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On the Caginalp for a Conserve Phase-Field with a Polynomial Potentiel of Order $2p-1$

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Abstract

Our aim in this paper is to study on the Caginalp for a conserved phase-field with a polynomial potentiel of order $2p-1$. In this part, one treats the conservative version of the problem of generalized phase field. We consider a regular potential, more precisely a polynomial term of the order $2p-1$ with edge conditions of Dirichlet type. Existence and uniqueness are analyzed. More precisely, we precisely, we prove the existence and uniqueness of solutions.

Keywords

A Conserved Phase-Field, Polynomial Potentiel of Order $2p-1$, Dirichlet Boundary Conditions, Maxwell-Cattaneo Law

1. Introduction

The Caginalp phase-field model

$$\frac{\partial u}{\partial t}-\Delta u+f(u)=\theta \tag{1}$$

$$\frac{\partial \theta}{\partial t}-\Delta \theta=-\frac{\partial u}{\partial t} \tag{2}$$

proposed in [1], has been extensively studied (see, e.g., [2]-[7] and [8]). Here, u denotes the order parameter and θ the (*relative*) temperature.

Furthermore, all physical constants have been set equal to one. This system models, e.g., melting-solidification phenomena in certain classes of materials.

The Caginalp system can be derived as follows. We first consider the (total) free energy

$$\psi(u,\theta)=\int_\Omega\left(\frac{1}{2}|\nabla u|^2+f(u)-u\theta-\frac{1}{2}\theta^2\right)\mathrm{d}x, \tag{3}$$

where Ω is the domain occupied by the materiel.

We then define the enthalpy H as

$$H = -\frac{\partial \psi}{\partial \theta} \tag{4}$$

where ∂ denotes a variational derivative, which gives

$$H = u + \theta. \tag{5}$$

The governing equations for u and θ are then given by (see [9])

$$\frac{\partial u}{\partial t} = -\frac{\partial \psi}{\partial u}, \tag{6}$$

$$\frac{\partial H}{\partial t} + divq = 0, \tag{7}$$

where q is the thermal flux vector. Assuming the classical Fourier Law

$$q = -\nabla \theta, \tag{8}$$

we find (1) and (2).

Now, a drawback of the Fourier Law is the so-called "paradox of heat conduction", namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see, e.g. [10] and [11]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo Law.

$$\left(1 + \frac{\partial}{\partial t}\right) q = -\nabla \theta, \tag{9}$$

In that case, it follows from (7) that

$$\left(1 + \frac{\partial}{\partial t}\right) \frac{\partial H}{\partial t} - \Delta \theta = 0,$$

hence,

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \Delta \theta = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}. \tag{10}$$

This model can also be derived by considering, as in [12] (see also [13]-[20]), the Caginalp phase-field model with the so-called Gurtin-Pipkin Law

$$q(t) = -\int_0^{+\infty} k(s) \nabla \theta (t - s) \mathrm{d}s. \tag{11}$$

for an exponentially decaying memory kernel k, namely,

$$k(s) = \mathrm{e}^{-s}. \tag{12}$$

Indeed, differentiating (11) with respect to t and integrating by parts, we recover the Maxwell-Cattaneo Law (9).

Now, in view of the mathematical treatment of the problem, it is more convenient to introduce the new variable

$$\alpha = \int_0^t \theta(s) \mathrm{d}s, \quad \theta = \frac{\partial \alpha}{\partial t}, \tag{13}$$

and we have, integrating (10) with respect to $s \in [0,1]$.

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{14}$$

where

$$\alpha(t,x)=\int_0^t T(\tau,x)\,\mathrm{d}\tau+\alpha_0(x) \tag{15}$$

is the conductive thermal displacement. Noting that $T=\frac{\partial \alpha}{\partial t}$, we finally deduce from (33) and (36)-(37) the following variant of the Caginalp phase-field system (see [17]):

$$\frac{\partial u}{\partial t}-\Delta u+f(u)=\frac{\partial \alpha}{\partial t} \tag{16}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{17}$$

In this paper, we consider the following conserved phase-field model:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta\frac{\partial \alpha}{\partial t} \tag{18}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{19}$$

These equations are known as the conserved phase-field model (see [21]-[30]) based on type II heat conduction and with two temperatures (see [3] and [4]), conservative in the sense that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. Indeed, in that case, integrating (18) over the spatial domain Ω, we have the conservation of mass,

$$\langle u(t)\rangle=\langle u(0)\rangle,\quad t\geq 0 \tag{20}$$

$$\langle\cdot\rangle=\frac{1}{vol\Omega}\int_\Omega \mathrm{d}x \tag{21}$$

denotes the spatial average. Furthermore, integrating (19) over, we obtain

$$\langle \alpha(t)\rangle=\langle \alpha(0)\rangle,\quad t\geq 0 \tag{22}$$

Our aim in this paper is to study the existence and uniqueness of solution of (17)-(39). We consider here only one type of boundary condition, namely, Dirichlet (see [31] [32] [33]).

2. Setting of the Problem

We consider the following initial and boundary value problem

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta\frac{\partial \alpha}{\partial t} \tag{23}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{24}$$

$$u\big|_\Gamma=\Delta u\big|_\Gamma=\alpha\big|_\Gamma=0,\quad \text{on } \partial\Omega, \tag{25}$$

$$u\big|_{t=0}=u_0,\ \alpha\big|_{t=0}=\alpha_0,\ \frac{\partial\alpha}{\partial t}=\alpha_1 \tag{26}$$

As far as the nonlinear term f is concerned, we assume that

$$f\in C^{\infty}(R),\ f(0)=0 \tag{27}$$

Consider the following polynomial potential of order $2p-1$

$$f(s)=\sum_{i=1}^{2p-1}a_i s^i,\ p\in N^*,\ p\geq 2; a_{2p-1}=2pb_{2p}\geq 0 \tag{28}$$

The function f satisfies the following properties

$$\frac{1}{2}a_{2p-1}s^{2p}-c_1\leq f(s)s\leq\frac{3}{2}a_{2p-1}s^{2p}+c_1, \tag{29}$$

$$f'(s)\geq\frac{1}{2}a_{2p-1}s^{2p-2}-c_2\geq -k,\forall s\in R,k\geq 0 \tag{30}$$

where

$$F(s)=\int_0^s f(\tau)\mathrm{d}\tau \tag{31}$$

such as

$$\frac{1}{4p}a_{2p-1}s^{2p}-c_3\leq F(s)\leq\frac{3}{4p}a_{2p-1}s^{2p}+c_3 \tag{32}$$

Remark 2.1. We take here, for simplicity, Dirichlet Boundary Conditions. However, we can obtain the same results for Neumann Boundary Conditions, namely,

$$\frac{\partial u}{\partial \nu}=\frac{\partial\Delta u}{\partial \nu}=\frac{\partial\varphi}{\partial \nu}\quad\text{on }\Gamma \tag{33}$$

where ν denotes the unit outer normal to Γ. To do so, we rewrite, owing to (23) and (24), the equations in the form

$$\frac{\partial\bar{u}}{\partial t}+\Delta^2\bar{u}-\Delta\big(f(u)-\langle f(u)\rangle\big)=-\Delta\frac{\partial\bar{\alpha}}{\partial t}$$

$$\frac{\partial^2\bar{\varphi}}{\partial t^2}+\frac{\partial\bar{\varphi}}{\partial t}-\Delta\bar{\varphi}=-\frac{\partial\bar{u}}{\partial t},$$

where $\bar{v}=v-\langle v\rangle$, $|\langle v_0\rangle|\leq M_1$, $|\langle v_0\rangle|\leq M_2$, for fixed positive constants M_1 and M_2. Then, we note that

$$v\rightarrow\left(\left\|(-\Delta)^{\frac{-1}{2}}v\right\|^2+\langle v\rangle^2\right)^{\frac{1}{2}}$$

where, here, $-\Delta$ denotes the minus Laplace operator with Neumann boundary conditions and acting on functions with null average and where it is understood that

$$\langle\cdot\rangle=\frac{1}{vol(\Omega)}\langle\cdot,1\rangle_{H^{-1}(\Omega),H^1(\Omega)}$$

Furthermore

$$v \mapsto \left(\|\bar{v}\|^2 + \langle v \rangle^2 \right)^{\frac{1}{2}},$$

$$v \mapsto \left(\|\nabla v\|^2 + \langle v \rangle^2 \right)^{\frac{1}{2}},$$

$$v \mapsto \left(\|\Delta v\|^2 + \langle v \rangle^2 \right)^{\frac{1}{2}}$$

are norms in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual ones.

We further assume that

$$|f(s)| \le \varepsilon F(s) + c_\varepsilon, \ \forall \varepsilon > 0, \ s \in R, \tag{34}$$

which allows to deal with term $\langle f(u) \rangle$.

3. Notations

We denote by $\|\cdot\|$ the usual L^2-norm (with associated product scalar (.,.) and set $\|\cdot\|_{-1} = \left\| (-\Delta)^{\frac{-1}{2}} \cdot \right\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet Boundary Conditions. More generally, $\|\cdot\|_X$ denote the norm of Banach space X.

Throughout this paper, the same letters c_1, c_2 and c_3 denote (generally positive) constants which may change from line to line, or even a same line.

4. A Priori Estimates

The estimates derived in this subsection will be formal, but they can easily be justified within a Galerkin scheme. We rewrite (23) in the equivalent form

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}. \tag{35}$$

We multiply (35) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x \right) + 2\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 = 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \tag{36}$$

We then multiply (24) by $\frac{\partial \alpha}{\partial t}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2\left\| \frac{\partial \alpha}{\partial t} \right\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \tag{37}$$

Summing (36) and (37), we find the differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2\left\| \frac{\partial \alpha}{\partial t} \right\|^2 = 0 \tag{38}$$

Integrating from 0 to t with $t \in [0;T]$ we obtain

$$\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2+2\int_\Omega F(u)\mathrm{d}x+\|\nabla\alpha(s)\|^2+\left\|\frac{\partial\alpha(s)}{\partial t}\right\|^2\right)\mathrm{d}s$$

$$+2\int\left\|\frac{\partial\alpha(s)}{\partial t}\right\|^2\mathrm{d}s+2\int\left\|\frac{\partial u(s)}{\partial t}\right\|_{-1}^2\mathrm{d}s=0$$

of (35) we deduce

$$F(u_0)\le\frac{3}{4p}a_{2p-1}u_0^{2p}+c_3$$

which involves

$$2\int_\Omega F(u_0)\mathrm{d}x\le\frac{3}{2p}a_{2p-1}\|u_0\|_{L^{2p}}^{2p}+2c_3|\Omega|$$

still of (35) we have

$$\frac{3}{4p}a_{2p-1}u_0^{2p}-c_3\le F(u)$$

which involves

$$\frac{1}{2p}a_{2p-1}\|u_0\|_{L^{2p}}^{2p}-2c_3|\Omega|\le F(u)$$

where

$$E(t)+2\int_0^t\left(\left\|\frac{\partial\alpha(s)}{\partial t}\right\|^2+\left\|\frac{\partial u(s)}{\partial t}\right\|_{-1}^2\right)\mathrm{d}s\le C$$

with

$$E(t)=\|\nabla u(t)\|^2+\frac{1}{2p}a_{2p-1}\|u(t)\|_{L^{2p}}^{2p}+\left\|\frac{\partial\alpha(t)}{\partial t}\right\|^2+\|\nabla\alpha(t)\|^2 \tag{39}$$

and $C=\|\nabla u_0\|^2+\dfrac{3}{2p}a_{2p-1}\|u_0\|_{L^{2p}}^{2p}+\|\alpha_1\|^2+\|\nabla\alpha_0\|^2+C_3$.

Finally, we conclude that
$u\in L^\infty\left(R^*;H_0^1(\Omega)\cap L^{2p}(\Omega)\right);\alpha\in L^2\left(0,T;H^{-1}(\Omega)\right)$;

$$\frac{\partial u}{\partial t}\in L^2\left(0,T;H^{-1}(\Omega)\right);\frac{\partial\alpha}{\partial t}\in L^\infty\left(R_+^*;L^2(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)\quad\forall T>0$$

Theorem 4.1. (Existence) We assume $(u_0,\alpha_0,\alpha_1)\in\left(H_0^1(\Omega)\cap L^{2p}(\Omega)\right)\times H_0^1(\Omega)\times L^2(\Omega)$ then the system (18)-(19) possesses at least one solution (u,α) such that

$$u\in L^\infty\left(R^*;H_0^1(\Omega)\cap L^{2p}(\Omega)\right);\alpha\in L^2\left(0,T;H^{-1}(\Omega)\right)$$

$$\frac{\partial u}{\partial t}\in L^2\left(0,T;H^{-1}(\Omega)\right);\frac{\partial\alpha}{\partial t}\in L^\infty\left(R_+^*;L^2(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)$$

$$\forall T>0$$

Theorem 4.2. (Uniqueness) Let the assumptions of Theorem 4.1 hold. Then, the system (18)-(19) possesses a unique solution (u,α) such that

$$u \in L^{\infty}\left(R^{*}; H_0^1(\Omega) \cap L^{2p}(\Omega)\right); \alpha \in L^2\left(0,T; H^{-1}(\Omega)\right)$$

$$\frac{\partial u}{\partial t} \in L^2\left(0,T; H^{-1}(\Omega)\right); \frac{\partial \alpha}{\partial t} \in L^{\infty}\left(R_+^{*}; L^2(\Omega) \cap L^2(0,T; L^2(\Omega)\right)$$

$$\forall T > 0$$

Let $\left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$ be two solutions (23)-(25) with initial data $\left(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}\right)$ and $\left(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}\right)$, respectively. We set

$$\left(u, \alpha, \frac{\partial \alpha}{\partial t}\right) = \left(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right) - \left(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$$

and

$$\left(u_0, \alpha_0, \alpha_1\right) = \left(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}\right) - \left(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}\right)$$

Then, (u, α) satisfies

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right)\right) = -\Delta \frac{\partial \alpha}{\partial t} \tag{40}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \tag{41}$$

$$u\big|_{\Gamma} = \Delta u\big|_{\Gamma} = \alpha\big|_{\Gamma} = 0, \quad \text{on } \partial\Omega, \tag{42}$$

$$u\big|_{t=0} = u_0,\ \alpha\big|_{t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t} = \alpha_1 \tag{43}$$

We multiply (40) by $(-\Delta)^{-1} \frac{\partial u}{\partial t}$, we have

$$\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left(\frac{\partial u}{\partial t}, -\Delta u\right) + \left(-\Delta\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right)\right), (-\Delta)^{-1} \frac{\partial u}{\partial t}\right) = \left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 = -2\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right), \frac{\partial u}{\partial t}\right) + 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right). \tag{44}$$

We multiply by (41) by $\frac{\partial \alpha}{\partial t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2\right) + 2\left\|\frac{\partial \alpha}{\partial t}\right\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right) \tag{45}$$

Now summing (44) and (45) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \|\nabla \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2\right) + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2\left\|\frac{\partial \alpha}{\partial t}\right\|^2 \\ &= -2\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right), \frac{\partial u}{\partial t}\right)\end{aligned} \tag{46}$$

We know that

$$f\left(u^1\right) - f\left(u^2\right) = \sum_{k=1}^{2p-1} a_k\left(u^{(1)k}\right) - \sum_{k=1}^{2p-1} a_k\left(u^{(2)k}\right) = \sum_{k=1}^{2p-1} a_k\left(u^{(1)k} - u^{(2)k}\right)$$

which involves

$$\left|f\left(u^1\right)-f\left(u^2\right)\right| \le \sum_{k=1}^{2p-1}\left|a_k\right|\left|u^{(1)k}-u^{(2)k}\right|$$
$$\le \sum_{k=1}^{2p-1}\left|a_k\right|\left|u^{(1)}-u^{(2)}\right|\left|u^{(1)}\right|^{k-1}+\sum_{j=1}^{k-2}\left|u^{(1)}\right|^{k-1-j}\left|u^{(2)}\right|^{j}+\left|u^{(2)}\right|^{k-1}.$$

Based on Young's inequality, we have

$$\sum_{j=1}^{k-2}\left|u^{(1)}\right|^{k-1-j}\left|u^{(2)}\right|^{j} \le \sum_{j=1}^{k-2}\left(\frac{k-j-1}{k-1}\left|u^{(1)}\right|^{k-1}+\frac{j}{k-1}\left|u^{(2)}\right|^{k-1}\right)$$

with $p=\frac{k-1}{k-j-1}$ and $q=\frac{k-1}{j}$ such as $\frac{1}{p}+\frac{1}{q}=1$. So

$$\sum_{j=1}^{k-2}\left|u^{(1)}\right|^{k-1-j}\left|u^{(2)}\right|^{j} \le \frac{1}{k-1}\sum_{j=1}^{k-2}(k-1)\left|u^{(1)}\right|^{k-1}+\frac{1}{k-1}\sum_{j=1}^{k-2} j\left(\left|u^{(2)}\right|^{k-1}-\left|u^{(1)}\right|^{k-1}\right).$$

As

$$\sum_{j=1}^{k-2} j=\frac{(k-2)(k-1)}{2}$$

then

$$\sum_{j=1}^{k-2}\left|u^{(1)}\right|^{k-1-j}\left|u^{(2)}\right|^{j} \le (k-2)\left|u^{(1)}\right|^{k-1}+\frac{k-2}{2}\left|u^{(2)}\right|^{k-1}-\frac{k-2}{2}\left|u^{(1)}\right|^{k-1}$$
$$\le \frac{k-2}{2}\left(\left|u^{(1)}\right|^{k-1}+\left|u^{(2)}\right|^{k-1}\right).$$

We know that

$$\forall k \in N;\ \ k-2 \le k \ \text{ then }\ \frac{k-2}{2} \le \frac{k}{2} \le k$$
$$\sum_{j=1}^{k-2}\left|u^{(1)}\right|^{k-1-j}\left|u^{(2)}\right|^{j} \le k\left(\left|u^{(1)}\right|^{k-1}+\left|u^{(2)}\right|^{k-1}\right)$$

which gives

$$\left|f\left(u^1\right)-f\left(u^2\right)\right| \le \sum_{j=1}^{k-2}\left|a_k\right|\left|u^{(1)}-u^{(2)}\right|\left((k+1)\left|u^{(1)}\right|^{k-1}+(k+1)\left|u^{(2)}\right|^{k-2}\right)$$
$$\le |u|\sum_{j=1}^{k-2}(k+1)\left|a_k\right|\left(\left|u^{(1)}\right|^{k-1}+\left|u^{(2)}\right|^{k-1}\right)$$

$\exists\, k>0$ such as

$$(k+1)\left|a_k\right| \le k;\ \ \forall\, k \in 1,2,\cdots,2p-1$$

so

$$\left|f\left(u^1\right)-f\left(u^2\right)\right| \le |u|\, k\sum_{k=1}^{k-2}\left(\left|u^{(1)}\right|^{k-1}+\left|u^{(2)}\right|^{k-1}\right).$$

Based on Young's inequality, we have $\forall\, k \ge 2$

$$\left|u^{(1)}\right|^{k-1} \le \frac{k-1}{2p-2}\left(\left|u^{(1)}\right|^{k-1}\right)^{\frac{2p-2}{k-1}}+\frac{2p-k-1}{2p-2}$$

and

$$\left|u^{(2)}\right|^{k-1} \le \frac{k-1}{2p-2}\left(\left|u^{(2)}\right|^{k-1}\right)^{\frac{2p-2}{k-1}} + \frac{2p-k-1}{2p-2}$$

that involve

$$\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right| \le |u|\frac{k}{2p-2}\sum_{k=1}^{2p-1}\left(\left(k-1\right)\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(2)}\right|^{2p-2}\right)+2\left(\frac{2p-k-1}{2p-2}\right)\right)$$
$$\le c|u|\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(2)}\right|^{2p-2}+1\right).$$

We finally

$$\int_{\Omega}\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right|\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le c\int_{\Omega}|u|\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(2)}\right|^{2p-2}+1\right)\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x. \tag{47}$$

The second member of (45) is increased in R^n for $n=1,2,3$.

If $n = 1$; $u^i \in H_0^1(\Omega) \subset H^1(\Omega) = W^{1,2}(\Omega)$ for $i=1,2$.

Thanks to the continuous injection $H^1(\Omega) \subset C(\bar{\Omega})$, then is $C>0$, by applying Holder's inegality, we get

$$\int_{\Omega}|u|\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(1)}\right|^{2p-2}+1\right)\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|\left\|\frac{\partial u}{\partial t}\right\|,$$

which involves using the compact injection $H^1(\Omega) \subset L^2(\Omega)$, we have

$$\int_{\Omega}\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right|\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{H^1}\left\|\frac{\partial u}{\partial t}\right\| \tag{48}$$

If $n = 2$ then $H^1(\Omega) \subset L^q(\Omega)$, $\forall\, q \in [1,\infty[$.

Based on Holder's inequality, we have

$$\int_{\Omega}|u|\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(1)}\right|^{2p-2}+1\right)\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{L^3}\left\|\frac{\partial u}{\partial t}\right\|.$$

Finally

$$\int_{\Omega}\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right|\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{H^1}\left\|\frac{\partial u}{\partial t}\right\|$$

If $n = 3$, then $H^1(\Omega) \subset L^q(\Omega)$ with $q \in [1,6]$

In this case, we also

$$\int_{\Omega}|u|\left(\left|u^{(1)}\right|^{2p-2}+\left|u^{(1)}\right|^{2p-2}+1\right)\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{L^6}\left\|\frac{\partial u}{\partial t}\right\|.$$

So

$$\int_{\Omega}\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right|\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{H^1}\left\|\frac{\partial u}{\partial t}\right\|.$$

We notice that in R^n for $n=1,2,3$, we have

$$\int_{\Omega}\left|f\left(u^{1}\right)-f\left(u^{2}\right)\right|\left|\frac{\partial u}{\partial t}\right|\mathrm{d}x \le C\|u\|_{H^1}\left\|\frac{\partial u}{\partial t}\right\|.$$

Using Young's inequality, we have

$$\int_\Omega \left| f\left(u^1\right) - f\left(u^2\right) \right| \left| \frac{\partial u}{\partial t} \right| \mathrm{d}x \le C \|u\|_{H^1}^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \tag{49}$$

Inserting (49) into (46), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} E_2 + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \le c' \|u\|_{H^1}^2 + \left\| \frac{\partial u}{\partial t} \right\|^2$$

and recalling the interpolation inequality $\left\| \frac{\partial u}{\partial t} \right\|^2 \le c \left\| \frac{\partial u}{\partial t} \right\|_{-1} \left\| \nabla \frac{\partial u}{\partial t} \right\|$

with $E_2 = \|\nabla u\|^2 + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2$

Finally

$$\frac{\mathrm{d}}{\mathrm{d}t} E_2 + c'' \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \le C E_2, \quad C > 0 \tag{50}$$

Theorem 4.3. (Second theorem of the solution's existence) The existence and uniqueness of the solution (23)-(25) problem being proven, now we seek the solution of (23)-(25) with more regularity.

Assume $\begin{array}{l}(u_0,\alpha_0,\alpha_1) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega) \\ \times (u_0,\alpha_0,\alpha_1) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega) \times H_0^1(\Omega)\end{array}$, then the (23)-(24) system admits a unique (u,α) solution such as

$$u \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right), \alpha \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega) \cap H_0^1(\Omega)\right),$$

and

$$\frac{\partial u}{\partial t} \in L^2\left(0,T;H^{-1}(\Omega)\right)$$

Theorems of existence (23) and uniqueness (24) being proven then $u \in L^\infty\left(0,T;H^2(\Omega) \cap L^{2p}(\Omega)\right)$, $\alpha \in L^\infty\left(0,T;H_0^1(\Omega)\right)$, $\frac{\partial \alpha}{\partial t} \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$ and $\frac{\partial u}{\partial t} \in L^\infty\left(0,T;H^{-1}(\Omega)\right)$, $\forall T > 0$.

We multiply (23) by $(-\Delta)^{-1} \frac{\partial u}{\partial t}$ and have, integrating over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 = 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \tag{51}$$

Multiplying (24) by $\frac{\partial \alpha}{\partial t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \tag{52}$$

Now summing (51) and (52) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 = 0 \tag{53}$$

where

$$E_3 = \|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\nabla \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2$$

finally

$$\|\nabla u(t)\|^2 + c\|u(t)\|_{L^{2p}}^{2p} + \|\nabla \alpha(t)\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\int_0^t \left(\left\|\frac{\partial \alpha(s)}{\partial t}\right\|^2 + \left\|\frac{\partial u(s)}{\partial t}\right\|_{-1}^2\right)\mathrm{d}s \le c_1.$$

We infer that

$$u \in L^\infty\left(0,T;H^2(\Omega)\cap L^{2p}(\Omega)\right),\quad \alpha \in L^\infty\left(0,T;H_0^1(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right) \text{ and } \frac{\partial u}{\partial t} \in L^\infty\left(0,T;H^{-1}(\Omega)\right).$$

We multiply (24) by $\frac{\partial^2 \alpha}{\partial t^2}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla \alpha\|^2\right) + \left\|\frac{\partial^2 \alpha}{\partial t^2}\right\|^2 \le \left\|\frac{\partial u}{\partial t}\right\|^2.$$

We infer from this that $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right)$.

5. Conclusion

In this work we have studied the existence and uniqueness of the solution of a conservative-type Caginalp system with Dirichlet-type boundary conditions. Finally we have also succeeded in this work to establish the existence theorems of the solution of this system with low regularity and more regularity. As a perspective, we plan to study this problem in a bounded or unbounded domain with different types of potentials and Neumann-type conditions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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