, it implies

$$\phi(r)\psi(d(x,Tx)) \le \psi(d(x,Tx)).$$

Since ψ is increase monotonic we have

$$\psi(d(x,Tx)) \le \psi(d(x,y))$$

and from the fact that T is a (r,ϕ,ψ) -Suzuki contraction, it follows that

$$\phi(H(Tx,Ty)) \le r \cdot T_\psi(x,y)$$

where,

$$T_\psi(x,y) = \max\left\{\psi(d(x,y)), \psi(d(x,Tx)), \psi(d(y,Ty)), \psi\left(\frac{d(x,Ty)+0}{2}\right)\right\}.$$

Substituting, the last inequality will be transformed in

$$\begin{aligned} &\phi(H(Tx,Ty)) \\ &\le r \cdot \max\left\{\psi(d(x,y)), \psi(d(x,Tx)), \psi(d(y,Ty)), \psi\left(\frac{d(x,Ty)+0}{2}\right)\right\} \end{aligned}$$

which implies

$$\begin{aligned} &\phi(H(Tx,Ty)) \\ &\le r \cdot \max\{\psi(d(x,y)), \psi(d(x,Tx)), \psi(d(y,Ty)), \psi(\max\{d(x,y), d(y,Ty)\})\} \quad (18) \\ &= r \cdot \max\{\psi(d(x,Tx)), \psi(d(y,Ty))\}. \end{aligned}$$

Remember that, from the definition of Hausdorff distance, we can write:

$$H(Tx,Ty) = \max\left\{\sup_{x' \in Tx} d(x',Ty), \sup_{y' \in Ty} d(y',Tx)\right\} \ge \sup_{x' \in Tx} d(x',Ty) \ge d(y,Tx)$$

which implies

$$\psi(d(y,Tx)) \le \psi(H(Tx,Ty)).$$

Using the fact that ψ is monotone increasing, it follows that

$$\psi(d(y,Ty)) \le \psi(H(Tx,Ty)) \le r \cdot \max\{\psi(d(x,y)), \psi(d(y,Ty))\}$$

But

$$\psi(d(y,Ty)) \le r \cdot \max\{\psi(d(x,y)), \psi(d(y,Ty))\},$$

and since $r \in [0,1)$, it follows that

$$\max\{\psi(d(x,y)), \psi(d(y,Ty))\} = \psi(d(x,y))$$

Indeed, using the fact that H is a Hausdorff distance, we can write:

$$\psi(d(y,Ty)) \le \psi(H(Tx,Ty)) \le r \cdot \max\{\psi d(x,y), \psi d(y,Ty)\}$$

and so,

$$\psi(d(y,Ty)) = r \cdot \max\{\psi(d(x,y)), \psi(d(y,Ty))\}.$$

If the maximum element of the set

$$\{\psi(d(x,y)), \psi(d(y,Ty))\}$$

it was $\psi(d(y,Ty))$ then,

$$d(y,Ty) \le r \cdot d(y,Ty)(!)$$

which is a contradiction, because $r < 1$. The last result with the result of inequality (18) implies

$$\psi(H(Tx,Ty)) \le r \cdot \psi(d(x,y))$$

Since y was arbitrary, the last result holds $\forall y \in Tx$.

Since ψ is absolutely continuous, by the Corollary 1.9, we can write:

$$\psi(d(z,Tx)) \le r \cdot \max\{\psi(d(z,x)), \psi(d(x,Tx))\} \tag{19}$$

2. The Main Result

Theorem 2.1 *Let* (X,d) *be a complete metric space and let the mapping* $T: X \to CB(X)$ *be a* (r,ϕ,ψ) *-Suzuki contraction. Then,* T *has a fixed point.*

Proof. Let z_0 be an arbitrary fixed point in *X*. Choose a real number $r < \overline{r} < 1$. If $d(z_0,Tz_0) = 0$, then $z_0 \in Tz_0$. Hence z_0 is a fixed point of T and the proof has finished.

Assume that $d(z_0,Tz_0) > 0$. Then, there exists $z_1 \in Tz_0$ with

$$d(z_0,z_1) \ge d(z_0,Tz_0) > 0.$$

Since $z_1 \in Tz_0$, using Lemma 1.10 we obtain:

$$\psi(d(z_1,Tz_1)) \le \max\{\psi(H(Tz_0,Tz_1))\} \le r \cdot \psi(d(z_0,z_1)) < \overline{r} \cdot \psi(d(z_0,z_1)). \tag{20}$$

We assume that $r > 0$ since:

$$r = 0 \Rightarrow \psi(d(z_1,Tz_1)) = 0 \Rightarrow d(z_1,Tz_1) = 0 \Rightarrow z_1 \in Tz_1. \tag{21}$$

As before, if $d(z_1,Tz_1) = 0$, for similarity, z_1 is a fixed point for the mapping T and the proof is done.

Assume that $d(z_1,Tz_1) = t_1 > 0$. Since ψ is continuous at t_1, given

$$\varepsilon_1 = \overline{r}\psi(d(z_0,z_1)) - \psi(t_1) > 0$$

there exists $\delta_1 > 0$, such that

$$t_1 \le t < t_1 + \delta_1 \Rightarrow \psi(t_1) \le \psi(t) < \overline{r}\psi(d(z_0, z_1)),$$

and, since there exists $z_2 \in Tz_1$ such that

$$t_1 \le d(z_1, z_2) < t_1 + \delta_1$$

it follows that

$$\psi(t_1) \le \psi(d(z_1, z_2)) < \overline{r}\psi(d(z_0, z_1)). \tag{22}$$

Inductively, assume now that we chose $z_n \in Tz_{n-1}$. The, by Lemma 1.10 we have

$$\overline{r}\psi(d(z_{n-1}, z_n)) - \psi(d(z_{n1}, Tz_n)) > 0.$$

If $d(z_n, Tz_n) = 0$, then z_n is a fixed point of (r, ϕ, ψ)-Suzuki contraction *T*, and the proof is done.

Assume that $d(z_n, Tz_n) > 0$. Since ψ is continuous at t_n, given

$$\varepsilon_n = \overline{r}\psi(d(z_{n-1}, z_n)) - \psi(t_n) > 0$$

there exists $\delta_n > 0$, such that

$$t_n \le t < t_n + \delta_n \Rightarrow \psi(t_n) \le \psi(t) < \overline{r}\psi(d(z_{n-1}, z_n)),$$

and, since there exists $z_{n+1} \in Tz_n$ such that

$$t_n \le d(z_n, z_{n+1}) < t_n + \delta n$$

it follows that

$$\psi(t_n) \le \psi(d(z_n, z_{n+1})) < \overline{r}\psi(d(z_{n-1}, z_n)). \tag{23}$$

Since $z_{n+1} \in Tz_n$, by Lemma 1.10, we have

$$\overline{r}\psi(d(z_n, z_{n+1})) - \psi(d(z_{n+1}, Tz_{n+1})) > 0.$$

By above construction, we obtain a sequence (z_n) with terms in *X* such that

$$\psi(d(z_n, z_{n+1})) < \overline{r}\psi(d(z_{n-1}, z_n)), \forall n \in \mathbb{N}. \tag{24}$$

Hence, we get

$$\psi(d(z_n, z_{n+1})) < \overline{r}^n\psi(d(z_0, z_1)), \forall n \in \mathbb{N}.$$

and since $\psi(d(z_n, z_{n+1})) > d(z_n, z_{n+1})$ it follows that

$$d(z_n, z_{n+1}) < \overline{r}^n\psi(d(z_0, z_1)), \forall n \in \mathbb{N}.$$

Summing side by side for $n = 1$ to ∞ it follows that

$$\sum_{n=1}^{\infty} d(z_n, z_{n+1}) \le \sum_{n=1}^{\infty} \overline{r}^n\psi(d(z_0, z_1)).$$

Since $\overline{r} < 1$ the last result yields that (z_n) is a Cauchy sequence in *X*, and by completeness of *X* it follows that (z_n) converges to a point $z \in X$. We are going to prove that *z* is a fixed point of *T*. Suppose the contrary, *i.e.*, $z \notin Tz$. It follows that two cases are possibles:

1) $0 < r < \frac{1}{2}$

2) $\frac{1}{2} \le r < 1$

We study each case as follows:

1) For an fixed arbitrary $\omega \in Tz$, since

$$d(z,Tz) \le d(z,T\omega) + H(T\omega,Tz)$$

we obtain

$$\psi(d(z,Tz)) \le \psi(d(z,T\omega)) + \psi(H(T\omega,Tz))$$

and since by Corollary 1.9 and Lemma 1.10 we have also

$$\psi(H(Tx,Tz)) \le r \cdot \psi(d(\omega,z))$$

$$\psi(d(Tz,T\omega)) \le r \cdot \max\{\psi(d(z,\omega)), \psi(d(\omega,T\omega))\}$$

and

$$\psi(d(\omega,T\omega)) \le \psi(d(\omega,T\omega)) \le \psi(H(Tz,T\omega))$$
$$\le r \cdot \psi(d(d(z,\omega))) < \psi(d(z,\omega))$$

it follows that

$$\psi(d(z,Tz)) < 2r\psi(d(z,\omega)). \tag{25}$$

Since ω was arbitrary, the last results yield

$$\psi(d(z,Tz)) \le 2r\psi(d(z,\omega))\omega \in Tz$$

for all $\omega \in Tz$. From equality

$$d(z,Tz) = \inf\{d(z,\omega) : \omega \in Tz\}$$

it follows that there exists sequence $\omega_n \subset Tz$ such that

$$\lim_{n\to\infty} d(z,\omega_n) = d(z,Tz)$$

Then, by inequality (25) we obtain

$$\psi(d(z,Tz)) < 2r\psi(d(z,Tz))$$

and since $0 < 2r < 1$ it follows that

$$\psi(d(z,Tz)) = 0 \Rightarrow d(z,Tz) = 0$$

or

$$\psi(d(z,Tz)) > 0 \Rightarrow \psi(d(z,Tz)) < \psi(d(z,Tz))$$

which is a contradiction and as consequence, $z \in Tz$.

2) Let be an arbitrary $x \in X \setminus x$. Since

$$d(x,Tx) \le d(x,z) + d(z,Tx)$$

we get

$$\begin{aligned}\psi(d(x,Tx)) &\le \psi(d(x,z)+d(z,Tx)) \\ &\le \psi(d(z,Tx)) - \psi[d(z,x)+d(z,Tx)] \\ &\le \psi(d(z,Tx)) + \psi(d(z,x))\end{aligned} \tag{26}$$

Since $x \neq z$, by Corollary 1.9, we have

$$\psi(d(z,Tx)) \le r \cdot \max\{\psi(d(z,x)), \psi(d(x,Tx))\}. \tag{27}$$

If $\psi(d(x,Tx)) \le \psi(d(z,x))$ then

$$\phi(r)\psi(d(x,Tx)) \le \psi(d(z,x)) \tag{28}$$

Otherwise, if $\psi(d(x,Tx)) > \psi(d(z,x))$ then we obtain by inequality (26) that

$$\psi(d(z,Tx)) \le r \cdot \psi(d(xTx))$$

and using inequality (26) again, we get

$$\phi(r)\psi(d(x,Tx)) = (1-r)\psi(d(x,Tx)) \le \psi(d(z,x)).$$

Since x was arbitrary, combining last result with inequality (28) yields

$$\phi(r)\psi(d(x,Tx)) \le \psi(d(z,x)), \forall x \in X \setminus z.$$

Then, by hypothesis, it follow that

$$\begin{aligned}\psi(H(Tx,Tz)) \le r \cdot \max\{&\psi(d(z,x)), \psi(d(x,Tz)), \psi(d(x,Tx)), \\ &\psi(\max\{d(z,Tx), d(x,Tz)\})\}.\end{aligned}$$

whenever $x \in X \setminus z$. Clearly, if $x = z$ then, the last inequality also holds. Thus, the last inequality holds for all $x \in X$. In particular, for $x = z_n$, we have

$$\begin{aligned}&\psi(d(z_{n+1},Tz)) \le H(Tz_n,Tz) \le r \cdot \max\{\psi(d(z,z_n)), \psi(d(z,Tz)), \\ &\psi(d(z_n,z_{n+1})), \psi(\max\{d(z,z_{n+1}), d(z_n,Tz)\})\}.\end{aligned}$$

Hence, by Theorem IX.4.1 in [3], it follows that

$$0 < \psi(d(z,Tz)) = \lim_{n\to\infty} \psi(d(z_{n+1},Tz)) \le r \cdot \psi(d(z,Tz)) \le \psi(d(z,Tz)).$$

This contradiction shows that $z \in Tz$ and the proof is done.

3. Conclusion

In this paper, we studied the (r,ϕ,ψ)-Suzuki contraction for multi-valued mappings in the complete metric spaces generated by the family of all nonempty closed bounded subsets of a set *X*, refereed as $CB(X)$. We proved that this contraction has a fixed point.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Đorić and Lazović (2011) Some Suzuki-Type Fixed Point Theorems for Multi-Valued Mappings and Applications. *Fixed Point Theory and Applications*, **2011**, Article Number: 40. https://doi.org/10.1186/1687-1812-2011-40

[2] Kaliaj, S.B. (2020) An Integral Suzuki-Type Fixed Point Theorem with Application. 2009.08643, arXiv, math.FA.

[3] Natanason, I.P. (1961) Theory of Functions of a Real Variables. Frederick Ungar Publishing Co., New York.