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The Traveling Wave Solutions of Space-Time Fractional Partial Differential Equations by Modified Kudryashov Method

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Abstract

In this paper, the modified Kudryashov method is employed to find the traveling wave solutions of two well-known space-time fractional partial differential equations, namely the Zakharov Kuznetshov Benjamin Bona Mahony equation and Kolmogorov Petrovskii Piskunov equation, and as a helping tool, the sense of modified Riemann-Liouville derivative is also used. The propagation properties of obtained solutions are investigated where the graphical representations and justifications of the results are done by mathematical software Maple.

Keywords

Traveling Wave Solutions, Modified Kudryashov Method, Zakharov Kuznetshov Benjamin Bona Mahony (ZKBBM) Equation, Kolmogorov Petrovskii Piskunov (KPP) Equation

1. Introduction

In the last century, many authors visualized the study on properties of fractional derivatives. Various types of methods have been used from that period. In recent decades, non-linear fractional differential equations (FDEs) have been charmed at large scale. New researchers are being attracted by this new concept due to its versatile applications. Variant applications of non-linear fractional differential equations are noticeable in various sciences such as physics, engineering, biological network, landscape evolution, fluid flow, elasticity, quantum mechanics, medical science and some other branches of pure and applied mathematics. Several mathematicians proposed different types of fractional derivatives. The most popular ones are Riemann-Liouville, Caputo, Grunwald-Letnikov, Hadamard, Erdelyi, Kober, Marchaud, and Riesz. One can see that these are theoretically much easier to handle and satisfy the classical properties. The exact solutions are important to describe the physical phenomenon of non-linear fractional partial differential equations. Again we see that fractional differential equations are the generalizations of classical differential equations of integer order. There are many techniques to obtain exact traveling wave solutions such as modified kudryashov method [1], fractional sub-equation method [2], residential power series method [3], fractional Riccati expansion method [4], the fractional sub-equation method [5], the first integral method [6], improved kudryashov method [7], the variational iteration method [8], Jumarie's modified Riemann-Liouville derivative [9] and other sciences (See [10] [11] [12] [13]). In this work, the modified kudryashov method [14] is used for solving the space-time fractional Zakharov Kuznetsov Benjamin Bona Mahony (ZKBBM) and Kolmogorov Petrovskii Piskunov (KPP) equation in the sense of modified Riemann-Liouville derivative [9] of order α, defined by the following expression

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} \left(f(\xi) - f(0) \right) \mathrm{d}\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left(f(\xi) - f(0) \right) \mathrm{d}\xi, & 0 < \alpha < 1 \\ \left[f^{(\alpha-n)}(x) \right]^{(n)}, & n \le \alpha < n+1, n \ge 1 \end{cases}$$

Some properties for the proposed modified Riemann-Liouville derivative are listed as follows:

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \gamma > 0$$

$$D_x^{\alpha} \left(f(x) g(x) \right) = g(x) D_x^{\alpha} f(x) + f(x) D_x^{\alpha} g(x)$$

$$D_x^{\alpha} f\left[g(x) \right] = f_g' \left[g(x) \right] D_x^{\alpha} g(x) = D_g^{\alpha} f\left[g(x) \right] \left(g'(x) \right)^{\alpha}$$

2. Methodology

We regard non-linear fractional partial differential equation with independent variables x, t and dependent variable u is given by

$$P\left(u, u_x, u_y, u_t, D_x^{\beta} u, D_y^{\gamma} u, D_t^{\alpha} u, \cdots \right) = 0, 0 < \alpha, \beta \le 1 \tag{2.1.1}$$

The main steps of this method are following as:

Step 1. We use the variable transformation

$$u(x,t) = u(\xi), \xi = \frac{Kx^{\beta}}{\Gamma(\beta+1)} + \frac{My^{\gamma}}{\Gamma(\gamma+1)} + \frac{Lt^{\alpha}}{\Gamma(\alpha+1)} \tag{2.1.2}$$

where K, L and M are non-zero arbitrary constants for transforming (2.1.1) to the following non-linear fractional ordinary differential equation with independent variable ξ is

$$P\left(u, Ku', Mu', Lu', K^{\beta} D_{\xi}^{\beta} u, M^{\gamma} D_{\xi}^{\gamma} u, L^{\alpha} D_{\xi}^{\alpha} u, \cdots \right) = 0 \tag{2.1.3}$$

Here prime denotes the derivative with respect to xi.

Step 2. We seek for the exact solution of Equation (2.1.3)

$$u(\xi)=\sum_{n=0}^{N} a_n Q(\xi)^n \tag{2.1.4}$$

where $a_n\left(n=0,1,\cdots,N\right)$ are constants to be determined such that $a_N\neq 0$ and $Q(\xi)$ is the solution of the equation

$$Q'(\xi)=\left[Q^2(\xi)-Q(\xi)\right]\ln a \tag{2.1.5}$$

Equation (2.1.5) has the solution

$$Q(\xi)=\frac{1}{1+a^{\xi}} \tag{2.1.6}$$

where $a>0, a\neq 1$ is a number.

Step 3. We determine the positive integer N in Equation (2.1.4) by considering the homogenous balance between the highest derivative and the non-linear term in (2.1.3).

Step 4. Substitute (2.1.4) along with (2.1.5) into (2.1.3), we calculate all necessary derivatives $u',u'',\cdots$ of the function $u(\xi)$. As a result of this substitution, we get a polynomial of $Q^i(\xi),(i=0,1,2,\cdots)$. In this polynomial we gather all terms of same powers of $Q^i(\xi)$ and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple to get the unknown parameters $a_n\left(n=0,1,\cdots,N\right)$, K and L. Consequently, we obtain the exact solutions of (2.1.1).

3. Applications of the Method

3.1. Solutions for the Space-Time Fractional Zakharov Kuznetshov Benjamin Bona Mahony (ZKBBM) Equation

In this section, we will manage the modified Kudryashov method to find the space-time fractional Zakharov Kuznetshov Benjamin Bona Mahony (ZKBBM) equation

$$D_t^{\alpha}u+D_x^{\alpha}u-2\omega u D_x^{\alpha}u-bD_t^{\alpha}\left(D_x^{2\alpha}u\right)=0 \tag{3.1.1}$$

Introducing the wave transformation

$$u(x,t)=u(\xi),\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)},$$

(3.1.1) becomes the following ODE:

$$(L+K)u-\omega K u^2-bLK^2u''=0 \tag{3.1.2}$$

We consider that Equation (3.1.2) has the travelling wave solution of the form

$$u(\xi)=\sum_{n=0}^{N} a_n Q(\xi)^n \tag{3.1.3}$$

where $a_n\left(n=0,1,\cdots,N\right)$ are constant to be determined by considering the homogenous balance between u'' and u^2 appearing in Equation (3.1.2), we have $N=2$ such that $a_N\neq 0$. We get the solution of (3.1.2) as

$$u(\xi)=a_2Q(\xi)^2+a_1Q(\xi)+a_0,\ a_2\neq 0 \tag{3.1.4}$$

where a_2,a_1,a_0 are constants to be determined afterwards and $Q=Q(\xi)$ satisfies the equation

$$Q'(\xi)=\left[Q^2(\xi)-Q(\xi)\right]\ln a \tag{3.1.5}$$

Equation (3.1.5) has the solution

$$Q(\xi)=\frac{1}{1+a^{\xi}} \tag{3.1.6}$$

where $a>0,a\neq 1$ is a number.

Substituting (3.1.4) and (3.1.5) into (3.1.2) and collecting the coefficients of each power of Q^i and then equating each of the coefficients to zero, a system of algebraic equations is found and the solution of this system gives the following traveling wave solutions.

Case-1:

$$\begin{gathered}K=K,\ L=\frac{K}{-1+bK^2\ln(a)^2},\ a_0=0,\\ a_1=\frac{6bK^2\ln(a)^2}{\left(-1+bK^2\ln(a)^2\right)\omega},\ a_2=-\frac{6bK^2\ln(a)^2}{\left(-1+bK^2\ln(a)^2\right)\omega}\end{gathered} \tag{3.1.7}$$

Setting (3.1.7) into (3.1.4), we reach at the following traveling wave solution

$$u_1(\xi)=-\frac{6bK^2\ln(a)^2}{\left(-1+bK^2\ln(a)^2\right)\omega\left(1+a^{\xi}\right)^2}+\frac{6bK^2\ln(a)^2}{\left(-1+bK^2\ln(a)^2\right)\omega\left(1+a^{\xi}\right)} \tag{3.1.8}$$

Case-2:

$$\begin{gathered}K=K,\ L=\frac{K}{1+bK^2\ln(a)^2},\ a_0=\frac{bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega},\\ a_1=-\frac{6bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega},\ a_2=-\frac{6bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega}\end{gathered} \tag{3.1.9}$$

Similarly setting (3.1.9) into (3.1.4), the following traveling wave solution is obtained

$$\begin{aligned}u_2(\xi)=&\frac{6bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega\left(1+a^{\xi}\right)^2}-\frac{6bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega\left(1+a^{\xi}\right)}\\ &+\frac{6bK^2\ln(a)^2}{\left(1+bK^2\ln(a)^2\right)\omega}\end{aligned} \tag{3.1.10}$$

3.2. Solutions for Kolmogorov Petrovskii Piskunov (KPP) Equation

Next we use the modified Kudryashov method to find the space-time fractional Kolmogorov Petrovskii Piskunov (KPP) equation

$$D_t^{\alpha}u-D_x^{2\alpha}u+\mu u+\omega u^2+\delta u^3=0 \tag{3.2.1}$$

where μ,ω,δ are arbitrary constants.

Making the wave transformation $u(x,t)=u(\xi),\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}$, (3.1.1) becomes the following ODE:

$$Lu'-K^2u''+\mu u+\omega u^2+\delta u^3=0 \tag{3.2.2}$$

Further we consider

$$u(\xi)=\sum_{n=0}^{N}a_nQ(\xi)^n \tag{3.2.3}$$

where $a_n\left(n=0,1,\cdots,N\right)$ are constants to be determined by considering the homogenous balance between the highest order derivatives and the non-linear terms appearing in Equation (3.2.2), we have $n=2$ such that $a_N\neq 0$. Putting the values of *n*, we get the solution of (3.2.2)

$$u(\xi)=a_2Q(\xi)^2+a_1Q(\xi)+a_0,\ a_2\neq 0 \tag{3.2.4}$$

where a_2,a_1,a_0 are constants to be determined.

Solving in similar procedure of example 1, we get the following families of values, then obtain traveling wave solutions.

Case-1:

$$K=\frac{\sqrt{\frac{\omega^2-4\delta\mu}{2\delta}}}{\ln(a)},\ L=-\frac{1}{2}\frac{\omega\left(2\delta\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+\omega\right)}{8\ln(a)},$$
$$a_0=\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}},\ a_1=\frac{\delta\omega\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+6\delta\mu-\omega^2}{\delta\left(3\delta\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+\omega\right)},\ a_2=0 \tag{3.2.5}$$

For (3.2.5), the traveling wave solution of (3.2.2) is

$$u_1(\xi)=-\frac{\frac{1}{2}\delta\omega\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}}+6\delta\mu-\omega^2}{\delta\left(\frac{3}{2}\delta\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}}+\omega\right)\left(1+a^{\xi}\right)}+\frac{1}{2}\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}} \tag{3.2.6}$$

where, $\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}$

Case-2:

$$K=\frac{\sqrt{\omega\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+\mu+2}}{\ln(a)},\ L=-\frac{1}{2}\frac{\omega\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+3\mu}{\ln(a)},$$
$$a_0=\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}},\ a_1=-\frac{2\omega\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+3\mu}{3\delta\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{2\delta}}+\omega},\ a_2=0 \tag{3.2.7}$$

For (3.2.7), the traveling wave solution of (3.2.2) is obtained as

$$u_2(\xi)=\frac{\omega\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}}+3\mu}{\left(\frac{3}{2}\delta\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}}+\omega\right)\left(1+a^{\xi}\right)}+\frac{1}{2}\sqrt{2}\sqrt{\frac{-\omega+\sqrt{\omega^2-4\delta\mu}}{\delta}} \quad (3.2.8)$$

where, $\xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}$.

3.3. Result and Direction

Here the three dimensional graph (**Figure 1**) of Equation (3.1.8) shows a bell shaped soliton profile. Similarly **Figure 2** displays a bell shaped three dimensional plot of Equation (3.1.10). Again, the traveling wave three dimensional sketch (**Figure 3**) of Equation (3.2.6) indicates kink shape. Likewise **Figure 4**

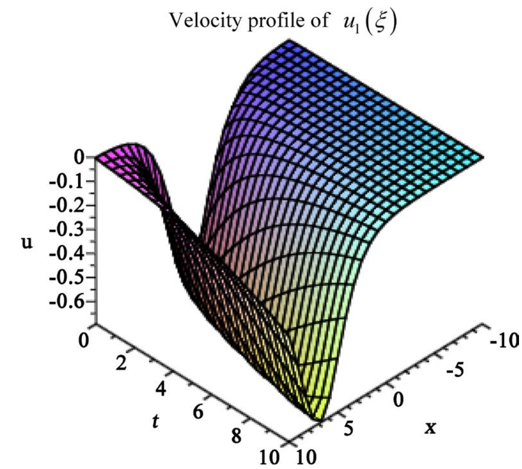

Figure 1. Profile of $u_1(\xi)$ for $b=1,k=1,a=2,\beta=1,\omega=2,\alpha=1/2$ within the interval $-10\le x\le 10, 0\le t\le 10$.

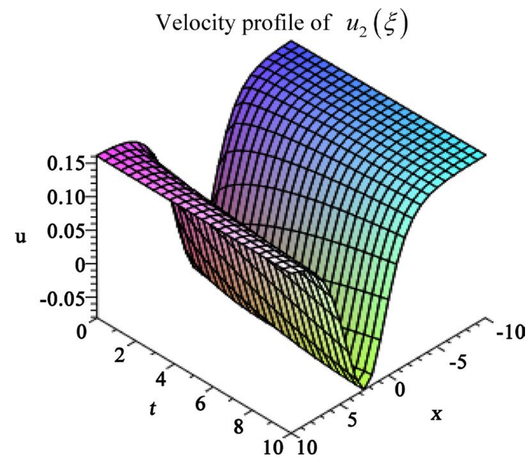

Figure 2. Profile of $u_2(\xi)$ for $b=1,k=1,a=2,\beta=1,\omega=2,\alpha=1/2$ within the interval $-10\le x\le 10, 0\le t\le 10$.

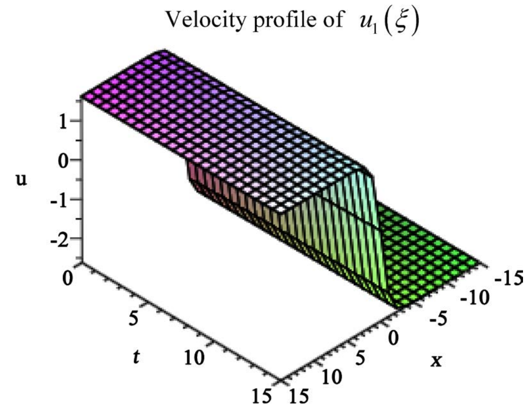

Figure 3. Profile of $u_1(\xi)$ for $\delta=1, \mu=1, a=2, \beta=1, \omega=-3, \alpha=1/2$ within the interval $-15 \le x \le 15, 0 \le t \le 15$.

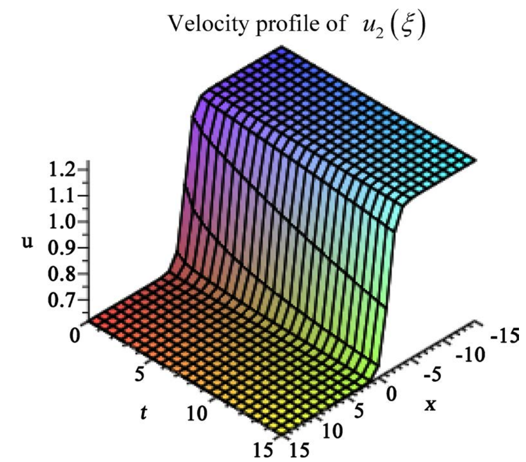

Figure 4. Profile of $u_2(\xi)$ for $\delta=-1, \mu=-1, a=2, \beta=1, \omega=3, \alpha=1/2$ within the interval $-15 \le x \le 15, 0 \le t \le 15$.

shows kink shape of Equation (3.2.8). We solve these equations with the help of Maple.

4. Conclusion

In our study, we solve such two equations (ZKBBM and KPP) which are very important for describing physical phenomenon. The first equation is used for modeling long surface gravity wave of small amplitude and the later one describes the phase transition problems. Here, the obtained analytical solutions are exact and the solutions expressed by the rational functions depict the propagations of all kinds of traveling wave. The process of finding exact results is very easy and effective. Therefore, it can also be applied to other fractional partial differential equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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