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Cesàro Bounded Weighted Backward Shift

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Abstract

In this paper, we give a criterion of the absolutely Cesàro bounded weighted backward shift in spirit of the comparison method. Our approach is to construct the proper product of weight functions $\prod_a^b w_n$ by the fraction of two monomials of the indexes, then we apply proper scaling to give Cesàro boundedness. In particular, we present a new example of non Cesàro bounded weighted backward shift on ℓ^p.

Keywords

Cesàro Boundedness, Absolutely Cesàro Boundedness, Backward Weighted Shift

1. Introduction

Let X be a complex Banach space and $B(X)$ be the Banach algebra of all linear bounded operators on X. Given $T \in B(X)$, the *Cesàro mean* of T is the family of operators $\{M_T(n)\}_{n\in\mathbb{N}} \subset B(X)$ which is defined by

$$M_T(n)x := \frac{1}{n+1}\sum_{j=0}^{n} T^j x$$

for $x \in X$. The operator $T \in B(X)$ is called *Cesàro bounded* if $\{M_T(n)\}_{n\in\mathbb{N}}$ is bounded in $B(X)$. That is $\sup_n \|M_T(n)\| < \infty$. The operator $T \in B(X)$ is called *absolutely Cesàro bounded* if there is $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n+1}\sum_{j=0}^{n}\|T^j x\| \le C\|x\|, \quad \forall x \in X.$$

It is clear that absolutely Cesàro bounded operators are Cesàro bounded. The concept of Cesàro boundness is highly connected with the dynamic of linear operators. It was firstly introduced by Hou and Luo in [1]. In their articles, they investigated that the unilateral weighted backward shift with weights

$w_k := \frac{2k}{2k-1}, (k \in \mathbb{N})$ is absolutely Cesàro bounded. Then it attracts lots of attentions by several mathematicians. Interested readers can refer [2] for the theory of linear chaos and [1] [3] [4] for some results of Cesàro boundedness. It was proved in [5] that the unilateral weighted backward shift operator T with weights $w_k := \left(\frac{k}{k-1}\right)^{\alpha}$ on the $\ell^p(\mathbb{N})$ ($1 \le p < \infty$) is absolutely Cesàro bounded for $0 < \alpha < \frac{1}{p}$, and operator T with weights $w_k := \left(\frac{k}{k-1}\right)^{1/p}$ is not Cesàro bounded. [6] generalized this work to the fractional case, constructed a weighted shift operator belonging to this class of operators, then they showed that the unilateral weighted backward shift operator T is absolutely (C,α) Cesàro bounded for $0 < \alpha \le 1$, and T is not (C,α) Cesàro bounded for any α. We can find more details about of the nth Cesàro mean of order α of the powers of T [7] [8] [9] [10]. Specially, when $\alpha = 1$, it is general Cesàro mean. The relation between (C,α) Cesàro mean and (C,α) strongly (weakly) ergodicity was given [11]. Example 5 in [12] proved that the unilateral weighted backward shift not have distributional unbounded orbit. [13] discussed that a distributionally unbounded orbit of the operator is not absolutely Cesàro bounded. Distributionally chaotic of type $DC2\frac{1}{2}$ is not absolutely Cesàro bounded. [14] gave some equivalent characterizations of absolutely Cesàro bounded operators. In [13] [15], firstly they selected the sequence of weights v, then showed that the unilateral backward shift B on $\ell^p(v)$ ($1 \le p < \infty$) is absolutely Cesàro bounded.

If $1 \le p < \infty$, denote by $\ell^p(\mathbb{N})$ the space of p-th summable sequences. Let $\{e_n : n = 1,2,\cdots\}$ be the canonical basis of $\ell^p(\mathbb{N})$. Any vector $x \in \ell^p(\mathbb{N})$ has the unique representation $x = \sum_{j=1}^{\infty} \alpha_j e_j$ where $\{\alpha_j : j = 1,\cdots\} \subset \mathbb{C}$. Let $w = (w_1, w_2, \cdots)$ be a weight sequence. We define the weighted backward shift operator B_w on $\ell^p(\mathbb{N})$ as $B_w e_1 = 0$ and $B_w e_k := w_k e_{k-1}$ for integer $k > 1$. That is for $x = (\alpha_1, \alpha_2, \cdots) \in \ell^p(\mathbb{N})$, $B_w(\alpha_1, \alpha_2, \cdots) = (w_2\alpha_2, w_3\alpha_3, \cdots)$.

The boundedness of the weighted backward shifts is studied intensively for decades. Our motivation is to characterize the Cesàro bounded weighted backward shift and give a practical method to distinguish the Cesàro bounded backward shifts. Our results include those concrete examples in [1] [5] [6].

2. A Criterion Based on the Comparison Principle

We first study the case when the weighted backward shift B_w is not Cesàro bounded. Our method is to estimate the products of the weights $\prod_i^j w_n$ by the a fraction of two monomials of the indexes. In the sequel, to make the argument more compact, we set $\prod_{n=i}^{j} a_n = 1$ whenever $j < i$.

Theorem 1. For $1 \le p < \infty$, let w be a weight sequence and B_w the weighted backward shift on $\ell^p(\mathbb{N})$. Suppose $\prod_{n=i}^{j} w_n \gtrsim \frac{j^{s/p}}{i^{t/p}}$ and the real pair (t,s) satisfies one of the following three conditions:

(1) $t>1$ and $s>1$;

(2) $t=1$ and $s\ge 1$;

(3) $t<1$ and $t<s$.

Then B_w is not Cesàro bounded on $\ell^p(\mathbb{N})$.

Proof. Let $y_{N+1} := \frac{1}{(N+1)^{1/p}} \sum_{n=1}^{N+1} e_n$, where N is an even integers. Therefore $\|y_{N+1}\|_p = 1$. We compute directly that

$$\begin{aligned}
\left\| \frac{1}{N+1} \sum_{j=0}^{N} T^j y_{N+1} \right\|_p^p &= \left\| \frac{1}{(N+1)^{1+1/p}} \sum_{k=1}^{N+1} \left(\sum_{j=k}^{N+1} \prod_{n=k+1}^{N-j+k+1} w_n \right) e_k \right\|_p^p \\
&= \frac{1}{(N+1)^{p+1}} \sum_{k=1}^{N+1} \left(\sum_{j=k}^{N+1} \prod_{n=k+1}^{N-j+k+1} w_n \right)^p \\
&\gtrsim \frac{1}{(N+1)^{p+1}} \sum_{k=1}^{N+1} \left(\sum_{j=k}^{N+1} \frac{(N-j+k+1)^{s/p}}{k^{t/p}} \right)^p \\
&\gtrsim \frac{1}{(N+1)^{p+1}} \sum_{k=1}^{N/2+1} \frac{1}{k^t} \left(\sum_{j=N/2+1}^{N+1} (N-j+k+1)^{s/p} \right)^p \\
&\ge \begin{cases} \frac{1}{(N+1)^{p+1}} \left(\sum_{k=1}^{N/2+1} \frac{1}{k^t} \right) \left(\sum_{j=N/2+1}^{N+1} (N-j+2)^{s/p} \right)^p, & \text{if } s>0; \\ \frac{1}{(N+1)^{p+1}} \left(\sum_{k=1}^{N/2+1} \frac{1}{k^t} \right) \left(\sum_{j=N/2+1}^{N+1} \left(\frac{3N}{2} - j + 1 \right)^{s/p} \right)^p, & \text{if } s \le 0. \end{cases}
\end{aligned}$$

For short, we define

$$I = \sum_{k=1}^{\frac{N}{2}+1} k^{-t},\ II = \sum_{j=\frac{N}{2}+1}^{N+1} (N-j+2)^{\frac{s}{p}} \text{ and } III = \sum_{j=\frac{N}{2}+1}^{N+1} \left(\frac{3N}{2} - j + 1 \right)^{\frac{s}{p}}.$$

If $t>1$,

$$I \gtrsim \int_1^{\frac{N}{2}+2} \frac{1}{x^t} \mathrm{d}x = \frac{1}{t-1} \left[1 - \frac{1}{\left(\frac{N}{2} + 2 \right)^{t-1}} \right] \ge \frac{2^{t-1}-1}{(t-1)2^{t-1}} \simeq 1.$$

Meanwhile $s>1$ implies

$$II = \sum_{j=1}^{\frac{N}{2}+1} j^{\frac{s}{p}} \ge \int_1^{\frac{N}{2}+1} x^{\frac{s}{p}} \mathrm{d}x = \frac{\left(\frac{N}{2}+1 \right)^{\frac{s}{p}+1} + \frac{s}{p}}{\frac{s}{p}+1} \simeq N^{\frac{s}{p}+1}.$$

Then

$$\frac{I \cdot II^{p}}{(N+1)^{p+1}} \gtrsim N^{s-1} \to \infty, \text{ as } N \to \infty.$$

If $t = 1$,

$$I \gtrsim \int_{1}^{\frac{N}{2}+2} \frac{1}{x} \mathrm{d}x = \log\left(\frac{N}{2}+2\right).$$

Meanwhile $s \geq 1$ implies $II \gtrsim N^{\frac{s}{p}+1}$. Then

$$\frac{I \cdot II^{p}}{(N+1)^{p+1}} \gtrsim N^{s-1} \log N \to \infty, \text{ as } N \to \infty.$$

If $t < 1$ and $s > \max\{t, 0\}$,

$$I \gtrsim \int_{1}^{\frac{N}{2}+2} x^{-t} \mathrm{d}x = \frac{\left(\frac{N}{2}+2\right)^{1-t} - 1}{1-t} \simeq N^{1-t}.$$

Meanwhile we have $II \gtrsim N^{\frac{s}{p}+1}$. Then

$$\frac{I \cdot II^{p}}{(N+1)^{p+1}} \gtrsim N^{s-t} \to \infty, \text{ as } N \to \infty.$$

If $t < 0$ and $t < s \leq 0$, $I \geq \int_{1}^{\frac{N}{2}+2} x^{-t} \mathrm{d}x \simeq N^{1-t}$. To estimate *III*, we have

$$III = \sum_{j=N/2}^{N} j^{s/p}.$$

If $t < s(\neq -p) \leq 0$,

$$III \geq \int_{\frac{N}{2}}^{N} x^{s/p} \mathrm{d}x = \frac{N^{s/p+1} - (N/2)^{s/p+1}}{1+s/p} = N^{s/p+1} \cdot \frac{1-2^{-s/p-1}}{1+s/p}.$$

Then

$$\frac{I \cdot III^{p}}{(N+1)^{p+1}} \gtrsim N^{s-t} \to \infty, \text{ as } N \to \infty.$$

If $t < s = -p$,

$$III \geq \int_{\frac{N}{2}}^{N} x^{-1} \mathrm{d}x = \log N - \log N/2 = \log 2.$$

Then

$$\frac{I \cdot III^{p}}{(N+1)^{p+1}} \gtrsim N^{-t-p} \to \infty, \text{ as } N \to \infty.$$

We use the same strategy to consider the absolutely Cesàro bounded weighted backward shift.

Theorem 2. For $1 \leq p < \infty$, let w be a weight sequence and B_w be the weighted backward shift on $\ell^p(\mathbb{N})$. Suppose $\prod_{m=i}^{j} |w_m| \lesssim \frac{j^{s/p}}{i^{t/p}}$ and the real pair (t, s) satisfies one of the following three conditions:

(1) $t>1$ and $s\le 1$,

(2) $t=1$ and $s<1$,

(3) $t<1$ and $s\le t$.

Then B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$.

Proof. For every $x\in\ell^p(\mathbb{N})$, denote by $x=\sum_{j=1}^{\infty}\alpha_j e_j$ for some complex sequence $\{\alpha_j\}$. Let N be a positive integer, we have

$$\begin{aligned}\sum_{n=0}^{N}\left\|B_w^n x\right\|_p^p &= \sum_{n=0}^{N}\left\|\sum_{j=n+1}^{\infty}\alpha_j\prod_{m=j+1-n}^{j}w_m e_{j-n}\right\|_p^p\\ &=\sum_{n=0}^{N}\sum_{j=n+1}^{\infty}\left|\alpha_j\right|^p\prod_{m=j+1-n}^{j}\left|w_m\right|^p=\sum_{j=1}^{\infty}\left|\alpha_j\right|^p\sum_{n=0}^{\min\{N,j-1\}}\prod_{m=j+1-n}^{j}\left|w_m\right|^p\\ &\lesssim\sum_{j=1}^{\infty}\left|\alpha_j\right|^p\sum_{n=0}^{\min\{N,j-1\}}\frac{j^s}{(j-n)^t}=\sum_{j=1}^{N}\left|\alpha_j\right|^p\sum_{n=0}^{j-1}\frac{j^s}{(j-n)^t}\\ &\quad+\sum_{j=N+1}^{2N}\left|\alpha_j\right|^p\sum_{n=0}^{N}\frac{j^s}{(j-n)^t}+\sum_{j=2N+1}^{\infty}\left|\alpha_j\right|^p\sum_{n=0}^{N}\frac{j^s}{(j-n)^t}\\ &:=S_1+S_2+S_3.\end{aligned}$$

In either case, we have $s\le t$. Whenever $j\ge 2N+1$ and $n\le N$ it is clear that $\left(\frac{j}{j-n}\right)^t\le 1$ for $t\le 0$ and $\left(\frac{j}{j-n}\right)^t\le 2^t$ for $t>0$. Then we get

$$\begin{aligned}S_3&\le\sum_{j=2N+1}^{\infty}\left|\alpha_j\right|^p\sum_{n=0}^{N}\left(\frac{j}{j-n}\right)^t(2N+1)^{s-t}\\ &\le(N+1)\cdot\max\{2^t,1\}\cdot\sum_{j=2N+1}^{\infty}\left|\alpha_j\right|^p\lesssim N\sum_{j=2N+1}^{\infty}\left|\alpha_j\right|^p.\end{aligned}$$

If $t>1$ and $s\le 1$, we estimate S_1 and S_2. For $j\le N$ we have

$$\sum_{n=0}^{j-1}\frac{j^s}{(j-n)^t}=j^s\sum_{n=1}^{j}n^{-t}\le j^s\left(1+\int_1^j x^{-t}\mathrm{d}x\right)=\frac{j^s}{t-1}\left(t-\frac{1}{j^{t-1}}\right)\lesssim N.$$

And for $j\in[N+1,2N]$,

$$\sum_{n=0}^{N}\frac{j^s}{(j-n)^t}\le j^s\sum_{n=0}^{j-1}\frac{1}{(j-n)^t}\lesssim(2N)^s\lesssim N.$$

Hence $S_1+S_2\lesssim N\sum_{j=1}^{2N}\left|\alpha_j\right|^p$.

Suppose $t=1$ and $s<1$. To estimate S_1 and S_2, for $j\le N$ we have

$$\sum_{n=0}^{j-1}\frac{j^s}{j-n}=j^s\sum_{n=1}^{j}\frac{1}{n}\le j^s\left(1+\int_1^j\frac{1}{x}\mathrm{d}x\right)=j^s\log j+j^s\lesssim N.$$

And for $j\in[N+1,2N]$,

$$\sum_{n=0}^{N}\frac{j^s}{j-n}\le j^s\sum_{n=0}^{j-1}\frac{1}{j-n}\lesssim j^s\log j\lesssim N.$$

Hence $S_1+S_2\lesssim N\sum_{j=1}^{2N}\left|\alpha_j\right|^p$.

Now we suppose $t<1$ and $s\le t$. For $j\le N$,

$$\sum_{n=0}^{j-1}\frac{j^s}{(j-n)^t}=j^s\sum_{n=1}^{j}n^{-t}\le j^s\left(1+\int_1^j x^{-t}\mathrm{d}x\right)=j^s\frac{j^{1-t}-t}{1-t}\lesssim N.$$

For $j\in[N+1,2N]$, we have

$$\sum_{n=0}^{N}\frac{j^s}{(j-n)^t}\le j^s\sum_{n=0}^{j-1}\frac{1}{(j-n)^t}\lesssim(2N)^{s-t+1}\lesssim N.$$

To complete the proof, we use all the inequalities above in each case, and use Jensen's inequality to get

$$\left(\frac{1}{N+1}\sum_{n=0}^{N}\left\|B_w^n x\right\|_p\right)^p\le\frac{1}{N+1}\sum_{n=0}^{N}\left\|B_w^n x\right\|_p^p\lesssim\|x\|_p^p.$$

That is B_w is absolutely Cesàro bounded.

We summarize the theorems above and give the following corollary.

Corollary 1. Suppose $1\le p<\infty$. Let w be a weight sequence such that $\prod_{m=i}^{j}|w_m|\simeq\frac{j^{s/p}}{i^{t/p}}$ for a real pair (t,s). The weighted backward shift B_w on $\ell^p(\mathbb{N})$ is absolutely Cesàro bounded if and only if the real pair (t,s) satisfies one of the following three conditions:

(1) $t>1$ and $s\le 1$,

(2) $t=1$ and $s<1$,

(3) $t<1$ and $s\le t$.

To give a visualization, we have the following Figure 1 to show the correspondence of the range of $(t,s)\in\mathbb{R}^2$ with the absolute Cesàro boundedness.

3. A Criterion around the Critical Point

It is clear that our result include the cases in [5]. We call the point $(t,s)=(1,1)$ in Figure 1 the *critical point*. In this section, we will give a criterion around the critical point. In the following conditions, we can treat $\log j$ to be non zero, that is $j\ne 1$. Because otherwise it is trivial or invalid. We consider the non Cesàro boundedness firstly.

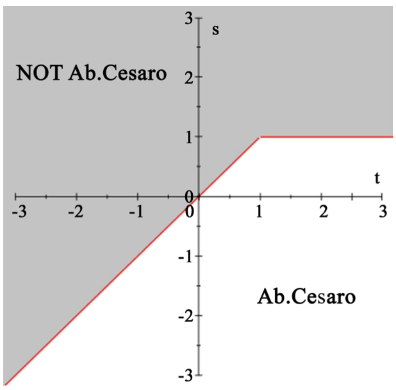

Figure 1. Except for the point (1,1), The red boundary line is Absolutely Cesàro bounded (Ab. Cesaro).

Theorem 3. For $1 \le p < \infty$, let w be a weight sequence and B_w the weighted backward shift on $\ell^p(\mathbb{N})$. Suppose $\prod_{n=i}^{j} w_n \gtrsim \left(\frac{j}{i} \cdot \frac{\log^t i}{\log^s j}\right)^{1/p}$ and the real pair (t,s) satisfies one of the following three conditions:

(1) $t > -1$ and $s < t+1$;

(2) $t = -1$ and $s \le 0$;

(3) $t < -1$ and $s < 0$.

Then B_w is not Cesàro bounded on $\ell^p(\mathbb{N})$.

Proof. Analogously to the proof of Theorem 1, let $y_{N+1} := \frac{1}{(N+1)^{1/p}} \sum_{n=1}^{N+1} e_n$, where N is a positive integer multiple of 4.

$$\left\| \frac{1}{N+1} \sum_{j=0}^{N} B_w^j y_{N+1} \right\|_p^p$$

$$\gtrsim \frac{1}{(N+1)^{p+1}} \sum_{k=1}^{N+1} \left(\sum_{j=k}^{N+1} \frac{(N-j+k+1)^{1/p}}{(k+1)^{1/p}} \frac{\log^{t/p}(k+1)}{\log^{s/p}(N-j+k+1)} \right)^p$$

$$\ge \frac{1}{(N+1)^{p+1}} \sum_{k=1}^{N/4+1} \frac{\log^t(k+1)}{k+1} \left(\sum_{j=N/4+1}^{3N/4+1} \frac{(N-j+k+1)^{1/p}}{\log^{s/p}(N-j+k+1)} \right)^p$$

$$\ge \begin{cases} \frac{1}{(N+1)^{p+1}} \left(\sum_{k=1}^{N/4+1} \frac{\log^t(k+1)}{k+1} \right) \left(\sum_{j=N/4+1}^{3N/4+1} \frac{(N-j+2)^{1/p}}{\log^{s/p}(5N/4-j+2)} \right)^p, & \text{if } s > 0; \\ \frac{1}{(N+1)^{p+1}} \left(\sum_{k=1}^{N/4+1} \frac{\log^t(k+1)}{k+1} \right) \left(\sum_{j=N/4+1}^{3N/4+1} \frac{(N-j+2)^{1/p}}{\log^{s/p}(N-j+2)} \right)^p, & \text{if } s \le 0. \end{cases}$$

For short, we define $M_1 = \sum_{k=1}^{\frac{N}{4}+1} \frac{\log^t(k+1)}{k+1}$, $M_2 = \sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}+1} \frac{(N-j+2)^{1/p}}{\log^{s/p}(5N/4-j+2)}$, and $M_3 = \sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}+1} \frac{(N-j+2)^{1/p}}{\log^{s/p}(N-j+2)}$.

If $-1 < t \le 0$,

$$M_1 = \sum_{k=2}^{\frac{N}{4}+2} \frac{\log^t k}{k} \ge \int_2^{\frac{N}{4}+3} \frac{\log^t x}{x} \mathrm{d}x$$
$$= \frac{\log^{t+1}\left(\frac{N}{4}+3\right) - \log^{t+1} 2}{t+1} \gtrsim \log^{t+1} N.$$

If $t > 0$,

$$M_1 = \sum_{k=2}^{\frac{N}{4}+2} \frac{\log^t k}{k} \gtrsim \frac{\log^t 2}{2} + \int_2^{\frac{N}{4}+2} \frac{\log^t x}{x} \mathrm{d}x$$
$$\gtrsim \frac{\log^{t+1}\left(\frac{N}{4}+2\right) - \log^{t+1} 2}{t+1} \gtrsim \log^{t+1} N.$$

Meanwhile $s > 0$ implies that

$$M_2 \ge \frac{1}{\log^{s/p}(N+1)} \sum_{\frac{N}{4}+1}^{\frac{3N}{4}+1} j^{1/p} \ge \frac{1}{\log^{s/p}(N+1)} \int_{\frac{N}{4}}^{\frac{3N}{4}+1} x^{1/p} \mathrm{d}x$$
$$= \frac{\left(\frac{3N}{4}+1\right)^{1/p+1} - \left(\frac{N}{4}\right)^{1/p+1}}{\left(\log^{s/p}(N+1)\right)\left(\frac{1}{p}+1\right)} \gtrsim \frac{N^{1/p+1}}{\log^{s/p} N}.$$

Hence, $t+1 > s$ implies

$$\frac{M_1 \cdot M_2^p}{(N+1)^{p+1}} \gtrsim \log^{t+1-s} N \to \infty, \quad \text{as } N \to \infty.$$

When $t > -1$, we have $M_1 \gtrsim \log^{t+1} N$. If $s \le 0$, then

$$M_3 \ge \frac{1}{\log^{s/p}\left(\frac{N}{4}+1\right)} \sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}+1} j^{1/p} \ge \frac{1}{\log^{s/p}\left(\frac{N}{4}+1\right)} \int_{j=\frac{N}{4}}^{\frac{3N}{4}+1} x^{1/p} \mathrm{d}x$$
$$= \frac{\left(\frac{3N}{4}+1\right)^{1/p+1} - \left(\frac{N}{4}\right)^{1/p+1}}{\log^{s/p}\left(\frac{N}{4}+1\right)\left(\frac{1}{p}+1\right)} \gtrsim \frac{N^{1/p+1}}{\log^{s/p} N}.$$

hence,

$$\frac{M_1 \cdot M_3^p}{(N+1)^{p+1}} \gtrsim \log^{t+1-s} N \to \infty, \quad \text{as } N \to \infty,$$

which proves the case **(1)**.

If $t = -1$,

$$M_1 = \sum_{k=2}^{\frac{N}{4}+2} \frac{1}{k \log k} \ge \int_2^{\frac{N}{4}+3} \frac{1}{x \log x} \mathrm{d}x$$
$$= \log\log\left(\frac{N}{4}+3\right) - \log\log 2 \gtrsim \log(\log N).$$

Meanwhile $s \le 0$ implies $M_3 \gtrsim \frac{N^{1/p+1}}{\log^{s/p} N}$. Hence,

$$\frac{M_1 \cdot M_3^p}{(N+1)^{p+1}} \gtrsim \frac{\log(\log N)}{\log^s N}$$

diverges when N goes to the infinity. That is the case **(2)**.

If $t < -1$,

$$M_1 = \sum_{k=2}^{\frac{N}{4}+2} \frac{\log^t k}{k} \gtrsim \int_2^{\frac{N}{4}+3} \frac{\log^t x}{x} \mathrm{d}x = \frac{\log^{t+1} 2 - \log^{t+1}\left(\frac{N}{4}+3\right)}{-t-1} \gtrsim 1.$$

Also $s < 0$ implies $M_3 \gtrsim \frac{N^{1/p+1}}{\log^{s/p} N}$. Hence,

$$\frac{M_1 \cdot M_3^p}{(N+1)^{p+1}} \gtrsim \log^{-s} N \to \infty, \text{ as } N \to \infty.$$

That is the case **(3)**.

Theorem 4. For $1 \le p < \infty$, let w be a weight sequence and B_w the weighted backward shift on $\ell^p(\mathbb{N})$. Suppose $\prod_{n=i}^{j} |w_n| \lesssim \left(\frac{j}{i} \cdot \frac{\log^t i}{\log^s j} \right)^{1/p}$ and the real pair (t, s) satisfies one of the following three conditions:

(1) $t > -1$ and $s \ge t+1$;

(2) $t = -1$ and $s > 0$;

(3) $t < -1$ and $s \ge 0$.

Then B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$. In the condition, we treat all $\log j$ to be positive. That is actually the case when $j \ge 3$. There are exceptions in our arguments. But the only cases are when $j = 1, 2$. We can concentrate to the cases when j large enough, because the exact values of w_1 and w_2 will not change the (absolute) Cesàro boundedness of the backward shift B_w. From this point of view, we avoid to consider the trivial cases and abuse to treat all the $\log j$ to be positive.

Proof. Analogously to the proof of Theorem 2, for $x = \sum_{j=1}^{\infty} \alpha_j e_j$, we have

$$\begin{aligned} \sum_{n=0}^{N} \left\| B_w^n x \right\|_p^p &\lesssim \sum_{j=1}^{N} |\alpha_j|^p \left(\sum_{n=0}^{j-1} \frac{j}{j-n} \cdot \frac{\log^t (j-n)}{\log^s j} \right) \\ &+ \sum_{j=N+1}^{2N} |\alpha_j|^p \left(\sum_{n=0}^{N} \frac{j}{j-n} \cdot \frac{\log^t (j-n)}{\log^s j} \right) \\ &+ \sum_{j=2N+1}^{\infty} |\alpha_j|^p \left(\sum_{n=0}^{N} \frac{j}{j-n} \cdot \frac{\log^t (j-n)}{\log^s j} \right) \\ &:= S_1 + S_2 + S_3. \end{aligned}$$

To estimate S_3, we note that $j \ge 2N+1$, $n \le N$. Then

$$\sum_{n=0}^{N} \frac{j}{j-n} \cdot \frac{\log^t (j-n)}{\log^s j} \le 2 \sum_{n=0}^{N} \frac{\log^t (j-n)}{\log^s j} \lesssim \begin{cases} \dfrac{N \log^t j}{\log^s j}, & \text{if } t > 0; \\ \dfrac{N \log^t (j-N)}{\log^s j}, & \text{if } t \le 0. \end{cases}$$

Let $L_1 = \frac{N \log^t j}{\log^s j}$ and $L_2 = \frac{N \log^t (j-N)}{\log^s j}$.

In either case **(1)**, **(2)** and **(3)**, we have $t - s \le -1$. If $t > 0$,

$$L_1 = N \log^{t-s} j \lesssim N \log^{t-s} N \lesssim N.$$

If $t \le 0$,

$$L_2 \lesssim \frac{N \log^t (N+1)}{\log^s j} \le \frac{N \log^t (N+1)}{\log^s (2N+1)} \lesssim N \log^{t-s} N \lesssim N.$$

Then, in either case **(1)**, **(2)** and **(3)**, we have

$$S_3 \lesssim \sum_{j=2N+1}^{\infty} |\alpha_j|^p \cdot 2N \lesssim N\|x\|_p^p .$$

We split **(1)** into two cases, that is when $-1 < t \le 0$ or $t > 0$, to estimate S_1 and S_2. If $-1 < t \le 0$ and $s \ge t+1$. To estimate S_1, we note that $j \le N$ and

$$\begin{aligned}\sum_{n=0}^{j-1} \frac{j}{j-n} \cdot \frac{\log^t (j-n)}{\log^s j} &= \frac{j}{\log^s j} \sum_{n=2}^{j} \frac{\log^t n}{n} \le \frac{j}{\log^s j}\left(\frac{\log^t 2}{2} + \int_2^j \frac{\log^t x}{x}\mathrm{d}x\right) \\ &= \frac{j}{\log^s j}\left(\frac{\log^t 2}{2} + \frac{\log^{t+1} j - \log^{t+1} 2}{t+1}\right) \\ &\lesssim N \log^{t+1-s} j \lesssim N.\end{aligned}$$

If $t > 0$ and $s \ge t+1$, we can estimate S_1 by the following computation

$$\begin{aligned}\frac{j}{\log^s j} \sum_{n=2}^{j} \frac{\log^t n}{n} &\le \frac{j}{\log^s j} \int_2^{j+1} \frac{\log^t x}{x}\mathrm{d}x \\ &= \frac{j}{\log^s j} \cdot \frac{\log^{t+1}(j+1) - \log^{t+1} 2}{t+1} \qquad (1) \\ &\lesssim N \log^{t+1-s} j \lesssim N.\end{aligned}$$

Thus, in the case **(1)** we have $S_1 \lesssim N \sum_{j=1}^{N} |\alpha_j|^p \lesssim N\|x\|_p^p$.

The estimate for S_2 is similar. We note that $j \ge N+1$ and

$$S_2 \le \sum_{j=N+1}^{2N} |\alpha_j|^p \left(\frac{j}{\log^s j} \sum_{n=2}^{j} \frac{\log^t n}{n} \right) \lesssim N\|x\|_p^p .$$

Now we consider the case **(2)**, that is $t = -1$ and $s > 0$. Since

$$\begin{aligned}\frac{j}{\log^s j} \sum_{n=2}^{j} \frac{1}{n \log n} &\le \frac{j}{\log^s j}\left(\frac{1}{2\log 2} + \int_2^j \frac{1}{x \log x}\mathrm{d}x\right) \\ &\lesssim N \frac{\log\log j}{\log^s j} \lesssim N \log^{-s} j \lesssim N,\end{aligned}$$

we have

$$S_1 \lesssim N \sum_{j=1}^{N} |\alpha_j|^p \lesssim N\|x\|_p^p .$$

And similarly,

$$S_2 \le \sum_{j=N+1}^{2N} |\alpha_j|^p \left(\frac{j}{\log^s j} \sum_{n=2}^{j} \frac{1}{n \log n} \right) \lesssim N\|x\|_p^p .$$

We have the last case **(3)** to consider. That is $t < -1$ and $s \ge 0$. Similarly to (1), we can obtain $S_1 \lesssim N\|x\|_p^p$ and $S_2 \lesssim N\|x\|_p^p$.

In the end of the proof, by the Jensen's inequality again, we have B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$.

We summarize the above two theorem as a corollary.

Corollary 2. Suppose $1 \le p < \infty$. Let w be a weight sequence such that

$\prod_{m=i}^{j} |w_m| \simeq \left(\frac{j}{i} \cdot \frac{\log^t i}{\log^s j} \right)^{1/p}$ for a real pair (t,s). The weighted backward shift B_w

on $\ell^p(\mathbb{N})$ is absolutely Cesàro bounded if and only if the real pair (t,s) satisfies one of the following three conditions:

(1) $t > -1$ and $s \ge t+1$,

(2) $t = -1$ and $s > 0$,

(3) $t < 1$ and $s \ge 0$.

We also give the following **Figure 2** to show our result around the critical point.

4. Examples

According to our result, we can construct lots of absolutely Cesàro bounded weighted backward shift.

Example 1. If $0 < s \le t < 1$, let

$$w = (w_1, \cdots, w_n, \cdots) = \left(\frac{(2+\log 2)^{s/p}}{(1+\log 1)^{t/p}}, \cdots, \frac{[n+1+\log(n+1)]^{s/p}}{(n+\log n)^{t/p}}, \cdots \right).$$

The operator B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$. It follows from $i+\log i \ge i$ and $\lim_{j\to\infty} \frac{j+\log j}{j} = 1$ and hence

$$\prod_{n=i}^{j} |w_n|^p = \frac{[j+1+\log(j+1)]^s}{(i+\log i)^t} \prod_{n=i+1}^{j} (n+\log n)^{s-t} \lesssim \frac{j^s}{i^t}$$

for any $j \ge i \in \mathbb{N}$.

Example 2. If $0 < s < t < 1$, let

$$w = (w_1, \cdots, w_n, \cdots) = \left(\frac{(2\log 3)^{s/p}}{(\log 2)^{t/p}}, \cdots, \frac{[(n+1)\log(n+2)]^{s/p}}{[n\log(n+1)]^{t/p}}, \cdots \right).$$

The operator B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$. It follows from $i\log(i+1) \gtrsim i$ and $\log(j+1) \lesssim j^{\frac{t-s}{2s}}$ and hence

$$\prod_{n=i}^{j} |w_n|^p = \frac{[(j+1)\log(j+2)]^s}{(i\log(i+1))^t} \prod_{n=i+1}^{j-1} [n\log(n+1)]^{s-t} \lesssim \frac{j^{\frac{s+t}{2}}}{i^t}$$

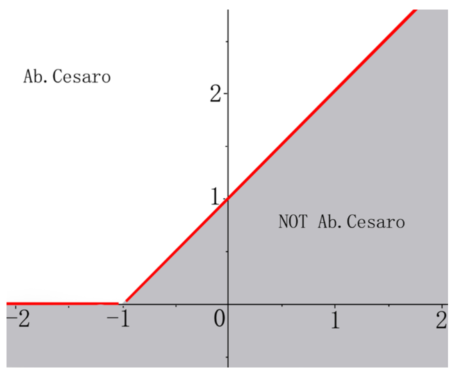

Figure 2. Except for the point (−1,0), the red boundary line is Absolutely Cesàro bounded (Ab. Cesaro).

for any $j \geq i \in \mathbb{N}$.

Example 3. If $t \geq 0$, let

$$w = (w_1, \cdots, w_m, \cdots) = \left(0, \cdots, \frac{(m+1)^{1/p}}{\log^{\frac{t+1}{p}}(m+1)} \cdot \frac{\log^{\frac{t}{p}} m}{m^{1/p}}, \cdots\right).$$

The operator B_w is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$ by Theorem 4. One can conduct the following computation that

$$\prod_{m=i}^{j} |w_m| = \frac{(j+1)^{\frac{1}{p}}}{i^{\frac{1}{p}}} \cdot \frac{\log^{\frac{t}{p}} i}{\log^{\frac{t+1}{p}}(j+1)} \cdot \prod_{k=i+1}^{j} \frac{1}{\log^{\frac{1}{p}} k} \leq \left(\frac{2}{\log 2}\right)^{\frac{1}{p}} \cdot \left(\frac{j}{i} \cdot \frac{\log^t i}{\log^{t+1} j}\right)^{\frac{1}{p}}.$$

We will also find a new example of non Cesàro bounded backward shift as follows.

Example 4. If $t \in \mathbb{R}$, let

$$w = (w_1, \cdots, w_m, \cdots) = \left(0, \cdots, \left(\frac{m+1}{\log^t(m+1)} \cdot \frac{\log^t m}{m}\right)^{1/p}, \cdots\right).$$

The operator B_w is not Cesàro bounded on $\ell^p(\mathbb{N})$ by Theorem 3.

5. Conclusion

In this paper, we proved Cesàro boundedness by constructing the proper product of weight functions $\prod_a^b w_n$ by the fraction of two monomials of the indexes. The method of proof is to obtain the characterization of absolutely Cesàro bounded and non Cesàro bounded by proper scaling and Jensen's inequality. we give some examples after our results.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Hou, B. and Luo, L. (2015) Some Remarks on Distributional Chaos for Bounded Linear Operators. *Turkish Journal of Mathematics*, **39**, 251-258. https://doi.org/10.3906/mat-1403-41

[2] Grosse-Erdmann, K.G. and Manguillot, A.P. (2011) Linear Chaos. Springer, London. https://doi.org/10.1007/978-1-4471-2170-1

[3] Abadias, L. (2016) A Katznelson-Tzafriri Type Theorem for Cesàro Bounded Operators. *Studia Mathematica*, **234**, 59-82. https://doi.org/10.4064/sm8436-5-2016

[4] Aleman, A. and Suciu, L. (2016) On Ergodic Operator Means in Banach Spaces. *Integral Equations and Operator Theory*, **85**, 259-287. https://doi.org/10.1007/s00020-016-2298-x

[5] Bermúdez, T., Bonilla, A., Müller, V. and Peris, A. (2020) Cesàro Bounded Operators in Banach Spaces. *Journal of Analyse Mathematique*, **140**, 187-206. https://doi.org/10.1007/s11854-020-0085-8

[6] Abadias, L. and Bonilla, A. (2019) Growth Orders and Ergodicity for Absolutely Cesàro Bounded Operators. *Linear Algebra and its Applications*, **561**, 253-267. https://doi.org/10.1016/j.laa.2018.10.002

[7] Abadias, L., Lizama, C., Miana, P.J. and Velasco, M.P. (2016) Cesàro Sums and Algebra Homomorphisms of Bounded Operators. *Israel Journal of Mathematics*, **216**, 471-505. https://doi.org/10.1007/s11856-016-1417-3

[8] Zygmund, A. (1959) Trigonometric Series. 2nd Edition, Vols. I, II, Cambridge University Press, New York.

[9] Lizama, C. (2017) The Poisson Distribution, Abstract Fractional Difference Equations, and Stability. *Proceedings of the American Mathematical Society*, **145**, 3809-3827. https://doi.org/10.1090/proc/12895

[10] Abadias, L. and Miana, P.J. (2018) Generalized Cesàro Operators, Fractional Finite Differences and Gamma Functions. *Journal of Functional Analysis*, **274**, 1424-1465. https://doi.org/10.1016/j.jfa.2017.10.010

[11] Ed-Dari, E. (2004) On the (C,α) Cesàro Bounded Operators. *Studia Mathematica*, **161**, 163-175. https://doi.org/10.4064/sm161-2-4

[12] Bernardes, N.C., Bonilla, A., Müller, V. and Peris, A. (2013) Distributional Chaos for Linear Operators. *Journal of Functional Analysis*, **265**, 2143-2163. https://doi.org/10.1016/j.jfa.2013.06.019

[13] Bernardes, N.C., Bonilla, A., Peris, A. and Wu, X. (2018) Distributional Chaos for Operators on Banach Spaces. *Journal of Mathematical Analysis and Applications*, **459**, 797-821. https://doi.org/10.1016/j.jmaa.2017.11.005

[14] Bernardes, N.C., Bonilla, A. and Peris, A. (2020) Mean Li-Yorke Chaos in Banach Spaces. *Journal of Functional Analysis*, **278**, Article ID: 108343. https://doi.org/10.1016/j.jfa.2019.108343

[15] Giménez, F.M., Oprocha, P. and Peris, A. (2013) Distributional Chaos for Operators with Full Scrambled Sets. *Mathematische Zeitschrift*, **274**, 603-612. https://doi.org/10.1007/s00209-012-1087-8