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# New Constructions of Nullnorms on Bounded Lattices

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## Abstract

We propose two more general methods to construct nullnorms on bounded lattices. By some illustrative examples, we demonstrate that the new method differ from the existing approaches.

## Keywords

Nullnorm, Triangular Norm, Triangular Subnorm, Bounded Lattice

## 1. Introduction

The notions of triangular norms (t-norms for short) and triangular conorms (t-conorms for short) were introduced by Schweizer and Sklar [1]. Nullnorms are generalizations of triangular norms and triangular conorms with a zero element in the interior of the unit interval, and have to satisfy some additional constraints. Nullnorms are important from a theoretical viewpoint but also because of their numerous potential applications, such as expert systems, fuzzy quantifiers, neural networks, fuzzy logic [2]. The constructions of nullnorms were first studied on the unit interval [2]-[9]. In the subsequent studies, the interval has extended to bounded lattices [10] [11] [12].

Some constructions of nullnorms on bounded lattices were demonstrated in previously papers. Based on the existence of t-norms and t-conorms on an arbitrary bounded lattice, Karaçal *et al.* [10] proposed three construction methods of nullnorms on bounded lattices with an arbitrary zero element $a \in L \setminus \{0,1\}$. Subsequently, Ümit Ertuğrul [11] proposed two construction methods of nullnorms on bounded lattices, which can be recognized as generalizations of two construction methods proposed in [10].

In this paper, we propose two more general construction methods of nullnorms on an arbitrary bounded lattice. The present study is organized as follows:

In Section 2, we recall some basic concepts and show some existing constructions of nullnorms on an arbitrary bounded lattice. In Section 3, we introduce the notions of t-subnorm and t-subcnonorm. By using these operations, we propose new methods to obtain nullnorms on $L$ under some additional constraints and their characteristics are examined. Finally, this summarization can be found in Section 4.

## 2. Preliminaries

A lattice is a partially ordered set $(L,\leq)$ in which each two-element subset $\{x,y\}$ has an infimum, denoted as $x\wedge y$, and a supremum, denoted as $x\vee y$. A bounded lattice $(L,\leq,0,1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively. We denote $(L,\leq,0,1)$ simply by $L$ in this article.

Let $(L,\leq,0,1)$ be a bounded lattice and $V_1,V_2:L^2\to L$ be two binary operations on $L$, we can define a partial order:

$$V_1\leq V_2 \Leftrightarrow V_1(x,y)\leq V_2(x,y) \text{ for all } x,y\in L.$$

Given a bounded lattice $(L,\leq,0,1)$ and $a,b\in L$, $a\leq b$, a subset $[a,b]$ of $L$ is defined as $[a,b]=\{x\in L \mid a\leq x\leq b\}$. Similarly, denote $[a,b)=\{a\leq x<b\}$, $(a,b]=\{x\in L \mid a<x\leq b\}$ and $(a,b)=\{x\in L \mid a<x<b\}$. If $a$ and $b$ are incomparable, we use the notation $a\parallel b$. The set of all elements which are incomparable with $a$ are denoted by $I_a$.

**Definition 2.1.** ([13] [14]) Let $(L,\leq,0,1)$ be a bounded lattice. An operation $T:L^2\to L$ is called a triangular norm (t-norm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $1\in L$ such that $T(1,x)=x$ for all $x\in L$.

**Definition 2.2.** ([13] [14]) Let $(L,\leq,0,1)$ be a bounded lattice. An operation $S:L^2\to L$ is called a triangular conorm (t-conorm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $0\in L$ such that $S(0,x)=x$ for all $x\in L$.

**Definition 2.3.** ([15]) Let $(L,\leq,0,1)$ be a bounded lattice. An operation $F:L^2\to L$ is called a t-subnorm on $L$ if it is commutative, associative, increasing with respect to both variables and $F(x,y)\leq x\wedge y$ for all $x,y\in L$.

**Definition 2.4.** ([15]) Let $(L,\leq,0,1)$ be a bounded lattice. An operation $R:L^2\to L$ is called a t-subconorm on $L$ if it is commutative, associative, increasing with respect to both variables and both $R(x,y)\geq x\vee y$ for all $x,y\in L$.

**Proposition 2.5.** ([15]) *If $F_1:L^2\to L$ is a t-subnorm on a bounded lattice $L$, then $T:L^2\to L$ defined by*

$$T(x,y)=\begin{cases}F_1(x,y), & \text{if } (x,y)\in(L\setminus\{1\})^2\\ x\wedge y, & \text{otherwise}\end{cases} \tag{1}$$

is a t-norm on $L$.

Dually, if $R_1:L^2\to L$ is a t-subconorm on a bounded lattice $L$, then

$S: L^2 \to L$ defined by

$$S(x,y)=\begin{cases} R_1(x,y), & \text{if } (x,y)\in (L\setminus\{0\})^2 \\ x\vee y, & \text{otherwise} \end{cases} \tag{2}$$

is a t-conorm on $L$.

**Definition 2.6.** ([10]) Let $(L,\le,0,1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $V: L^2 \to L$ is called a nullnorm if an element $a \in L$ exists such that $V(x,0)=x$ for all $x \le a$ and $V(x,1)=x$ for all $x \ge a$.

It is easy to see that $V(x,a)=a$ for all $x\in L$, and thus $a$ is the zero element for $V$ [10].

**Proposition 2.7.** ([16]) *Let* $(L,\le,0,1)$ *be a bounded lattice and* $V: L^2 \to L$ *be a nullnorm on L with the zero element a. Then,* [(*i*)]

(i) $V|_{[0,a]^2}: [0,a]^2 \to [0,a]$ is a t-conorm on $[0,a]$;

(ii) $V|_{[a,1]^2}: [a,1]^2 \to [a,1]$ is a t-norm on $[a,1]$.

Let $(L,\le,0,1)$ be a bounded lattice and $a\in L\setminus\{0,1\}$. Let $T:[a,1]^2\to[a,1]$ be a t-norm on $[a,1]$ and $S:[0,a]^2\to[0,a]$ be a t-conorm on $[0,a]$. Based on the knowledge of the existence of t-norms and t-conorms on an arbitrary given bounded lattice, many construction methods of nullnorms were presented in previous papers. Generally speaking, these construction methods on an arbitrary bounded lattice under no additional constraints can be divided into two groups. One is $V_a^{(T,S)}(x,y): L^2 \to L$ proposed by Karaçal *et al.* in [10], which is defined as

$$V_a^{(T,S)}(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ a, & \text{otherwise.} \end{cases} \tag{3}$$

The structures of $V_a^{(T,S)}$ is shown in **Figure 1**.

The other group is $V_T^S$ and its dual, *i.e.*, $V_S^T: L^2\to L$, which are proposed by Ümit Ertuğrul [11] and defined as

$$V_T^S(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ S(x\wedge a, y\wedge a), & \text{if } (x,y)\in[0,a]\times I_a \cup I_a\times[0,a]\cup I_a\times I_a \\ a, & \text{otherwise} \end{cases} \tag{4}$$

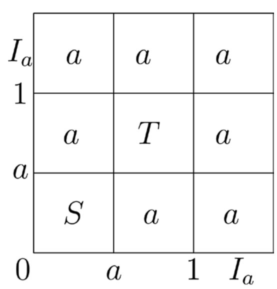


**Figure 1.** The frame of $V_a^{(T,S)}$.

and

$$V_S^T(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ T(x\vee a, y\vee a), & \text{if } (x,y)\in[a,1]\times I_a \cup I_a\times[a,1]\cup I_a\times I_a \\ a, & \text{otherwise.} \end{cases} \quad (5)$$

The structures of $V_T^S$ and $V_S^T$ are shown in **Figure 2** and **Figure 3**, respectively. In these figures, we denote $S_\wedge = S(x\wedge a, y\wedge a)$ and $T_\vee = T(x\vee a, y\vee a)$.

## 3. New Methods for Constructing Nullnorms on Bounded Lattices

In order to reduce the complexity in the proof of associativity, we introduce the following proposition.

**Proposition 3.1.** ([17]) *Let S be a nonempty set and* $A,B,C$ *be subsets of S. Let H be a commutative binary operation on S. Then H is associative on* $A\cup B\cup C$ *if both of the following statements hold*:

1) $H(H(x,y),z)=H(x,H(y,z))$ for all

$$(x,y,z)\in(A,A,A)\cup(B,B,B)\cup(C,C,C)\cup(A,A,B)\cup(A,B,B)\cup(A,A,C)\cup(A,C,C)\cup(B,B,C)\cup(B,C,C);$$

2) $H(H(x,y),z)=H(x,H(y,z))=H(H(x,z),y)$ for all $(x,y,z)\in(A,B,C)$.

Now, we introduce two construction methods which can be regard as generalizations of existing methods.

**Theorem 3.2.** *Let* $(L,\le,0,1)$ *be a bounded lattice and* $a\in L\setminus\{0,1\}$. *Let*

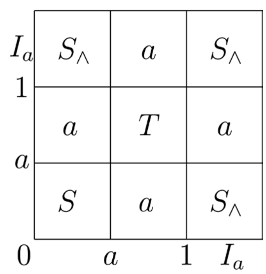


**Figure 2.** The frame of $V_T^S$.

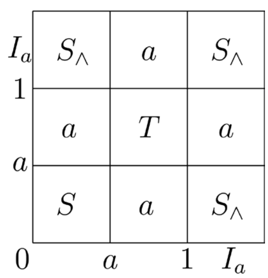


**Figure 3.** The frame of $V_S^T$.

*$T:[a,1]^2 \to [a,1]$ be a t-norm on $[a,1]$, $S:[0,a]^2 \to [0,a]$ be a t-conorm on $[0,a]$ and $R:[0,a]^2 \to [0,a]$ be a t-subconorm on $[0,a]$. If $S \le R$ and*

$$S(x,R(y,z)) = R(R(x,y),z) = R(S(x,y),z) \text{ forall } x,y,z \in [0,a], \quad (6)$$

then $V_T^{S,R}: L^2 \to L$ is a nullnorm on *L* with the zero element *a*, where

$$V_T^{S,R}(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [0,a]^2 \\ T(x,y), & \text{if } (x,y) \in [a,1]^2 \\ R(x \wedge a, y \wedge a), & \text{if } (x,y) \in [0,a] \times I_a \cup I_a \times [0,a] \cup I_a \times I_a \\ a, & \text{otherwise.} \end{cases} \quad (7)$$

*Proof.* The commutativity of $V_T^{S,R}$ can be proven directly based on its description. Similarly, we can express $V_T^{S,R}(x,0) = S(x,0) = x$ for all $x \in [0,a]$ and $V_T^{S,R}(x,1) = T(x,1) = x$ for all $x \in [a,1]$.

Monotonicity: Let us prove that if $x \le y$, then $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$ for all $z \in L$. If $x,y \in [0,a]$, or $x,y \in I_a$, or $x,y \in (a,1]$, then it is clear that $U(x,z) \le U(y,z)$ because $(x,z)$ and $(y,z)$ are in the same piece of *U* and *U* is monotonic in each piece. Moreover, $(x,y) \in (a,1] \times [0,a] \cup I_a \times [0,a] \cup I_a \times (a,1]$ contradicts the assumption that $x \le y$. Therefore, there are only three cases left to consider, namely, $(x,y) \in [0,a] \times (a,1]$, $(x,y) \in [0,a] \times I_a$, and $(x,y) \in I_a \times (a,1]$.

(I) Assume that $x \in [0,a]$ and $y \in (a,1]$.

(i) If $z \in [0,a]$, then $V_T^{S,R}(x,z) = S(x,z)$ and $V_T^{S,R}(y,z) = a$. As $S(x,z) \le a$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(ii) If $z \in (a,1]$, then $V_T^{S,R}(x,z) = a$ and $V_T^{S,R}(y,z) = T(y,z)$. As $a \le T(y,z)$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(iii) If $z \in I_a$, then $V_T^{S,R}(x,z) = R(x \wedge a, z \wedge a)$ and $V_T^{S,R}(y,z) = a$. As $R(x \wedge a, z \wedge a) \le a$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

Therefore, $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$ holds for $(x,y) \in [0,a] \times [a,1]$.

(II) Assume that $x \in [0,a]$ and $y \in I_a$ such that $x \le y$.

(i) If $z \in [0,a]$, then $V_T^{S,R}(x,z) = S(x,z)$ and $V_T^{S,R}(y,z) = R(y \wedge a, z \wedge a)$. As $S(x,z) = S(x \wedge a, z \wedge a) \le R(x \wedge a, z \wedge a) \le R(y \wedge a, z \wedge a)$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(ii) If $z \in [a,1]$, then $V_T^{S,R}(x,z) = a$ and $V_T^{S,R}(y,z) = a$, and thus $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(iii) If $z \in I_a$, then $V_T^{S,R}(x,z) = R(x \wedge a, z \wedge a)$ and $V_T^{S,R}(y,z) = R(y \wedge a, z \wedge a)$. As $R(x \wedge a, z \wedge a) \le R(y \wedge a, z \wedge a)$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

Therefore, $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$ holds for $(x,y) \in [0,a] \times I_a$.

(III) Assume that $x \in I_a$ and $y \in (a,1]$ such that $x \le y$.

(i) If $z \in [0,a]$, then $V_T^{S,R}(x,z) = R(x \wedge a, z \wedge a)$ and $V_T^{S,R}(y,z) = a$. As $R(x \wedge a, z \wedge a) \le a$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(ii) If $z \in [a,1]$, then $V_T^{S,R}(x,z) = a$ and $V_T^{S,R}(y,z) = T(y,z)$. As $a \le T(y,z)$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

(iii) If $z \in I_a$, then $V_T^{S,R}(x,z) = R(x \wedge a, z \wedge a)$ and $V_T^{S,R}(y,z) = a$. As

$R(x \wedge a, z \wedge a) \le a$, we have $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$.

Therefore, $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$ holds for $(x,y) \in I_a \times (a,1]$.

Combining the above cases, we obtain that $V_T^{S,R}(x,z) \le V_T^{S,R}(y,z)$ holds for $x,y,z \in L$ such that $x \le y$. Therefore, $V_T^{S,R}$ is monotonic.

Associativity: It can be shown that $V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(V_T^{S,R}(x,y),z)$ for all $x,y,z \in L$. By Proposition 3.1, We only need to consider the following cases:

(i) If $x,y,z \in [0,a]$, then since $S$ is associative, we have
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(ii) If $x,y,z \in [a,1]$, then since $T$ is associative, we have
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(iii) If $x,y,z \in I_a$, then
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(R(x \wedge a, y \wedge a),z) = R(R(x \wedge a, y \wedge a), z \wedge a)$,
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x, R(y \wedge a, z \wedge a)) = R(x \wedge a, R(y \wedge a, z \wedge a))$. As $R$ is an associative function on $[0,a]$, we have
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(iv) If $x,y \in [0,a]$ and $z \in [a,1]$, then
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(S(x,y),z) = a$ and
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x,a) = a$, and thus
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(v) If $x \in [0,a]$ and $y,z \in [a,1]$, then $V_T^{S,R}(V_T^{S,R}(x,y),z) = a$ and
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x, T(y,z)) = a$. Thus
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(vi) If $x,y \in [0,a]$ and $z \in I_a$, then
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(S(x,y),z) = R(S(x,y), z \wedge a)$ and
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x, R(y \wedge a, z \wedge a)) = S(x, R(y \wedge a, z \wedge a))$. It follows from (6) that $V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(vii) If $x \in [0,a]$ and $y,z \in I_a$, then
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(R(x \wedge a, y \wedge a),z) = R(R(x \wedge a, y \wedge a), z \wedge a)$ and
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x, R(y \wedge a, z \wedge a)) = S(x, R(y \wedge a, z \wedge a))$. It follows from (6) that $V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(viii) If $x,y \in [a,1]$ and $z \in I_a$, then
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(T(x,y),z) = a$ and $V_T^{S,R}(x, V_T^{S,R}(y,z)) = a$. Thus
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(ix) If $x \in [a,1]$ and $y,z \in I_a$, then $V_T^{S,R}(V_T^{S,R}(x,y),z) = a$ and
$V_T^{S,R}(x, V_T^{S,R}(y,z)) = V_T^{S,R}(x, R(y \wedge a, z \wedge a)) = a$. Thus
$V_T^{S,R}(V_T^{S,R}(x,y),z) = V_T^{S,R}(x, V_T^{S,R}(y,z))$.

(x) If $x \in [0,a]$, $y \in (a,1]$, $z \in I_a$, then $V_T^{S,R}\left(V_T^{S,R}(x,y),z\right) = a$, $V_T^{S,R}\left(x,V_T^{S,R}(y,z)\right) = a$ and $V_T^{S,R}\left(V_T^{S,R}(x,z),y\right) = V_T^{S,R}\left(R(x \wedge a, z \wedge a),y\right) = a$. Thus $V_T^{S,R}\left(V_T^{S,R}(x,y),z\right) = V_T^{S,R}\left(x,V_T^{S,R}(y,z)\right) = V_T^{S,R}\left(V_T^{S,R}(x,z),y\right)$.

From (i) to (x), we obtain that $V_T^{S,R}\left(x,V_T^{S,R}(y,z)\right) = V_T^{S,R}\left(V_T^{S,R}(x,y),z\right)$ for all $x,y,z \in L$ by Proposition 3.1. Therefore, $V_T^{S,R}$ is a nullnorm on $L$ with the zero element $a$.□

**Theorem 3.3.** *Let* $(L,\le,0,1)$ *be a bounded lattice and* $a \in L\setminus\{0,1\}$. *Let* $T:[a,1]^2 \to [a,1]$ *be a t-norm on* $[a,1]$, $F:[a,1]^2 \to [a,1]$ *be a t-subnorm on* $[a,1]$ *and* $S:[0,a]^2 \to [0,a]$ *be a t-conorm on* $[0,a]$. *If* $F \le T$ *and* $T\left(x,F(y,z)\right) = F\left(F(x,y),z\right) = F\left(T(x,y),z\right)$ *for all* $x,y,z \in L$, *then* $V_S^{T,F}: L^2 \to L$ *is a nullnorm on L with the zero element a, where*

$$V_S^{T,F}(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [0,a]^2 \\ T(x,y), & \text{if } (x,y) \in [a,1]^2 \\ F(x \vee a, y \vee a), & \text{if } (x,y) \in [a,1]\times I_a \cup I_a \times [a,1] \cup I_a \times I_a \\ a, & \text{otherwise.} \end{cases} \quad (8)$$

*Proof.* This can be proved similarly as Theorem 3.2.□

The structures of $V_T^{S,R}$ and $V_S^{T,F}$ from Formula (7) and Formula (8) are shown in **Figure 4** and **Figure 5**, respectively. We denote $R_\wedge = R(x \wedge a, y \wedge a)$ and $F_\vee = F(x \vee a, y \vee a)$ in these figures.

Let $(L,\le,0,1)$ be a bounded lattice and $a \in L\setminus\{0,1\}$. Let $T:[a,1]^2 \to [a,1]$ be a t-norm on $[a,1]$, $S:[0,a]^2 \to [0,a]$ be a t-conorm on $[0,a]$. Taking $R(x,y) = S(x,y)$ in Formula (7), we obtain that

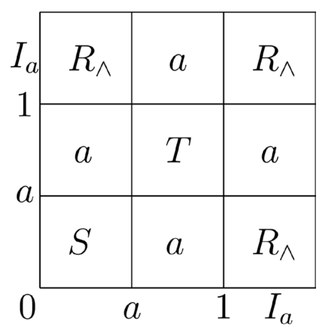


**Figure 4.** The frame of $V_S^{T,F}$.

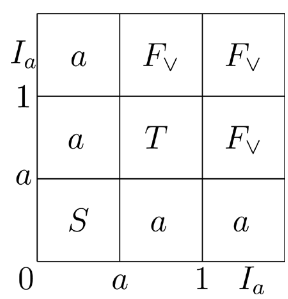


**Figure 5.** The frame of $V_S^{T,F}$.

$$V_T^{S,S}(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ S(x\wedge a, y\wedge a), & \text{if } (x,y)\in[0,a]\times I_a \cup I_a\times[0,a]\cup I_a\times I_a \\ a, & \text{otherwise,} \end{cases} \tag{9}$$

which is equal to $V_T^S(x,y)$ given by Formula (4).

Dually, taking $F(x,y)=T(x,y)$ in Formula (8), we obtain that

$$V_S^{T,T}(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ T(x\vee a, y\vee a), & \text{if } (x,y)\in[a,1]\times I_a \cup I_a\times[a,1]\cup I_a\times I_a \\ a, & \text{otherwise} \end{cases} \tag{10}$$

which is equal to $V_S^T(x,y)$ given by Formula (5).

Taking $R(x,y)=a$ for all $(x,y)\in[0,a]^2$ in Formula (7), then

$$V_T^{S,a}(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ a, & \text{otherwise,} \end{cases} \tag{11}$$

which is equal to $V_a^{(T,S)}$ given by Formula (3).

Taking $R(x,y)=a$ for all $(x,y)\in[0,a]^2$ in Formula (8), then it is clear that $V_S^{T,a}(x,y)$ also coincides with $V_a^{(T,S)}$, which is given by Formula (3). Therefore, the two methods proposed in this study are more generalized than the methods proposed previously by [10] [11]. Now we give an example to show that we can obtain new nullnorms by the construction methods proposed in this paper.

**Example 3.4.** Let $(L,\le,0,1)$ be a bounded lattice and let $a\in L\setminus\{0,1\}$.

(i) Let $T:[a,1]^2\to[a,1]$ be a t-norm on $[a,1]$ and $b,c\in L$ be such that $c\le b\le a$. Let $S:[0,a]^2\to[0,a]$ and $R:[0,a]^2\to[0,a]$ be two functions on $[0,a]$ defined by

$$S(x,y)=\begin{cases} x, & \text{if } y=0 \text{ and } x\in[0,a] \\ y, & \text{if } x=0 \text{ and } y\in[0,a] \\ x\vee y\vee c, & \text{if } (x,y)\in[0,a)^2 \end{cases} \tag{12}$$

and

$$R(x,y)=x\vee y\vee b. \tag{13}$$

Then $S$ is a t-conorm and $R$ is a t-subconorm on $[0,a]$. It is easy to verify $S\le R$ and the condition (6) holds. Therefore,

$$V_1(x,y)=\begin{cases} x, & \text{if } y=0 \text{ and } x\in[0,a] \\ y, & \text{if } x=0 \text{ and } y\in[0,a] \\ x\vee y\vee c, & \text{if } (x,y)\in(0,a]^2 \\ T(x,y), & \text{if } (x,y)\in[a,1]^2 \\ (x\wedge a)\vee(y\wedge a)\vee b, & \text{if } (x,y)\in[0,a]\times I_a\cup I_a\times[0,a]\cup I_a\times I_a \\ a, & \text{otherwise} \end{cases} \tag{14}$$

is a nullnorm on $L$ with the zero element $a$ by Theorem 3.2.

(ii) Dually, let $S:[0,a]^2 \to [0,a]$ be a t-conorm on $[0,a]$ and $j,k \in L$ be such that $a \le j \le k$. Then $V_2: L^2 \to L$ is a nullnorm on $L$ with the zero element $a$ by Theorem 3.3, where

$$V_2(x,y)=\begin{cases} x, & \text{if } y=1 \text{ and } x\in[a,1] \\ y, & \text{if } x=1 \text{ and } y\in[a,1] \\ x\wedge y\wedge k, & \text{if } (x,y)\in[a,1)^2 \\ S(x,y), & \text{if } (x,y)\in[0,a]^2 \\ (x\vee a)\wedge(y\vee a)\wedge j, & \text{if } (x,y)\in[a,1]\times I_a\cup I_a\times[a,1]\cup I_a\times I_a \\ a, & \text{otherwise.} \end{cases} \quad (15)$$

## 4. Conclusion

In this study, based on the existing constructions of nullnorms on $L$, we continue to study construction methods of nullnorms on bounded lattices. Two methods for obtaining nullnorms on $L$ are presented in this paper. Some examples were provided to show that the construction methods proposed in this paper generalized the methods presented in previous studies.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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