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A General Method of Researching the N-Ordered Fixed Point on a Metric Space with a Graph

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Abstract

In this paper, we propose a new perspective to discuss the N-order fixed point theory of set-valued and single-valued mappings. There are two aspects in our work: we first define a product metric space with a graph for the single-valued mapping whose conversion makes the results and proofs concise and straightforward, and then we propose an SG-contraction definition for set-valued mapping which is more general than some recent contraction's definition. The results obtained in this paper extend and unify some recent results of other authors. Our method to discuss the N-order fixed point unifies N-order fixed point theory of set-valued and single-valued mappings.

Keywords

N-Order Fixed Point, Product Metric Space Endowed with a Graph, Set-Valued Mapping, SG-Contraction

1. Introduction

Since Banach Contraction Principle was first shown in 1922 [1], some mathematicians have improved and generalized this principle to discuss fixed point theorems of various mappings on different spaces. In 1969, S. B. Nadler [2] applied the Pompeiu-Hausdorff metric to study the fixed point theory of set-valued mapping in metric space. Then, many fixed point theorems of set-valued mappings appeared, such as [3] [4] [5] [6]. In 2004, Ran and Reurings [7] built the fixed point theories on partially ordered metric spaces. In 2008, J. Jachymski [8] established some fixed point results on a metric space with a graph. The results extended and unified some results in metric spaces endowed with a partial order. Since then, many scholars began to study the fixed point theory on a metric space with a graph (one can refer [4] [5] [6] [9] [10] [11] and references therein).

The definition of coupled fixed point was given by Bhaskar and Lakshmikantham [12]. Since coupled fixed point theory has wide applications in nonlinear differential, nonlinear integral equations and partially ordered metric space, some coupled, tripled and N-order fixed point theorems appearing in [12]-[18] and reference therein.

Based on the results obtained in [2] and [8], I. Beg, etc. [4] proposed a contraction principle for set-valued mappings on a metric space with a graph in 2010. In 2015, M. R. Alfuraidan [5] proposed monotone set-valued mappings on a metric space with a graph which is somewhat different from that in [4]. In 2018, M. R. Alfuraidan and M. A. Khamsi [5] discussed the coupled fixed point theorems of single-valued and set-valued mappings, in which there was an important situation: $\exists k \in [0,1)$, such that for $x, y, u, v \in X$ with $(x,u),(v,y) \in E(G)$, it holds that

$$d\left(F(x,y),F(u,v)\right) \le k\left(\frac{d(x,u)+d(y,v)}{2}\right) \tag{1.1}$$

The inequalities similar with (1.1) appeared in many literatures about fixed point theories such as [12] [13] [15] [19] [20]. Most of the recent literatures considered N-order fixed point theorems and the inequalities similar with (1.1) only from the aspect of (X,d,G) (or $(X,d,\preceq)$) rather than $(X^N,D,\mathcal{G})$ which is a product metric space (X^N,D) endowed with a graph $\mathcal{G}$, thus, the proofs are tediously long. When we turn our attention to $(X^N,D,\mathcal{G})$ and the mapping $\mathcal{F}: X^N \to X^N$ (resp., $CB(X^N)$), we find that the N-order fixed point theories are very concise and straightforward for both set-valued mappings and single-valued mappings.

We focus on two aspects: one is to construct a concise method to study the fixed point theory; the other is to construct the N-order fixed point theories of single-valued and set-valued mappings. First, we define $(X^N,D,\mathcal{G})$ which is induced by (X,d,G). We transfer the N-order fixed point of $F: X^N \to X^N$ (resp., $CB(X^N)$), to the fixed point of $\mathcal{F}: X^N \to X^N$ (resp., $CB(X^N)$). Next, we construct the N-order fixed point theories of single-valued and set-valued mappings.

2. Preliminaries

We first introduce a terminology of graph theory. Consider a complete metric space (X,d) endowed with a directed graph G, denoted by (X,d,G). The vertices set $V(G)$ coincides X and the edges set $E(G)$ contains the diagonal elements of $X \times X$. Assume that G has no parallel edges. The metric between two vertices can be treated as the weight so that G is a weighted graph. Moreover, let G^{-1} denote the conversion of the graph G, *i.e.*, $E(G^{-1}) = \{(y,x):(x,y) \in E(G)\}$. Simultaneously, let $\bar{G}$ denote the undirected graph obtained from G, that is $E(\bar{G}) = E(G) \cup E(G^{-1})$. We say that G is weakly connected if is $\bar{G}$ connected.

Let X^N denote the product space of N X's and $\mathcal{X} = X^N$. We say that $(\mathcal{X}, D, \mathcal{G})$ is induced by (X, d, G) if $V(\mathcal{G}) = \mathcal{X}$, and $\forall x = (\xi_1, \xi_2, \cdots, \xi_n), y = (\eta_1, \eta_2, \cdots, \eta_n) \in \mathcal{X}$, $E(\mathcal{G})$ and D are defined respectively by:

$$(x, y) \in E(\mathcal{G}) \Leftrightarrow (\xi_i, \eta_i) \in E(G), i = 1, 2, \cdots, N, \tag{2.2}$$

and

$$D(x, y) = d(\xi_1, \eta_1) + \cdots + d(\xi_N, \eta_N) \tag{2.3}$$

In addition, we use $\mathcal{D}$ to denote the Pompeiu-Hausdorff distance between two sets.

In this paper, we mainly consider the following two properties of (X, d, G) and the mapping $f: X \to Y$ where Y is a metric space with a graph.

(P_1): For any sequence $(x_n)_{n \in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for any $n \in N$, then $(x_n, x) \in E(G)$.

(P_2): for given x and a sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ and $\lim_{n \to \infty} x_n = x$, we have $\lim_{n \to \infty} f(x_n) = f(x)$.

3. *N*-Order Fixed Point Theory of Single-Valued Mapping

In this section, We make use of f, F, $\mathcal{F}$ to denote the following mappings, respectively, $f: X \to X, F: \mathcal{X} \to X$ and $\mathcal{F}: \mathcal{X} \to \mathcal{X}$.

For simplicity of description, we denote $X_f = \{x \in X : (x, f(x)) \in E(G)\}$.

3.1. Basic Notations

Definition 1. [16] *Let X be a nonempty set and* $F: \mathcal{X} \to X$ *be a given mapping. An element* $x = (\xi_1, \xi_2, \cdots, \xi_N) \in X^N$ *is called an N-order fixed point of F if*

$$\xi_1 = F(\xi_1, \xi_2, \cdots, \xi_N), \xi_2 = F(\xi_2, \cdots, \xi_N, \xi_1), \cdots, \xi_N = F(\xi_N, \xi_1, \cdots, \xi_{N-1}).$$

When $N = 1, 2, 3$*, we call it a fixed point, coupled fixed point and tripled fixed point respectively.*

Definition 2. *Let* $F: \mathcal{X} \to X$ *be a mapping. We say that F is NG-contractive if it satisfies the following two conditions:*

1) *NG-monotone property, i.e., for all* $x, y \in \mathcal{X}$ *with* $(x, y) \in E(\mathcal{G})$, *it holds that* $(F(x), F(y)) \in E(G)$; (3.1)

2) *There exists* $k \in (0,1)$ *such that* $d(F(x), F(y)) \le kD(x, y)$ *for all* $x, y \in X^N$ *with* $(x, y) \in E(\mathcal{G})$.

Particularly, when $F: X \to X$ *satisfies* (1) *and* (2) *for* $N = 1$, *it is called a G-contraction.*

The definition "*NG*-contractive" was first introduced in [6]. In order to study the *N*-order fixed point theory, we here extend it to *N*-dimensional space.

We consider the definition of *N*-order fixed point of the mapping $F: \mathcal{X} \to X$ and find that the *N*-order fixed point of the mapping F is just the fixed point of the mapping $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ defined by

$$\forall x\in\mathcal{X}, F(x)=\left(F(\xi_1,\xi_2,\cdots,\xi_N),F(\xi_2,\xi_3,\cdots,\xi_1)\cdots,F(\xi_N,\xi_1,\cdots,\xi_{N-1})\right), \quad (3.2)$$

which implies that the study of the *N*-order fixed point can be converted to that of fixed point of $\mathcal{F}$. Simultaneously, $\mathcal{F}$ has many properties relative to *F*. According to Definition 2 and by (2.2), (2.3) and (3.2), if $\mathcal{F}$ is *G*-contractive, then we have:

1) for $x,y\in\mathcal{X}$ with $(x,y)\in E(\mathcal{G})\Rightarrow(\mathcal{F}(x),\mathcal{F}(y))\in E(\mathcal{G})\Rightarrow(F(x),F(y))\in E(G)$;

2) there exists $\alpha\in(0,1)$ such that $d(\mathcal{F}(x),\mathcal{F}(y))\le\alpha D(x,y)$ for all $x,y\in\mathcal{X}$ with $(x,y)\in E(\mathcal{G})$, and then $d(F(x),F(y))\le\alpha D(x,y)$.

Thus, if $\mathcal{F}$ is *G*-contractive, *F* must be *NG*-contractive. However, if *F* is *NG*-contractive, $\mathcal{F}$ is not necessarily *G*-contractive. We can show it by the following example.

Example 3.1. Let $X = R$ be endowed with the Euclidean distance $|\cdot|$. For any $\xi,\eta\in X$, we say that $(\xi,\eta)\in E(G)$ if $\xi\le\eta$. $(R^3,D,\mathcal{G})$ is induced by $(R,|\cdot|,G)$. For $x,y\in R^3$ with $(x,y)\in E(\mathcal{G})$, by (2.3), we have $D(x,y)=|\xi_1-\eta_1|+|\xi_2-\eta_2|+|\xi_3-\eta_3|$. Let $F:R^3\to R$ be defined by $F(x)=\frac{1}{2}(|\xi_1|+|\xi_2|+|\xi_3|)$.

It is obvious that

$$d(F(x),F(y))=|F(x)-F(y)|\le\frac{1}{2}(|\xi_1-\eta_1|+|\xi_2-\eta_2|+|\xi_3-\eta_3|)=\frac{1}{2}D(x,y)$$

which implies that *F* is *NG*-contractive. However,

$$\begin{aligned}D(\mathcal{F}(x),\mathcal{F}(y))&=d(F(x),F(y))+d(F(\xi_2,\xi_3,\xi_1),F(\eta_2,\eta_3,\eta_1))\\&\quad+d(F(\xi_3,\xi_1,\xi_2),F(\eta_3,\eta_1,\eta_2))\\&=|F(x)-F(y)|+|F(\xi_2,\xi_3,\xi_1)-F(\eta_2,\eta_3,\eta_1)|\\&\quad+|F(\xi_3,\xi_1,\xi_2)-F(\eta_3,\eta_1,\eta_2)|\\&\le\frac{3}{2}(x,y).\end{aligned}$$

Notice that when $x=(1,1,1)$ and $y=(2,2,2)$, then $D(\mathcal{F}(x),\mathcal{F}(y))=\frac{3}{2}D(x,y)$ which means $\mathcal{F}$ is not *G*-contractive.

3.2. Main Results

Lemma 3.1. [8] *Suppose that (X,d,G) satisfies (P_1). Let the mapping $f:X\to X$ be G-contractive and $X_f\ne\varnothing$. Then*

1) *for any $x\in X_f$, $f|_{[x]G}$ has a unique fixed point;*

2) *if G is weakly connected, then f has a unique fixed point in G;*

3) *if $X'=\cup\{[x]_{\tilde{G}}:x\in X_f\}$, then $f|_{X'}$ has a fixed point in X;*

4) *if $f(X)\subset E(G)$ then f has a fixed point;*

5) *fix $f\ne\varnothing$ if and only if $X_f\ne\varnothing$.*

Lemma 3.2. [8] *Let the mapping $f:X\to X$ be G-contractive and satisfy (P_2). Then f has a fixed point if there exists $x_0\in X$ such that*

$(x_0, f(x_0)) \in E(G)$. *Moreover, if G is weakly connected, the fixed point is unique.*

In the sequel, we study the *N*-order fixed point results of $F:\mathcal{X} \to X$ by finding the fixed point of $\mathcal{F}:\mathcal{X} \to \mathcal{X}$ and we give a concise proof. We try to ensure the *G*-contractive property of $\mathcal{F}$ in Theorem 4.3 and Corollary 1.

Theorem 3.1. *Suppose that* $(\mathcal{X}, D, \mathcal{G})$ *is induced by* (X, d, G) *and* $(\mathcal{X}, D, \mathcal{G})$ *satisfies* (P_1). *Suppose that* $F:\mathcal{X} \to X$ *satisfies the NG-monotone property* (3.4) *and there are some constants* $k_1, \cdots, k_N$ *with* $0 < k_1 + \cdots + k_N < 1$ *such that*

$$d(F(x), F(y)) \le k_1 d(\xi_1, \eta_1) + \cdots + k_N d(\xi_N, \eta_N) \tag{3.3}$$

with $(\xi_i, \eta_i) \in E(G)$ *for* $i = 1, \cdots, N$

Suppose that there is a point $x^0 = (\xi_1^0, \xi_2^0, \cdots, \xi_N^0) \in \mathcal{X}$ *such that*

$$\left(\xi_i^0, F\left(\xi_i^0, \xi_{i+1}^0, \cdots, \xi_N^0, \xi_1^0, \cdots, \xi_{i-1}^0\right)\right) \in E(G), \forall i = 1, 2, \cdots, N. \tag{3.4}$$

Define $\mathcal{F}:\mathcal{X} \to \mathcal{X}$ *by* (3.2). *Then,*

1) *for any* $x \in \mathcal{X}_{\mathcal{F}}$, $F|_{[x]\tilde{\mathcal{G}}}$ *has a unique N-order fixed point;*

2) *if G is weakly connected, then F has a unique N-order fixed point in* $\mathcal{G}$;

3) *if* $\mathcal{X}' = \cup\{[x]_{\tilde{\mathcal{G}}} : x \in \mathcal{X}_{\mathcal{F}}\}$, *then* $F|_{\mathcal{X}'}$ *has an N-order fixed point in* $\mathcal{X}$;

4) *if* $\mathcal{F}(\mathcal{X}) \subset E(\mathcal{G})$ *then F has an N-order fixed point;*

5) *fix* $\mathcal{F} = \varnothing$ *if and only if* $\mathcal{X}_{\mathcal{F}} \ne \varnothing$.

Moreover, if $x^* = (\xi_1^*, \xi_2^*, \cdots, \xi_N^*)$ *is an N-order fixed point of F with* $\xi_1^* = \xi_2^* = \cdots = \xi_N^*$.

Proof. (1) Since $(\mathcal{X}, D, \mathcal{G})$ is induced by (X, d, G) and (X, d) is complete, we know that $(\mathcal{X}, D)$ is complete, and for all $x, y \in \mathcal{X}$ with $x = (\xi_1, \cdots, \xi_N), y = (\eta_1, \cdots, \eta_N)$, $\mathcal{G}$ is defined as (2.2) and *D* is defined as (2.3). Then, for $x, y \in E(\mathcal{G})$, we obtain that

$$(\xi_2, \cdots, \xi_N, \xi_1; \eta_2, \cdots, \eta_N, \eta_1) \in E(\mathcal{G}), \cdots, (\xi_N, \cdots, \xi_{N-1}; \eta_N, \cdots, \eta_{N-1}) \in E(\mathcal{G}). \tag{3.5}$$

Considering (2.2), (2.3), (3.1), we have $(\mathcal{F}(x), \mathcal{F}(y)) \in E(\mathcal{G})$ and

$$\begin{aligned}
&D(\mathcal{F}(x), \mathcal{F}(y)) \\
&= d(F(x), F(y)) + d(F(\xi_2, \cdots, \xi_N, \xi_1), F(\eta_2, \cdots, \eta_N, \eta_1)) + \cdots \\
&\quad + d(F(\xi_N, \cdots, \xi_{N-1}), F(\eta_N, \cdots, \eta_{N-1})) \\
&\le (k_1 d(\xi_1, \eta_1) + \cdots k_N d(\xi_N, \eta_N)) + (k_1 d(\xi_2, \eta_2) + \cdots k_N d(\xi_1, \eta_1)) + \cdots \\
&\quad + (k_1 d(\xi_N, \eta_N) + \cdots k_N d(\xi_{N-1}, \eta_{N-1})) \\
&= (k_1 + \cdots k_N)(d(\xi_1, \eta_1) + \cdots + k_N d(\xi_N, \eta_N)) = (k_1 + \cdots k_N) D(x, y).
\end{aligned}$$

Thus $\mathcal{F}$ is a *G*-contraction on $\mathcal{X}$ satisfying $(x_0, \mathcal{F}(x_0)) \in E(\mathcal{G})$ by (4.13). Then, by Lemma 3.1, we obtain that $\mathcal{F}$ has a unique fixed point $x^* = (\xi_1^*, \xi_2^*, \cdots, \xi_N^*)$ in $[x_0]_{\mathcal{G}}$ which is just the *N*-order fixed point of *F*.

At the same time, we have

$$\xi_1^* = F(\xi_1^*, \xi_2^*, \cdots, \xi_N^*), \xi_2^* = F(\xi_2^*, \cdots, \xi_N^*, \xi_1^*), \cdots, \xi_N^* = F(\xi_N^*, \xi_1^*, \cdots, \xi_{N-1}^*)$$

which implies that $(\xi_2^*,\cdots,\xi_N^*,\xi_1^*),\cdots,(\xi_N^*,\xi_1^*,\cdots,\xi_{N-1}^*)$ are all *N*-order fixed points of *F*. By the uniqueness, we can get $\xi_1^*=\xi_2^*=\cdots=\xi_N^*$.

(2)-(5) can be proved by Lemma 3.1 similarly. We omit them.

Theorem 3.2. *If we replace the condition* "$(\mathcal{X},D,\mathcal{G})$ *satisfies* (P_1)" *by* "$\mathcal{F}:\mathcal{X}\to\mathcal{X}$ *satisfies* (P_2)" *and the other conditions in Theorem* 3.1 *are satisfied, we can get the same results with Theorem* 3.1.

Corollary 1. *Suppose that* $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ *satisfies* (P_2) *and* $\mathcal{F}:\mathcal{X}\to X$ *satisfies the NG-monotone property* (3.1). *And*

$$d(F(x),F(y))\le\frac{k}{N}D(x,y),\forall x,y\in\mathcal{X}\ \ \text{with}\ \ (x,y)\in E(\mathcal{G})\tag{3.6}$$

where $0<k<1$. *If there exists* $x_0=(\xi_1^0,\xi_2^0,\cdots,\xi_N^0)\in\mathcal{X}$ *such that*

$$\forall i,\left(\xi_i^0,F\left(\xi_i^0,\xi_{i+1}^0,\cdots,\xi_N^0,\xi_1^0,\cdots,\xi_{i-1}^0\right)\right)\in E(G).$$

Then there is an N-order fixed point $x^*=(\xi_1^*,\xi_2^*,\cdots,\xi_N^*)$ *of F. If G is weakly connected,* x^* *is the unique N-order fixed point with* $\xi_1^*=\xi_2^*=\cdots=\xi_N^*$.

Proof. We can easily prove that $\mathcal{F}$ defined as (3.2) is *G*-contractive. By Theorem 3.2, we can get the proof.

Corollary 2. *Suppose that a mapping* $F:\mathcal{X}\to X$ *has an N-order fixed point* $x^*=(\xi_1^*,\xi_2^*,\cdots,\xi_N^*)$. *Then* x^* *is the unique N-order fixed point and satisfies* $\xi_1^*=\xi_2^*=\cdots=\xi_N^*$ *if*

(P_3): *for each* $x,y\in\mathcal{X}$, *there exists* $z\in\mathcal{X}$ *such that* $(x,z)\in E(\overline{G})$ *and* $(x,y)\in E(\overline{G})$

Proof. It is clear that $\overline{G}$ is connected. By theorem 3.2, we can obtain the result.

In partially ordered set, the assumption (P_3) implies the comparable property which is widely used in the uniqueness of fixed point theorems.

Remark The conditions (3.5) appeared in many documents with $N=1,2,3$, such as, Bhaskar and Lakshmikantham [12], V. Berinde and M. Borcut [15], Agarwal, EI-Gebeily, D. O'Regan [19] and Amini-Harandi [13], M. R. Alfuraidan, M. A. Khamsi [5], etc. Simultaneously, comparable property is used to study the uniqueness of fixed points. Thus, our results extend and unify a more general version. Moreover, the introduction of $\mathcal{F}$ provides a new idea to study the *N*-order fixed point with $N>1$.

3.3. Application

The followings are excerpted from [12] and [6]. They can be seen as the corollaries to Theorem 3.2 and Corollary 2.

Corollary 3. [12] *Let* $(X,\le,d)$ *be a complete metric space endowed with a partial order* $\le$. *The mapping* $F:X\times X\to X$ *is a continuous mapping. Suppose that*

1) *F has mixed monotone property: for any* $\xi_1,\xi_2,\eta_1,\eta_2,\xi,\eta\in X$

$$\xi_1\le\xi_2\Rightarrow F(\xi_1,\eta)\le F(\xi_2,\eta),\eta_1\le\eta_2\Rightarrow F(\xi,\eta_2)\le F(\xi,\eta_1);$$

2) *there is a* $k\in[0,1)$ *such that for all* $\xi_1,\xi_2,\eta_1,\eta_2\in X$ *with* $\xi_1\le\xi_2$ *and*

$\eta_2 \le \eta_1$,

$$d\left(F\left(\xi_1,\xi_2\right),F\left(\eta_1,\eta_2\right)\right) \le k\left(\frac{d\left(\xi_1,\eta_1\right)+d\left(\xi_2,\eta_2\right)}{2}\right)$$

If there exists $\left(\xi_1^0,\xi_2^0\right)$ *with* $\xi_1^0 \le F\left(\xi_1^0,\xi_2^0\right)$ *and* $F\left(\xi_2^0,\xi_1^0\right) \le \xi_2^0$, *then F has a coupled fixed* point. Assume that the comparable property holds, we can obtain the uniqueness of the coupled fixed point.

Proof. Set a graph G on X defined as: for $x=\left(\xi_1,\xi_2\right), y=\left(\eta_1,\eta_2\right) \in X^2$, we say

$$\left(x,y\right) \in E\left(\mathcal{G}\right) \Leftrightarrow x \preceq y \Leftrightarrow \xi_1 \le \eta_1 \text{ and } \eta_2 \le \xi_2 .$$

Then by mixed monotone property, we know

$$\begin{aligned}&\left(x,y\right) \in E\left(\mathcal{G}\right) \Rightarrow F\left(x\right)=F\left(\xi_1,\xi_2\right) \le F\left(\eta_1,\xi_2\right) \le F\left(\eta_1,\eta_2\right)=F\left(y\right)\\&\Rightarrow \left(F\left(x\right),F\left(y\right)\right) \in E\left(G\right).\end{aligned}$$

By Theorem 3.2 and Corollary 2, we can get the proof.

Corollary 4 [6] *Let* $\left(X,d,G\right)$ *be a complete metric space endowed with a graph G. The mapping* $F: X \times X \to X$ *be a continuous and NG-monotone mapping satisfying that there is a* $k \in [0,1)$ *such that*

$$d\left(F\left(\xi_1,\xi_2\right),F\left(\eta_1,\eta_2\right)\right) \le k\left(\frac{d\left(\xi_1,\eta_1\right)+d\left(\xi_2,\eta_2\right)}{2}\right)$$

for $\left(\xi_1,\eta_1\right),\left(\eta_2,\xi_2\right) \in E\left(G\right)$. *If there exists* $\left(\xi_1^0,\xi_2^0\right)$ *with and* $\left(\xi_1^0,F\left(\xi_1^0,\xi_2^0\right)\right) \in E\left(G\right)$, *then F has a coupled fixed point.*

Proof. We can get the proof by Theorem 3.2.

4. N-Order Fixed Point Theory of Set-Valued Mapping

Let $\left(X,d,g\right)$ and $\left(\mathcal{X},D,\mathcal{G}\right)$ be defined in Section 2 and

$$CB\left(X\right)=\left\{S : S \text{ is a nonempty closed and bounded subset of } X\right\}.$$

We make use of *h*, *H*, $\mathcal{H}$ to denote the following different set-valued mappings, *i.e.*,

$$h: X \to CB\left(X\right), H: \mathcal{X} \to CB\left(X\right) \text{ and } \mathcal{H}: \mathcal{X} \to CB\left(\mathcal{X}\right).$$

For simplicity of description, we denote

$$X_h=\left\{x \in X : \left(x,u\right) \in E\left(G\right) \text{ for some } u \in h\left(x\right)\right\}.$$

4.1. Basic Notations

Let $\mathcal{D}$ denote the Pompeiu-Hausdorff metric between two sets on $CB\left(X\right)$, *i.e.*, for $A,B \in CB\left(X\right)$,

$$\mathcal{D}\left(A,B\right)=\max\left\{\sup_{a\in A} d\left(a,B\right),\sup_{b\in B} d\left(b,A\right)\right\}$$

where $d\left(a,B\right)=inf_{b\in B} d\left(a,b\right)$.

Definition 3. *Let X be a non-empty set and* $h: X \to CB\left(X\right)$ *be a set-valued* *mapping. An element* $x \in X$ *is called a fixed point of h provided* $x \in h(x)$.

In order to discuss the fixed point theory of set-valued mappings, many contraction's notations have been introduced such as [2] [4] [5].

Definition 4. [2] *Let* (X,d) *be a non-empty metric space. We say that the set-valued mapping* $h: X \to CB(X)$ *is set-valued contractive if there exists* $k \in (0,1)$ *such that* $\mathcal{D}(h(x),h(y)) \le kd(x,y)$ *for all* $x, y \in X$.

Definition 5. [4] *Let* (X,d) *be a non-empty metric space. We say that the set-valued mapping* $h: X \to CB(X)$ *is G-contractive if there exists* $k \in (0,1)$ *such that* $\mathcal{D}(h(x),h(y)) \le kd(x,y)$ *for all* $x, y \in X$ *and, if* $u \in h(x)$ *and* $v \in h(y)$ *with* $d(u,v) \le kd(x,y) + \alpha$ *for each* $\alpha > 0$, *then* $(u,v) \in E(G)$.

Definition 6. [6] *Let* (X,d) *be a non-empty metric space with a graph G. Let* $h: X \to CB(X)$ *be a set-valued mapping. There exists* $k \in (0,1)$ *such that, for* $\forall x, y \in X$ *with* $(x,y) \in E(G)$ *and any* $u \in h(x)$, *there exists* $v \in h(y)$ *such that* $(u,v) \in E(G)$ *and* $d(u,v) \le kd(x,y)$. *We say that the mapping* $h: X \to CB(X)$ *is monotone increasing G-contractive.*

For all $x, y \in X$, we have $h(x), h(y) \in CB(X)$ and, for a given number $\alpha > 0$, for any $u \in h(x)$, there is a point $v \in h(y)$ such that $d(u,v) \le \mathcal{D}(h(x),h(y)) + \alpha$ ([3]). Thus, we propose the "*SG*-contraction" as follows.

Definition 7. *Let* (X,d) *be a non-empty metric space with a graph G. Let* $h: X \to CB(X)$ *be a set-valued mapping. There exists* $k \in (0,1)$ *such that, for* $\forall x, y \in X$ *with* $(x,y) \in E(G)$ *and any* $u \in h(x)$, *for any* $\alpha > 0$, *there exists* $v \in h(y)$ *such that* $(u,v) \in E(G)$ *and* $d(u,v) \le kd(x,y) + \alpha$. *We say that the mapping* $h: X \to CB(X)$ *is SG-contractive.*

If h is a single-valued mapping, then *SG*-contractive property of h implies that, $\forall x, y \in X$ with $(x,y) \in E(G)$, it holds that $h(x), h(y) \in E(G)$ and $d(h(x),h(y)) \le kd(x,y)$ which is called *G*-contractive (see Definition 2).

Let's investigate the above definitions. "Set-valued contractive" in Definition 4 imlies "*SG*-contractive" in Definition 7. In Definition 5, the condition "if $u \in h(x)$ and $v \in h(y)$ with $d(u,v) \le kd(x,y) + \alpha$ for each $\alpha > 0$, then $(u,v) \in E(G)$ " is very hard. It is obvious that "*G*-contractive" in Definition 5 implies "*SG*-contractive" in Definition 7. And, "monotone increasing *G*-contractive" in Definition 6 implies "*SG*-contractive" in Definition 7. Since, for $u \in h(x)$, there may not be a point $v \in h(y)$ such that $d(u,v) \le \mathcal{D}(h(x),h(y))$ (see [2]), we can get "set-valued contractive" in Definition 4 doesn't imply "monotone increasing *G*-contractive" in Definition 6. Hence, "*SG*-contractive" in Definition 7 is more general.

Definition 8. The point $x = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathcal{X}$ is said to be an *N*-order fixed point of $H: \mathcal{X} \to CB(X)$ if

$$\xi_i \in H(\xi_i, \xi_{i+1}, \cdots, \xi_N, \xi_1, \cdots, \xi_{i-1}), i = 1, 2, \cdots, N.$$

For a set-valued mapping $H: \mathcal{X} \to CB(X)$, we define $\mathcal{H}: \mathcal{X} \to CB(\mathcal{X})$ as: for $x = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathcal{X}$,

$$\mathcal{H}(x)=\{y=(\eta_1,\eta_2,\cdots,\eta_N):\eta_i\in F(\xi_i,\xi_{i+1},\cdots,\xi_N,\xi_1,\cdots,\xi_{i-1})\}. \tag{4.1}$$

We can see that:

x is an N-order fixed point of H

$\Leftrightarrow \xi_i\in H(\xi_i,\xi_{i+1},\cdots,\xi_N,\xi_1,\cdots,\xi_{i-1}), i=1,2,\cdots,N$

$\Leftrightarrow x\in\mathcal{H}(x)$

$\Leftrightarrow$ x is a fixed point of $\mathcal{H}$

Thus, we can research the N-order fixed point theory by studying the fixed point theory.

4.2. Main Results

Theorem 4.1. *Suppose that* (X,d,G) *has the property* (P_1). *Let* $h:X\to CB(X)$ *be a set-valued SG-contraction. If* $X_h\neq\varnothing$, *then the following statements hold*:

1) *for any* $x\in X_h$, $h|_{[x]\tilde{G}}$ *has a fixed point*;
2) *if G is weakly connected, then h has a fixed point in G*;
3) *if* $X'=\cup\{[x]_{\tilde{G}}:x\in X_h\}$, *then* $h|X'$ *has a fixed point in X*;
4) *if* $h(x)\subset E(G)$ *then h has a fixed point*;
5) *fix* $h\neq\varnothing$ *if and only if* $X_h\neq\varnothing$.

Proof. (1) Considering $X_h\neq\varnothing$, we take a point $x_0\in X_h$. Then there is a point $x_1\in h(x_0)$ such that $(x_0,x_1)\in E(G)$. Noting that $h:X\to CB(X)$ is an SG-contractive mapping, for $k\in[0,1)$, there is a point $x_2\in h(x_1)$ such that

$$(x_1,x_2)\in E(G) \text{ and } d(x_1,x_2)\le kd(x_0,x_1)+k$$

By the similar deduction, there is a point x_3 such that

$$(x_2,x_3)\in E(G) \text{ and } d(x_2,x_3)\le kd(x_1,x_2)+k^2\le k^2d(x_1,x_2)+2k^2.$$

$$\cdots$$

We can construct a sequence $\{x_n\}$ such that for $n\ge1$

$$x_{n+1}\in h(x_n),(x_n,x_{n+1})\in E(G)$$

and

$$d(x_n,x_{n+1})\le kd(x_{n-1},x_n)+k^n\le k^nd(x_1,x_2)+nk^n \tag{4.2}$$

Since $\sum_{n=0}^{\infty}d(x_n,x_{n+1})<\infty$, we get that $\{x_n\}$ is a Cauchy sequence and converges a point $x^*\in X$.

Now, We claim that $x^*\in h(x^*)$ under the condition (P_1). By (P_1), we have $(x_n,x^*)\in E(G)$. For $x_{n+1}\in h(x_n)$, by Definition 7, for $\alpha=\frac{1}{n}$, there exists a point $y_n\in h(x^*)$ such that $(x_{n+1},y_n)\in E(G)$ and $d(x_{n+1},y_n)\le kd(x_n,x)+\frac{1}{n}$. Thus, for any $n\ge1$,

$$d(x,y_n)\le d(x,x_{n+1})+d(x_{n+1},y_n)\le d(x,x_{n+1})+kd(x_n,x)+\frac{1}{n}$$

which implies $\lim_{n\to\infty}=x^*$. Noting that $h(x^*)\in CB(X)$, we get $x^*\in h(x^*)$.

We can get the proof of (2)-(5) by the similar deduction of Theorem 3.1 in [5].

Theorem 4.2. *Let* (X,d,G) *be the complete metric space endowed with a graph G. Suppose that h has the property* (P_2). *Let* $h: X \to CB(X)$ *be a set-valued SG-contraction. Suppose that h has the property* (P_2). *If* $X_h \neq \varnothing$, *then conclusions obtained in Theorem* 4.1 *remain true.*

By the Similar deduction in Section 3.1, in order to discuss the *N*-order fixed point of *H*, we try to ensure the *SG*-contraction of $\mathcal{H}$.

Theorem 4.3. *Suppose that* (X,d,G) *has the property* (P_1). *Let* $H: \mathcal{X} \to CB(X)$ *be a set-valued mapping. Suppose that there are some constants* $k_1,\cdots,k_N$ *with* $0 < k_1 + \cdots + k_N < 1$ *such that for any* $u \in H(x)$, *for any* $\alpha > 0$, *there is a point* $v \in H(y)$ *such that*

$$d(u,v) \le k_1 d(\xi_1,\eta_1) + \cdots + k_N(\xi_N,\eta_N) + \alpha \tag{4.3}$$

for $(x,y) \in E(\mathcal{G})$. *Suppose that there is a point* $x^0 = (\xi_1^0,\xi_2^0,\cdots,\xi_N^0) \in \mathcal{X}$ *such that there is a point* $u^0 = (u_1^0,u_2^0,\cdots,u_N^0) \in \mathcal{X}$ *with* $(\xi_i^0,u_i^0) \in E(G)$ *and*

$$u_i^0 \in H(\xi_i^0,\xi_{i+1}^0,\cdots,\xi_N^0,\xi_1^0,\cdots,\xi_{i-1}^0), \forall i = 1,2,\cdots,N. \tag{4.4}$$

Then

1) *for any* $x \in \mathcal{X}_{\mathcal{H}}$, $H|_{[x]\tilde{\mathcal{G}}}$ *has an N-order fixed point;*

2) *if G is weakly connected, then H has an N-order fixed point in* $\mathcal{G}$;

3) *if* $\mathcal{X}' = \cup\{[x]_{\tilde{\mathcal{G}}} : x \in \mathcal{X}_{\mathcal{H}}\}$, *then* $H|_{\mathcal{X}'}$ *has an N-order fixed point in* $\mathcal{X}$;

4) *if* $\mathcal{H}(\mathcal{X} \subset) E(\mathcal{G})$ *then H has an N-order fixed point;*

5) *fix* $\mathcal{H} \neq \varnothing$ *if and only if* $\mathcal{X}_{\mathcal{H}} \neq \varnothing$.

Proof. (1) Since (X,d) is complete, we know that $(\mathcal{X},D)$ is complete. Let $\mathcal{H}: \mathcal{X} \to CB(X)$ be defined by (4.10).

We first show that $\mathcal{H}$ is *SG*-contractive. For $x = (\xi_1,\cdots,\xi_N), y = (\eta_1,\cdots,\eta_N)$ with $(x,y) \in E(\mathcal{G})$, we obtain that

$$(\xi_2,\cdots,\xi_N,\xi_1;\eta_2,\cdots,\eta_N,\eta_1) \in E(\mathcal{G}),\cdots,(\xi_N,\cdots,\xi_{N-1};\eta_N,\cdots,\eta_{N-1}) \in E(\mathcal{G}) \tag{4.5}$$

Let $u = (u_1,\cdots,u_N) \in \mathcal{H}(x)$ and $\alpha > 0$ be arbitrary. Considering (4.3), for $u_1 \in H(x)$, there is a point $v_1 \in H(y)$ such that

$$d(u_1,v_1) \le k_1 d(\xi_1,\eta_1) + \cdots + k_N d(\xi_N,\eta_N) + \frac{\alpha}{N}$$

$$\cdots$$

For $u_n \in H(\xi_N,\xi_1,\cdots,\xi_{N-1})$, there is a point $v_n \in H(\eta_N,\eta_1,\cdots,\eta_{N-1})$ such that

$$d(u_n,v_n) \le k_N d(\xi_N,\eta_N) + k_1 d(\xi_1,\eta_1) + \cdots + k_{N-1} d(\xi_{N-1},\eta_{N-1}) + \frac{\alpha}{N}$$

Then, the point $v = (v_1,v_2,\cdots,v_N)$ belongs to $\mathcal{H}(y)$ and

$$\begin{aligned} d(u,v) &= d(u_1,v_1) + d(u_2,v_2) + \cdots + d(u_N,v_N) \\ &\le \left(k_1 d(\xi_1,\eta_1) + \cdots + k_N d(\xi_N,\eta_N) + \frac{\alpha}{N}\right) \\ &\quad + \left(k_1 d(\xi_2,\eta_2) + \cdots + k_N d(\xi_1,\eta_1) + \frac{\alpha}{N}\right) + \cdots \end{aligned}$$

$$+\left(k_1 d\left(\xi_N,\eta_N\right)+\cdots+k_N d\left(\xi_{N-1},\eta_{N-1}\right)+\frac{\alpha}{N}\right)$$
$$=\left(k_1+\cdots+k_N\right)\left(d\left(\xi_1,\eta_1\right)+\cdots+k_N d\left(\xi_N,\eta_N\right)\right)+\alpha$$
$$=\left(k_1+\cdots+k_N\right)D\left(x,y\right)+\alpha$$

Thus $\mathcal{H}$ is an *SG*-contraction on $\mathcal{X}$.

Next $\left(\mathcal{X},D,\mathcal{G}\right)$ has the property $\left(P_1\right)$. In fact, for any sequence $\left(x_n\right)_{n\in N}$ in $\mathcal{X}$, if $\left(\xi_1^n,\cdots,\xi_N^n\right)=x_n\to x=\left(\xi_1,\cdots,\xi_N\right)$ and $\left(x_n,x_{n+1}\right)\in E\left(\mathcal{G}\right)$ for any $n\in N$, then

$$\xi_i^n\to\xi_i,\left(\xi_i^n,\xi_i^{n+1}\right)\in E\left(G\right),i=1,\cdots,N.$$

By the property $\left(P_1\right)$ of $\left(X,d,G\right)$, we get

$$\left(\xi_i^n,\xi\right)\in E\left(G\right),$$

then $\left(x_n,x\right)\in E\left(\mathcal{G}\right)$.

Last, we can see that $\left(x^0,u^0\right)\in\mathcal{X}_{\mathcal{H}}$ which implies that $\mathcal{X}_{\mathcal{H}}\neq\varnothing$. By Theorem 4.3, we get the proof.

Theorem 4.4. *Let* $H:\mathcal{X}\to CB\left(X\right)$ *be a set-valued mapping and H satisfy the property* $\left(P_2\right)$.

Suppose that there are some constants $k_1,\cdots,k_N$ *with* $0<k_1+\cdots+k_N<1$ *such that for any* $u\in H\left(x\right)$, *for any* $\alpha>0$, *there is a point* $v\in H\left(y\right)$ *such that* (4.12) *holds for* $\left(x,y\right)\in E\left(G\right)$. *Suppose that there is a point* $x^0=\left(\xi_1^0,\xi_2^0,\cdots,\xi_N^0\right)\in X^N$ *such that* (4.13) *holds for some* $u_0=\left(u_1^0,u_2^0,\cdots,u_N^0\right)$ *with* $\left(\xi_i^0,u_i^0\right)\in E\left(G\right)$. *Then the conclusions in Theorem* 4.3 *hold.*

Corollary 5. [6] *Let* $H:X\times X\to CB\left(X\right)$ *be a continuous set-valued mapping having the mixed G-monotone property on X and satisfying* (MBL) *condition. If there exist* $x_0,y_0\in X$ *and* $x_1\in H\left(x_0,y_0\right),y_1\in H\left(y_0,x_0\right)$ *such that* $\left(\left(x_0,x_1\right),\left(y_0,y_1\right)\right)\in E\left(\mathcal{G}\right)$, *then there exists* $\left(x,y\right)$ *a coupled fixed point of H.*

Proof. The "the mixed *G*-monotone property" and the "(MBL) condition" on the page 9 in [6] can imply equality (4.3). By Theorem 4.4, we get the proof.

5. Applications

Let $f:I\times R\times R\times R\to R$. We discuss the differential equations

$$\begin{cases}\dot{x}\left(t\right)=f\left(t,x\left(t\right),y\left(t\right),z\left(t\right)\right)\\ \dot{y}\left(t\right)=f\left(t,y\left(t\right),z\left(t\right),x\left(t\right)\right),t\in I=\left[0,1\right]\\ \dot{z}\left(t\right)=f\left(t,z\left(t\right),x\left(t\right),y\left(t\right)\right)\end{cases}\tag{5.1}$$

with the initial condition

$$x\left(0\right)=y\left(0\right)=z\left(0\right)=x_0\tag{5.2}$$

Suppose that there are three continuous mapping $\alpha\left(t\right),\beta\left(t\right)$ and $\gamma\left(t\right)$ with the initial condition $\alpha\left(0\right)=\beta\left(0\right)=\gamma\left(0\right)=x_0$. If there is a constant $l>0$ such that

$$\max_{t\in I}\left|\alpha\left(t\right)-\left(x_0+\int_0^t f\left(t,\alpha\left(s\right),\beta\left(s\right),\gamma\left(s\right)\right)\mathrm{d}s\right)\right|\le l$$

$$\max_{t\in I}\left|\beta(t)-\left(x_0+\int_0^t f(t,\beta(s),\gamma(s),\alpha(s))\mathrm{d}s\right)\right|\le l$$

$$\max_{t\in I}\left|\gamma(t)-\left(x_0+\int_0^t f(t,\gamma(s),\alpha(s),\beta(s))\mathrm{d}s\right)\right|\le l$$

then $(\alpha(t),\beta(t),\gamma(t))$ is called an *l*–lower solution of differential Equations (5.1) with the initial condition (5.2). Although we use the lower solution definition, we can't judge $(\alpha(t),\beta(t),\gamma(t))$ is bigger or smaller than the solution of (5.1) since *l* is arbitrary bigger than 0. This notation of *l*-lower solution only shows the relative distance between $(\alpha(t),\beta(t),\gamma(t))$ and the solution of (5.1).

Theorem 5.1. *Let* $f\in C(I\times R\times R\times R,R)$. *Consider the differential* Equations (5.1) *with the initial condition* (5.2) *and a constant* $l>0$. *Suppose that the following conditions hold*:

1) *for* $x,y,z,u,v,w\in X$ *with* $|x-u|\le l,|y-v|\le l,|z-w|\le l$, *we have*

$$|f(t,x,y,z)-f(t,u,v,w)|\le k_1|x-u|+k_2|y-v|+k_3|z-w|$$

where $0\le k_1,k_2,k_3<1$ *with* $k_1+k_2+k_3<1$;

2) *there is an l-lower solution of* (5.1).

Then the differential Equations (5.1) *with initial condition* (5.2) *has a unique solution* (x^*,x^*,x^*) *in* $C(I,R)\times C(I,R)\times C(I,R)$. *That is,* x^* *is the solution of the differential*

$$\dot{x}(t)=f(t,x(t),x(t),x(t)),\ t\in[0,1] \tag{5.3}$$

with the initial condition $x(0)=x_0$.

Proof Let $X=C(I,R)$. It is clear that (X,d) is a complete metric space with the metric d defined as

$$\forall x,y\in X,\ d(x,y)=\max_{t\in I}|x(t)-y(t)|.$$

Denote

$$B_l=\left\{x\in X:\max_{t\in I}|x(t)|\le l\right\}.$$

We know that B is a closed subset of *X*. Let G be a graph on X with $V(G)=X$ and

$$\forall x,y\in X,(x,y)\in E(G)\Leftrightarrow x-y\in B_l \tag{5.4}$$

Clearly, $E(G)$ is an equivalence relation and $(x,x)\in E(G),\forall x\in X$. And besides, G is weakly connected.

Define $F:X^3\to X$ by

$$F(x,y,z)(t)=x_0+\int_0^t f(t,x(s),y(s),z(s))\mathrm{d}s \tag{5.5}$$

By *f*'s continuity and (5.4), we know that *F* is continuous. Then the solution of the differential

Equations (5.1) with initial condition (5.2) are just the tripled fixed point of *F*. In the following, we discuss the tripled fixed point of *F*. We will testify that *F* satisfies the all conditions in Theorem 4.3.

For $x,y,z,u,v,w\in X$, define a graph $\mathcal{G}$ and a metric D on X^3 by

$$((x,y,z),(u,v,w))\in E(\mathcal{G})\Leftrightarrow x-u\in B_l,\ y-v\in B_l,\ z-w\in B_l$$

and

$$D((x,y,z),(u,v,w))=d(x,u)+d(y,v)+d(z,w).$$

Obviously, $(X^3,D,\mathcal{G})$ is induced by (X,d,G). We now prove $(F(x,y,z),F(u,v,w))\in E(G)$ if $((x,y,z),(u,v,w))\in E(\mathcal{G})$. Since

$$\begin{aligned}&((x,y,z),(u,v,w))\in E(\mathcal{G})\\ &\Leftrightarrow x-u\in B_l,\ y-v\in B_l,\ z-w\in B_l\\ &\Leftrightarrow \max_{t\in I}|x(t)-u(t)|\le l,\max_{t\in I}|y(t)-v(t)|\le l,\max_{t\in I}|z(t)-w(t)|\le l\\ &\Leftrightarrow d(x,u)\le l,d(y,v)\le l,d(z,w)\le l,\end{aligned}$$

by (i), we have for $t\in I$,

$$\begin{aligned}&|F(x,y,z)(t)-F(u,v,w)(t)|\\ &=\left|\int_0^t f(t,x(s),y(s),z(s))\mathrm{d}s-\int_0^t f(t,u(s),v(s),w(s))\mathrm{d}s\right|\\ &\le\int_0^t|f(t,x(s),y(s),z(s))-f(t,u(s),v(s),w(s))|\mathrm{d}s\\ &\le\int_0^t k_1|x(s)-u(s)|+k_2|y(s)-v(s)|+k_3|z(s)-w(s)|\mathrm{d}s\\ &\le\int_0^t k_1d(x,u)+k_2d(y,u)+k_3d(z,w)\mathrm{d}s\\ &\le k_1d(x,u)+k_2d(y,u)+k_3d(z,w)\quad\text{(which is a constant)}\\ &\le(k_1+k_2+k_3)\max\{d(x,u),d(y,v),d(z,w)\}\\ &\le l\end{aligned}$$

which implies $F(x,y,z)-F(u,v,w)\in B_l$ and then $(F(x,y,z),F(u,v,w))\in E(G)$.

Moreover, by the above deduction, we have

$$\begin{aligned}&d(F(x,y,z),F(u,v,w))\\ &=\max_{t\in I}|F(x,y,z)(t)-F(u,v,w)(t)|\\ &\le k_1d(x,u)+k_2d(y,v)+k_3d(z,w)\end{aligned}$$

By (2), the differential Equations (5.1) has an *l*-lower solution denoted $(\alpha(t),\beta(t),\gamma(t))$, we know

$$(\alpha,F(\alpha,\beta,\gamma))\in E(G),(\beta,F(\beta,\gamma,\alpha))\in E(G),(\gamma,F(\gamma,\alpha,\beta))\in E(G).$$

Thus by Theorem 4.3, F has a unique fixed point (x^*,y^*,z^*) with $x^*=y^*=z^*$.

Thus, the different Equations (5.1) with initial condition (5.2) has a unique solution (x^*,x^*,x^*) and which means

$$x^*(t)=F(x^*,x^*,x^*)(t)=x_0+\int_0^t f(t,x^*(s),x^*(s),x^*(s))\mathrm{d}s$$

which is equivalent to the different Equations (5.3) with the initial condition $x(0)=x_0$.

Theorem 5.2. Let $f:I\times R\times R\times R\to R$ satisfy: forgiven $x,y,z\in R$ and a sequence $\{x_n, y_n, z_n\}$ with $|x_n - x_{n+1}| \le l, |y_n - y_{n+1}| \le l, |z_n - z_{n+1}| \le l$ and $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$, we have $\lim_{n\to\infty} f(t, x_n, y_n, z_n) = f(t, x, y, z)$. Consider the differential Equations (5.1) with the initial condition (5.2) and a constant $l > 0$. Suppose that the conditions (1) and (2) in Theorem 5.1 are satisfied. Then we can get the same result with Theorem 5.1.

Proof. Define F as (5.4), then F satisfies (P_2) with $N = 3$. By the similar induction of Theorem 4.3, we can get the proof.

6. Conclusion

We establish *N*-order fixed point theorems of set-valued and single-valued mapping on product metric space with a graph. We build a unified method to study the *N*-order fixed point theory.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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