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Blow-Up for a Periodic Two-Component Camassa-Holm Equation with Generalized Weakly Dissipation

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Abstract

In this paper we study a periodic two-component Camassa-Holm equation with generalized weakly dissipation. The local well-posedness of Cauchy problem is investigated by utilizing Kato's theorem. The blow-up criteria and the blow-up rate are established by applying monotonicity. Finally, the global existence results for solutions to the Cauchy problem of equation are proved by structuring functions.

Keywords

Periodic Two-Component Camassa-Holm Equation, Local Well-Posedness, Blow-Up, Global Existence, Monotonicity

1. Introduction

In this paper, we consider the Cauchy problem of periodic two-component Camassa-Holm equation with a generalized weakly dissipation:

$$\begin{cases} u_t - u_{xxt} + ku_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \lambda(u - u_{xx}) + \sigma\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x); \rho(0,x) = \rho_0(x), & x \in R, \\ u(t,x) = u(t,x+1); \rho(t,x) = \rho(t,x+1), & t \geq 0, x \in R, \end{cases} \tag{1.1}$$

where $\lambda \geq 0$ and k is a fixed constant; σ is a free parameter.

It is well known that the two-component integrable Camassa-Holm equation is

$$\begin{cases} u_t - u_{xxt} + ku_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} - \rho\rho_x = 0, & t > 0, x \in R \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in R \end{cases} \tag{1.2}$$

which is a model for wave motion on shallow water, where $u(t,x)$ standing for the fluid velocity at time $t \ge 0$ in the spatial x direction [1], $\rho(t,x)$ is in connection with the horizontal deviation of the surface from equilibrium (*i.e.* amplitude). Equation (1.2) possesses a bi-Hamiltonian structure [2] and the solution interaction of peaked travelling waves and wave breaking [1] [2] [3]. It is completely integrable [3] and becomes the Camassa-Holm equation when $\rho = 0$.

Equation (1.2) was derived physically by Constantin and Ivanov [4] in the context of shallow water theory. As soon as this equation was put forward, it attracted attention of a large number of researchers. Escher *et al.* [5] established the local well-posedness and present the blow-up scenarios and several blow-up results of strong solutions to Equation (1.2). Constantin and Ivanov [6] investigated the global existence and blow-up phenomena of strong solutions of Equation (1.2). Guan and Yin [7] obtained a new global existence result for strong solutions to Equation (1.2) and several blow-up results, which improved the results in [6]. Gui and Liu [8] established the local well-posedness for Equation (1.2) in a range of the Besov spaces, they also characterized a wave breaking mechanism for strong solutions. Hu and Yin [9] [10] studied the blow-up phenomena and the global existence of Equation (1.2).

Dissipation is an inevitable phenomenon in real physical word. It is necessary to study periodic two-Camassa-Holm equation with a generalized weakly dissipation. Hu and Yin [11] study the blow-up of solutions to a weakly dissipative periodic rod equation. Hu considered global existence and blow-up phenomena for a weakly dissipative two-component Camassa-Holm system [12] [13]. The purpose of this paper is to study the blow-up phenomenon of the solutions of Equation (1.1). The results show that the behavior of solutions to the periodic two-component Camassa-Holm equation with a generalized weakly dissipation is similar to Equation (1.2) and the blow-up rate of Equation (1.1) is not affected by the dissipative term when $\sigma > 0$.

The paper is organized as follows. Section 2 gives the local well-posedness of the Cauchy problem associated with Equation (1.1). The blow-up criteria for solutions and two conditions for wave breaking in finite time are given in Section 3. Furthermore, we also learn the blow-up rate of solutions. In Section 4, we address the global existence of Equation (1.1).

2. Local Well-Posedness

Let us introduce some notations, the $S = R/Z$ is the circle of unit length, the $[x]$ stands for the integer part of $x \in R$, the $*$ stands for the convolution, the $\|\cdot\|_X$ is used to represent the norm of Banach space *X*.

In this section, we investigate the local well-posedness for the Cauchy problem of Equation (1.1) by applying Kato's theory [14] in $H^s(S) \times H^{s-1}(S)$, $s \ge 2$.

For convenience we recall the Kato's theorem in the suitable form for our purpose. Consider the following abstract quasilinear evolution equation:

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t} + A(z)z = f(z), t \ge 0 \\ z(0) = z_0 \end{cases} \tag{2.1}$$

There are two Hilbert's spaces X and Y, Y is continuously and densely embedded in X and $Q: Y \to X$ is a topological isomorphism, the $L(Y, X)$ stands for the space of all bounded linear operator from Y to X.

Theorem 2.1 [14] 1) $A(y) \in L(Y, X)$, for $\forall y \in X$ with

$$\left\|\left(A(y) - A(z)\right)w\right\|_X \le \mu_1 \|y - z\|_X \|w\|_Y \tag{2.2}$$

where $z, y, w \in Y$, $A(y) \in G(X, 1, \beta)$, *i.e.* $A(y)$ is quasi-m-accretive, uniformly on bounded sets in Y.

2) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is uniformly bounded on a bounded sets in Y

$$\left\|\left(B(y) - B(z)\right)w\right\|_X \le \mu_2 \|y - z\|_Y \|w\|_X \tag{2.3}$$

where $z, y \in Y$, $w \in X$.

3) $f: Y \to Y$ is a bounded map on bounded sets in Y

$$\left\| f(y) - f(z) \right\|_Y \le \mu_3 \|y - z\|_Y \tag{2.4}$$

$$\left\| f(y) - f(z) \right\|_X \le \mu_4 \|y - z\|_X \tag{2.5}$$

where $z, y \in Y$, $\mu_1, \mu_2, \mu_3, \mu_4$ are constants which only depending $\left\{\|y\|_Y, \|z\|_Y\right\}$.

If the 1), 2), 3) hold, given $u_0 \in Y$, there is a maximal $T > 0$ depending only on $\|u_0\|_Y$ and a unique solution u of Equation (2.1) such that

$$u = u(\cdot, u_0) \in C([0, T); Y) \cap C^1([0, T); X) \tag{2.6}$$

Moreover, the map $u \to u(\cdot, u_0)$ is continuous from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

Note that $g(x) := \dfrac{\cosh\left(x - [x] - \frac{1}{2}\right)}{2\sinh\frac{1}{2}}$, $x \in R$, $\left(1 - \partial_x^2\right)^{-1} f = g * f$ for all $f \in L^2(S)$ and $g * (u - u_{xx}) = u$. Then Equation (1.1) can be rewritten as

$$\begin{cases} u_t + uu_x = -\partial_x g * \left(u^2 + \frac{1}{2}u_x^2 + ku + \frac{\sigma}{2}\rho^2\right) - \lambda u \\ \rho_t + (\rho u)_x = 0 \\ u(0, x) = u_0(x); \rho(0, x) = \rho_0(x) \\ u(t, x) = u(t, x + 1); \rho(t, x) = \rho(t, x + 1) \end{cases} \tag{2.7}$$

Theorem 2.2 Let $z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s \ge 2$, there exists a maximal time $T > 0$ which is independent on s and exists a unique solution (u, ρ) of Equation (1.1) in the interval $[0, T)$ with initial data z_0, such that the solution depends continuously on the initial data.

The remainder of this section is devoted the proof of Theorem 2.2. Let

$z=\begin{pmatrix} u \\ \rho \end{pmatrix}$, $T=H^s\times H^s$, $X=H^{s-1}\times H^{s-1}$, $\wedge=\left(1-\partial_x^2\right)^{\frac{1}{2}}$, $Q=\begin{pmatrix} \wedge & 0 \\ 0 & \wedge \end{pmatrix}$, and

$$A(z)=\begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix} \tag{2.8}$$

The [15] shows that Q is an isomorphism from $H^s\times H^s$ onto $H^{s-1}\times H^{s-1}$. It is sufficiently to verify $A(z)$, $B(z)$, $f(z)$ satisfy 1), 2), 3) to prove the theorem 2.2. For this purpose, the following lemmas are necessary.

Lemma 2.1 [15] The operator $A(z)$ is defined in (2.8) with $z\in H^s\times H^s$, $s>\frac{3}{2}$ belongs to $G\left(L^2\times L^2,1,\beta\right)$.

Lemma 2.2 [15] The operator $A(z)$ is defined in (2.8) with $z\in H^s\times H^s$, $s>\frac{3}{2}$ belongs to $G\left(H^{s-1}\times H^{s-1},1,\beta\right)$.

Lemma 2.3 [15] The operator $A(z)$ is defined in (2.8) with $z\in H^s\times H^s$, $s>\frac{3}{2}$ belongs to $L\left(H^s\times H^s,H^{s-1}\times H^{s-1}\right)$, moreover,

$$\left\|\left(A(y)-A(z)\right)w\right\|_{H^{s-1}\times H^{s-1}}\le\mu_1\|y-z\|_{H^s\times H^s}\|w\|_{H^s\times H^s} \tag{2.9}$$

where $y,z,w\in H^s\times H^s$.

Lemma 2.4 [15] Let $B(z)=QA(z)Q^{-1}-A(z)$ with $z\in H^s\times H^s$, $s>\frac{3}{2}$, then the operator $B(z)\in L\left(H^{s-1}\times H^{s-1}\right)$ and

$$\left\|\left(B(y)-B(z)\right)w\right\|_{H^{s-1}\times H^{s-1}}\le\mu_2\|y-z\|_{H^s\times H^s}\|w\|_{H^{s-1}\times H^{s-1}} \tag{2.10}$$

for $y,z\in H^s\times H^s$, and $w\in H^{s-1}\times H^{s-1}$.

Lemma 2.5 Let $z\in H^s\times H^s$, $s>\frac{3}{2}$, and

$$f(z)=-\begin{pmatrix} \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2+\frac{1}{2}u_x^2+ku+\frac{\sigma}{2}\rho^2\right)+\lambda u \\ \rho u_x \end{pmatrix}$$

Then f is bounded on bounded sets in $H^s\times H^s$ and satisfies

$$1)\ \left\|f(y)-f(z)\right\|_{H^s\times H^s}\le\mu_3\|y-z\|_{H^s\times H^s},\quad y,z\in H^s\times H^s \tag{2.11}$$

$$2)\ \left\|f(y)-f(z)\right\|_{H^{s-1}\times H^{s-1}}\le\mu_4\|y-z\|_{H^{s-1}\times H^{s-1}},\quad y,z\in H^s\times H^s \tag{2.12}$$

Proof: For any $z,y\in H^s\times H^s$, $s>\frac{3}{2}$,

$$\begin{aligned}&\left\|f(y)-f(z)\right\|_{H^s\times H^s}\\ &\le\left\|-\partial_x\left(1-\partial_x^2\right)^{-1}\left[\left(y_1^2-u^2\right)+\frac{1}{2}\left(y_{1x}^2-u_x^2\right)+k\left(y_1-u\right)+\frac{\sigma}{2}\left(y_2^2-\rho^2\right)\right]\right\|_{H^s}\\ &\quad+\left\|\lambda\left(y_1-u\right)\right\|_{H^s}+\left\|u_x\rho-y_{1x}y_2\right\|_{H^s}\end{aligned}$$

$$\leq \left\| \left(y_1^2 - u^2\right) + \frac{1}{2}\left(y_{1x}^2 - u_x^2\right) + k\left(y_1 - u\right) \right\|_{H^{s-1}} + \frac{|\sigma|}{2}\left\| y_2^2 - \rho^2 \right\|_{H^{s-1}}$$

$$+ |\lambda| \left\| y_1 - u \right\|_{H^s} + \left\| \left(u_x - y_{1x}\right)\rho \right\|_{H^s} + \left\| y_{1x}\left(\rho - y_2\right) \right\|_{H^s}$$

$$\leq \left\| \left(y_1 - u\right)\left(y_1 + u\right) \right\|_{H^{s-1}} + \frac{1}{2}\left\| \left(y_{1x} - u_x\right)\left(y_{1x} + u_x\right) \right\|_{H^{s-1}} + |k| \left\| y_1 - u \right\|_{H^{s-1}}$$

$$+ |\lambda| \left\| y_1 - u \right\|_{H^s} + \frac{|\sigma|}{2}\left\| y_2 - \rho \right\|_{H^{s-1}} \left\| y_2 + \rho \right\|_{H^{s-1}}$$

$$+ \left\| y_1 - u \right\|_{H^s} \left\| \rho \right\|_{H^s} + \left\| y_1 \right\|_{H^s} \left\| \rho - y_2 \right\|_{H^s}$$

$$\leq \left\| \left(y_1 - u\right)\left(y_1 + u\right) \right\|_{H^{s-1}} + \frac{1}{2}\left\| \left(y_{1x} - u_x\right)\left(y_{1x} + u_x\right) \right\|_{H^{s-1}} + |k| \left\| y_1 - u \right\|_{H^{s-1}}$$

$$+ |\lambda| \left\| y_1 - u \right\|_{H^s} + \frac{|\sigma|}{2}\left\| y_2 - \rho \right\|_{H^{s-1}} \left\| y_2 + \rho \right\|_{H^{s-1}}$$

$$+ \left\| y_1 - u \right\|_{H^s} \left\| \rho \right\|_{H^s} + \left\| y_1 \right\|_{H^s} \left\| \rho - y_2 \right\|_{H^s}$$

Let $\mu_3 = \frac{5 + |\sigma|}{2}\|y\|_{H^s \times H^s} + \frac{3 + |\sigma|}{2}\|z\|_{H^s \times H^s} + |k| + |\lambda|$, then

$$\left\| f(y) - f(z) \right\|_{H^s \times H^s} \leq \mu_3 \left\| y - z \right\|_{H^s \times H^s}, \quad y, z \in H^s \times H^s$$

Making $y = 0$ in the above inequality, it shows that f is bounded on bounded sets in $H^s \times H^s$, the proof of 1) is complete.

Similarly, the inequality (2.12) also can be proved.

Proof of Theorem 2.2: The 1) is true for $A(z)$ from the inequality (2.9), the 2) is true for $B(z)$ from the inequality (2.10), the 3) is true for $f(z)$ from the inequalities (2.11) (2.12). According to the Theorem 2.1, the proof of the Theorem 2.2 is complete.

3. Blow-Up

This section will establish a blow-up criterion for solution of Equation (1.1) when $\sigma > 0$.

Theorem 3.1 [8] [16] Let $\sigma \neq 0$ and (u, ρ) be the solution of (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$, T is the maximal time of existence of the solution, then

$$T < \infty \Rightarrow \int_0^T \left\| u_x(\tau) \right\|_{L^\infty} \mathrm{d}\tau = \infty \tag{3.1}$$

Consider the following equation of trajectory:

$$\begin{cases} \dfrac{\mathrm{d}q(t,x)}{\mathrm{d}t} = u\left(t, q(t,x)\right), & t \in [0,T) \\ q(0,x) = x, & x \in S \end{cases} \tag{3.2}$$

The (3.2) shows $q(t,\cdot): S \to S$ is the differential homeomorphism for every $t \in [0,T)$

$$q_x(t,x) = \mathrm{e}^{\int_0^t u_x(\tau, q(\tau,x))\mathrm{d}\tau} > 0, \quad \forall (t,x) \in [0,T) \times S \tag{3.3}$$

Hence

$$\left\| v(t,\cdot) \right\|_{L^\infty} = \left\| v(t, q(t,\cdot)) \right\|_{L^\infty} \tag{3.4}$$

Lemma 3.1 [17] Let $T > 0$ and $v \in C^1\left([0,T); H^1(R)\right)$, then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in R$ with

$$m(t) := \inf_{x \in R} \left[v_x(t,x) \right] = v_x\left(t, \xi(t)\right)$$

The function $m(t)$ is absolutely continuous in $(0,T)$ with

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = v_{tx}\left(t,\xi(t)\right) \text{ a.e. in } (0,T).$$

Lemma 3.2 Let $z_0 = (u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$ with $s \geq 2$, there exist a maximal time $T > 0$ and a unique solution (u, ρ) of Equation (1.1) with initial data z_0, then we have

$$\|u\|_{H^1}^2 + \sigma \|\rho - 1\|_{L^2}^2 \leq \|u_0\|_{H^1}^2 + \sigma \|\rho_0 - 1\|_{L^2}^2 \tag{3.5}$$

Proof: Multiply the first equation of Equation (1.1) by u and integrate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \left(u^2 + u_x^2\right)\mathrm{d}x + 2\lambda \int_S \left(u^2 + u_x^2\right)\mathrm{d}x + 2\sigma \int_S \rho\rho_x u \mathrm{d}x = 0 \tag{3.6}$$

The second equation of Equation (1.1) can be rewritten as

$$(\rho - 1)_t + \rho_x u + \rho u_x = 0$$

Multiply the above equation by $(\rho - 1)$ and integrate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S (\rho-1)^2 \mathrm{d}x + 2\int_S u\rho\rho_x \mathrm{d}x - 2\int_S u\rho_x \mathrm{d}x + 2\int_S u_x \rho^2 \mathrm{d}x - 2\int_S u_x \rho \mathrm{d}x = 0 \tag{3.7}$$

According to (3.6) and (3.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \left(u^2 + u_x^2 + \sigma(\rho - 1)^2 + 2\lambda \int_0^t \left(u^2 + u_x^2\right)\mathrm{d}\tau\right)\mathrm{d}x = 0$$

Then

$$\begin{aligned}
&\int_S \left(u^2 + u_x^2 + \sigma(\rho-1)^2 + 2\lambda \int_0^t \left(u^2 + u_x^2\right)\mathrm{d}\tau\right)\mathrm{d}x \\
&= \int_S \left(u_0^2 + u_{0x}^2 + \sigma(\rho_0 - 1)^2\right)\mathrm{d}x = \|u_0\|_{H^1}^2 + \sigma\|\rho_0 - 1\|_{L^2}^2
\end{aligned}$$

Notice that $2\lambda \int_0^t \left(u^2 + u_x^2\right)\mathrm{d}x \geq 0$, then

$$\begin{aligned}
\|u\|_{H^1}^2 + \sigma\|\rho - 1\|_{L^2}^2 &= \int_S \left(u^2 + u_x^2 + \sigma(\rho-1)^2\right)\mathrm{d}x \\
&\leq \|u_0\|_{H^1}^2 + \sigma\|\rho_0 - 1\|_{L^2}^2
\end{aligned}$$

Lemma 3.3 [18] [19] 1) For every $f \in H^1(S)$, we have

$$\max_{x \in [0,1]} f^2(x) \leq \frac{e+1}{2(e-1)} \|f\|_{H^1}^2 \tag{3.8}$$

where the constant $\frac{e+1}{2(e-1)}$ is the best constant.

2) For every $f \in H^3(S)$, we have

$$\max_{x \in [0,1]} f^2(x) \leq c\|f\|_{H^1}^2 \tag{3.9}$$

where the best constant c is $\frac{e+1}{2(e-1)}$.

3) For every $f \in H^3(S)$, we have

$$\max_{x\in[0,1]} f_x^2(x) \le \frac{1}{12}\|f\|_{H^2}^2 \tag{3.10}$$

Lemma 3.4 Suppose $\sigma > 0$, and (u,ρ) be the solution of Equation (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s \ge 2$, and T be the maximal time of existence, then

$$\sup_{x\in S} u_x(t,x) \le \|u_{0x}\|_{L^\infty} + \sqrt{\lambda^2 + \sigma\|\rho_0\|_{L^\infty}^2 + C_1^2}$$

where $C_1 = \sqrt{\frac{(3\sigma+2)(e+1)}{2(e-1)} + \left(\frac{e+1}{e-1} + \frac{k^2+1}{2}\right)\left(\|u_0\|_{H^1}^2 + \sigma\|\rho_0 - 1\|_{L^2}^2\right)}$.

Proof: The theorem 2.2 and a density argument imply that it is sufficient to prove the desired estimates for $s = 3$.

Differentiate the first equation of Equation (2.7) with respect to x

$$u_{tx} = u^2 - \frac{1}{2}u_x^2 - \lambda u_x + \frac{\sigma}{2}\rho^2 - k\partial_x^2 g * u - g * \left(u^2 + \frac{1}{2}u_x^2 + \frac{\sigma}{2}\rho^2\right) - uu_{xx} \tag{3.11}$$

Define

$$\bar{m}(t) = u_x(t,\eta(t)) = \sup_{x\in S}(u_x(t,x)), m(t) = \inf_{x\in S}(u_x(t,x)) \tag{3.12}$$

From the Fermat's lemma, we know

$$u_{xx}(t,\eta(t)) = 0,\ \ a.e.\ t \in [0,T)$$

there exists $x_1(t) \in S$ such that

$$q(t,x_1(t)) = \eta(t), t \in [0,T) \tag{3.13}$$

Set

$$\bar{\zeta}(t) = \rho(t, q(t,x_1)),\ \ t \in [0,T) \tag{3.14}$$

From (3.11) and the second equation of Equation (1.1), we obtain

$$\begin{cases} \bar{m}'(t) = -\frac{1}{2}\bar{m}^2(t) - \lambda\bar{m}(t) + \frac{\sigma}{2}\bar{\zeta}^2(t) + f(t,q(t,x_1)) \\ \bar{\zeta}'(t) = -\bar{\zeta}(t)\bar{m}(t) \end{cases} \tag{3.15}$$

where $f = u^2 - k\partial_x^2 g * u - g * \left(u^2 + \frac{1}{2}u_x^2 + \frac{\sigma}{2}\rho^2\right)$.

Notice that $\partial_x^2 g * u = \partial_x g * \partial_x u$, then

$$\begin{aligned} f &= u^2 - k\partial_x^2 g * u - g * \left(u^2 + \frac{1}{2}u_x^2\right) - \frac{\sigma}{2} g * (\rho^2) \\ &= u^2 - k\partial_x^2 g * u - g * \left(u^2 + \frac{1}{2}u_x^2\right) - \frac{\sigma}{2} g * 1 - \sigma g * (\rho - 1) - \frac{\sigma}{2} g * (\rho-1)^2 \\ &\le u^2 + k|\partial_x g * \partial_x u| + \frac{\sigma}{2}|g * 1| + \sigma|g * (\rho - 1)| \end{aligned}$$

From (3.8) (3.9) and (3.10), we have

$$u^2 \le \frac{e+1}{2(e-1)}\|u\|_{H^1}^2$$

$$k\left|\partial_x g * \partial_x u\right| \le k\|g_x\|_{L^2}\|u_x\|_{L^2} \le \frac{e+1}{2(e-1)} + \frac{1}{4}k^2\|u_x\|_{L^2}^2$$

$$\left|g*\left(u^2+\frac{1}{2}u_x^2\right)\right| \le \frac{e+1}{2(e-1)}\|u\|_{L^2}^2 + \frac{e+1}{4(e-1)}\|u_x\|_{L^2}^2$$

$$\frac{\sigma}{2}|g*1| \le \frac{\sigma}{2}\|g\|_{L^\infty} \le \frac{\sigma(e+1)}{4(e-1)}$$

$$\sigma\left|g*(\rho-1)\right| \le \sigma\|g\|_{L^2}\|\rho-1\|_{L^1} \le \frac{\sigma(e+1)}{2(e-1)} + \frac{\sigma}{4}\|\rho-1\|_{L^2}^2$$

$$\frac{\sigma}{2}\left|g*(\rho-1)^2\right| \le \frac{\sigma}{2}\|g\|_{L^\infty}\|(\rho-1)\|_{L^1} \le \frac{\sigma(e+1)}{4(e-1)}\|\rho-1\|_{L^2}^2$$

Therefore we get the upper bound of f

$$\begin{aligned} f &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{e+1}{2(e-1)} + \frac{k^2}{4}\right)\|u\|_{H^1}^2 + \frac{1}{4}\sigma\|\rho-1\|_{L^2}^2 \\ &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{e+1}{2(e-1)} + \frac{k^2+1}{4}\right)\left(\|u\|_{H^1}^2 + \sigma\|\rho-1\|_{L^2}^2\right) \\ &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{e+1}{2(e-1)} + \frac{k^2+1}{4}\right)\left(\|u_0\|_{H^1}^2 + \sigma\|\rho_0-1\|_{L^2}^2\right) \\ &= \frac{1}{2}C_1^2 \end{aligned} \tag{3.16}$$

Similarly, we turn to the lower bound of f

$$\begin{aligned} -f &\le u^2 + k\left|\partial_x g * \partial_x u\right| + \left|g*\left(u^2+\frac{1}{2}u_x^2\right)\right| + \frac{\sigma}{2}|g*1| \\ &\quad + \sigma\left|g*(\rho-1)\right| + \frac{\sigma}{2}\left|g*(\rho-1)^2\right| \\ &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \frac{e+1}{e-1}\|u\|_{L^2}^2 + \left(\frac{3(e+1)}{4(e-1)} + \frac{k^2}{4}\right)\|u_x\|_{L^2}^2 \\ &\quad + \left(\frac{e+1}{4(e-1)} + \frac{1}{4}\right)\sigma\|\rho-1\|_{L^2}^2 \\ &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{7(e+1)}{4(e-1)} + \frac{k^2+1}{4}\right)\left(\|u\|_{H^1}^2 + \sigma\|\rho-1\|_{L^2}^2\right) \\ &\le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{7(e+1)}{4(e-1)} + \frac{k^2+1}{4}\right)\left(\|u_0\|_{H^1}^2 + \sigma\|\rho_0-1\|_{L^2}^2\right) \end{aligned} \tag{3.17}$$

According to (3.16) and (3.17)

$$|f| \le \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{7(e+1)}{4(e-1)} + \frac{k^2+1}{4}\right)\left(\|u_0\|_{H^1}^2 + \sigma\|\rho_0-1\|_{L^2}^2\right) \tag{3.18}$$

From Sobolev's embedding theorem, we have $u \in C_0^1(S)$, due to the periodic of Equation (1.1), then

$$\inf_{x\in S} u_x(t,x) \le 0, \sup_{x\in S} u_x(t,x) \ge 0, t \in [0,T) \quad (3.19)$$

hence

$$\bar{m}(t) \ge 0, t \in [0,T) \quad (3.20)$$

From the second Equation of (3.15), we have

$$\bar{\zeta}(t) = \bar{\zeta}(0) e^{-\int_0^t \bar{m}(\tau)\mathrm{d}\tau}$$

then

$$\left|\rho(t, q(t, x_1))\right| = \left|\bar{\zeta}(t)\right| \le \left|\bar{\zeta}(0)\right| \le \|\rho_0\|_{L^\infty}$$

For any given $x \in S$, define

$$P_1(t) = \bar{m}(t) - \|u_{0x}\|_{L^\infty} - \sqrt{\lambda^2 + \sigma\|\rho_0\|_{L^\infty}^2 + C_1^2} \quad (3.21)$$

then $P_1(t)$ is C^1-function in $[0,T)$ and satisfies

$$\begin{aligned} P_1(0) &= \bar{m}(0) - \|u_{0x}\|_{L^\infty} - \sqrt{\lambda^2 + \sigma\|\rho_0\|_{L^\infty}^2 + C_1^2} \\ &\le \bar{m}(0) - \|u_{0x}\|_{L^\infty} \le 0 \end{aligned}$$

Next, we will show $P_1(t) \le 0, t \in [0,T)$.

By contradictory arguement, there exists $t_0 \in [0,T)$ such that $P_1(t_0) > 0$. Making $t_1 = \max\{t < t_0 : P_1(t) = 0\}$, we have

$$P_1(t_1) = 0, P_1'(t_1) \ge 0$$

then

$$\bar{m}(t_1) = \|u_{0x}\|_{L^\infty} + \sqrt{\lambda^2 + \sigma\|\rho_0\|_{L^\infty}^2 + C_1^2}.$$

From (3.21), we know

$$\bar{m}'(t_1) = P_1'(t_1) \ge 0 \quad (3.22)$$

On the other hand, from the first Equation of (3.15), we have

$$\begin{aligned} \bar{m}'(t_1) &= -\frac{1}{2}\bar{m}^2(t_1) - \lambda\bar{m}(t_1) + \frac{\sigma}{2}\bar{\zeta}^2(t_1) + f(t_1, q(t_1, x_1)) \\ &\le -\frac{1}{2}(m(t_1) + \lambda)^2 + \frac{1}{2}\lambda^2 + \frac{\sigma}{2}\|\rho_0\|_{L^\infty}^2 + \frac{1}{2}C_1^2 \\ &\le -\frac{1}{2}\left(\|u_{0x}\|_{L^\infty} + \sqrt{\lambda^2 + \sigma\|\rho_0\|_{L^\infty}^2 + C_1^2} + \lambda\right)^2 + \frac{\sigma}{2}\|\rho_0\|_{L^\infty}^2 + \frac{1}{2}C_1^2 \\ &< 0 \end{aligned}$$

It yields a contradiction, then the proof of the Lemma 3.4 is complete.

Lemma 3.5 Suppose $\sigma > 0$, and (u, ρ) be the solution of Equation (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s \ge 2$, and T is the maximal time of the solution. If there exists $M \ge 0$ such that

$$\inf_{(t,x)\in[0,T)\times S} u_x \ge -M \quad (3.23)$$

then

$$\|\rho(t,\cdot)\|_{L^\infty(S)} \le \|\rho_0\|_{L^\infty(S)} e^{Mt} \tag{3.24}$$

Proof: For any given $x \in S$, define

$$U(t) = u_x(t, q(t, x_1)), \gamma(t) = \rho(t, q(t, x))$$

the second equation of Equation (1.1) becomes

$$\gamma'(t) = -\gamma U$$

then

$$\gamma(t) = \gamma(0) e^{-\int_0^t U(\tau)d\tau}.$$

From (3.23), we know $U(t) \ge -M, t \in [0, T)$. Hence

$$|\rho(t, q(t,x))| = |\gamma(t)| \le |\gamma(0)| e^{-\int_0^t U(\tau)d\tau} \le |\gamma(0)| e^{Mt} \le \|\rho_0\|_{L^\infty} e^{Mt}$$

which together with (3.4), then the proof of lemma 3.5 is complete.

Theorem 3.2 Suppose $\sigma > 0$, and (u, ρ) be the solution of Equation (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s \ge 2$, and T is the maximal time of existence of the solution, then the solution of Equation (1.1) blows up in finite time if and only if

$$\liminf_{t \to T^-} \inf_{x \in S} u_x(t, x) = -\infty \tag{3.25}$$

Proof: Suppose that $T < \infty$ and (3.25) is invalid, then there exists $M > 0$ satisfies

$$u_x(t, x) \ge -M, \forall (t, x) \in [0, T) \times S$$

The Lemma 3.4 shows that $u_x(t, x)$ is bounded on $[0, T)$, *i.e.* $|u_x(t, x)| \le C$, where $C = C\left(k, M, \sigma, \lambda, \|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}\right)$. Then from the Theorem 3.1, we have $T = \infty$, which contradicts the assumption $T < \infty$.

On the other hand, Sobolev embedding theorem $H^s \mapsto L^\infty$ with $s > \frac{1}{2}$ implies that if (3.25) holds, then the corresponding solution blows up in finite time, the proof of Theorem 3.2 is complete.

Next we give two blow-up conditions in finite time.

Theorem 3.3 Suppose $\sigma > 0$, and (u, ρ) be the solution of Equation (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s \ge 2$, and T is the maximal time of existence of the solution. If there exists $x_0 \in S$ satisfies

$$\rho_0(x_0) = 0, u_{0x}(x_0) = \inf_{x \in R} u_{0x}(x) \tag{3.26}$$

and

$$\|u_0\|_{H^1}^2 + \sigma \|\rho_0 - 1\|_{L^2}^2 \le \left(\frac{(8\sigma - 1)(e+1)}{36(e-1)} - \frac{1}{2}\lambda^2 \right) \frac{4(e+1)}{3(e+1) + 18(k^2+1)(e-1)} \tag{3.27}$$

then the corresponding solution u of Equation (1.1) blows up in finite time when $0 < T < T^*$, where

$$T^* = \frac{2\left(1+\left|u_{0x}\left(x_0\right)\right|\right)}{\frac{(8\sigma-1)(e+1)}{18(e-1)}+\left(\frac{3(e+1)+18\left(k^2+1\right)(e-1)}{2(e-1)}\right)\left(\left\|u_0\right\|_{H^1}^2+\sigma\left\|\rho_0-1\right\|_{L^2}^2)-\lambda^2\right)} + \frac{2}{1-\lambda}$$

Proof: Without loss of generality, assume $s=3$, and choose $x_2(t)$ such that $q\left(t,x_2(t)\right)=\xi(t)$, $t\in[0,t)$, along the trajectory $q\left(t,x_2\right)$, we rewrite the transport Equation of ρ in (2.7) as

$$\frac{\mathrm{d}\rho\left(t,\xi(t)\right)}{\mathrm{d}t}=-\rho\left(t,\xi(t)\right)u_x\left(t,\xi(t)\right) \tag{3.28}$$

From (3.26), we have

$$m(0)=u_x\left(0,\xi(0)\right)=\inf_{x\in S}u_{0x}(x)=u_{0x}\left(x_0\right)$$

Let $x_0=\xi(0)$, then $\rho_0\left(\xi(0)\right)=\rho_0\left(x_0\right)$, from (3.28)

$$\rho\left(t,\xi(t)\right)=0,\forall t\in[0,T) \tag{3.29}$$

From (3.11), (3.29) and $u_{xx}\left(t,\xi(t)\right)=0$, we obtain

$$\begin{aligned}\overline{m}'(t)&=-\frac{1}{2}m^2(t)-\lambda m(t)+u^2\left(t,\xi(t)\right)-k\partial_x g*\partial_x u\\&\quad-g*\left(u^2+\frac{1}{2}u_x^2+\frac{\sigma}{2}\rho^2\right)\left(t,\xi(t)\right)\\&=-\frac{1}{2}m^2(t)-\lambda m(t)+f^*\left(t,q\left(t,x_2\right)\right)\\&=-\frac{1}{2}\left(m(t)+\lambda\right)^2+\frac{1}{2}\lambda^2+f^*\left(t,q\left(t,x_2\right)\right)\end{aligned} \tag{3.30}$$

where

$$f^*=u^2\left(t,\xi(t)\right)-k\partial_x g*\partial_x u-g*\left(u^2+\frac{1}{2}u_x^2+\frac{\sigma}{2}\rho^2\right)\left(t,\xi(t)\right) \tag{3.31}$$

Modify the estimates:

$$k\left|\partial_x g*\partial_x u\right|\le k\left\|g_x\right\|_{L^2}\left\|u_x\right\|_{L^2}\le\frac{e+1}{36(e-1)}+\frac{9}{2}k^2\left\|u_x\right\|_{L^2}^2$$

$$\sigma\left|g*(\rho-1)\right|\le\sigma\left\|g\right\|_{L^2}\left\|\rho-1\right\|_{L^2}\le\left(\frac{e+1}{36(e-1)}+\frac{9}{2}\right)\sigma\left\|\rho-1\right\|_{L^2}^2$$

The similar process to (3.16) leads to

$$f^*\le\frac{(1-8\sigma)(e+1)}{36(e-1)}+\frac{3(e+1)+18\left(1+k^2\right)(e-1)}{4(e-1)}\left(\left\|u_0\right\|_{H^1}^2+\sigma\left\|\rho-1\right\|_{L^2}^2\right)=-C_2$$

From the above inequality and (3.27), we have $\frac{1}{2}\lambda^2-C_2<0$, then

$$m'(t)\le-\frac{1}{2}\left(m(t)+\lambda\right)^2+\frac{1}{2}\lambda^2-C_2\le\frac{1}{2}\lambda^2-C_2<0,t\in[0,T) \tag{3.32}$$

So $m(t)$ is strictly decreasing in $[0,T)$.

If there exist global solutions, we will show that this leads to a contradiction. Let

$$t_1=\frac{2\left(1+\left|u_{0x}(x_0)\right|\right)}{2C_2-\lambda^2}$$

integrating (3.32) over $[0,t_1]$ yields

$$m(t_1)=m(0)+\int_0^{t_1}m'(t)\mathrm{d}t\le\left|u_{0x}(x_0)\right|+\left(\frac{1}{2}\lambda^2-C_2\right)t_1=-1 \tag{3.33}$$

Hence we know $m(t)\le m(t_1)\le -1, t\in[t_1,T)$.

From (3.32), we have

$$m'(t)\le-\frac{1}{2}\left(m(t)+\lambda\right)^2 \tag{3.34}$$

Integrating (3.34) over $[t_1,T)$ and knowing $m(t_1)\le-1$, we get

$$-\frac{1}{m(t)+\lambda}+\frac{1}{\lambda-1}\le-\frac{1}{m(t)+\lambda}+\frac{1}{\lambda+m(t_1)}\le-\frac{1}{2}(t-t_1), t\in[t_1,T)$$

then

$$m(t)\le\frac{1}{\frac{1}{2}(t-t_1)+\frac{1}{\lambda-1}}-\lambda\to-\infty, \text{ as } t\to t_1+\frac{2}{1-\lambda}$$

Thus $T\le t_1+\frac{2}{1-\lambda}$ is a contradiction with $T=\infty$.

The proof of the Theorem 3.3 is complete.

Theorem 3.4 Let $\sigma>0$, and (u,ρ) be the solution of Equation (1.1) with initial data $(u_0,\rho_0-1)\in H^s(S)\times H^{s-1}(S)$, $s\ge 2$, and T is the maximal time of existence of the solution. If there exists $x_0\in S$ satisfies

$$\rho_0(x_0)=0, u_{0x}(x_0)=\inf_{x\in R}u_{0x}(x) \tag{3.35}$$

and

$$u_{0x}(x_0)<-\sqrt{\lambda^2+C_1^2}-\lambda \tag{3.36}$$

then the corresponding solution u of Equation (1.1) blows up in finite time when $0<T<T^{**}$,where

$$T^{**}=-\frac{2\left(\lambda+u_{0x}(x_0)\right)}{\left(\lambda+u_{0x}(x_0)\right)^2-\left(\lambda^2+C_1^2\right)}$$

Proof: From (3.16), we have

$$m'(t)\le-\frac{1}{2}\left(m(t)+\lambda\right)^2+\frac{1}{2}\left(\lambda^2+C_1^2\right), t\in[0,T)$$

From (3.36), we have $m'(0)<0$, $m(t)$ is strictly decreasing on $[0,T)$ and set

$$\delta=\frac{1}{2}-\frac{1}{\left(\lambda+u_{0x}(x_0)\right)^2}\left(\frac{1}{2}\lambda^2+\frac{1}{2}C_1^2\right)\in\left(0,\frac{1}{2}\right).$$

Because $m(t) < m(0) = u_{0x}(x_0) < -\lambda$, then

$$m'(t) \le -\frac{1}{2}\left(m(t)+\lambda\right)^2 + \frac{1}{2}\left(\lambda^2 + C_1^2\right) \le -\delta\left(m(t)+\lambda\right)^2$$

Similar discussion of the Theorem 3.3

$$m(t) \le \frac{\lambda + u_{0x}(x_0)}{1+\delta t\left(\lambda + u_{0x}(x_0)\right)} - \lambda \to -\infty, \quad t \to -\frac{1}{\lambda\delta + \delta u_{0x}(x_0)}$$

Hence

$$0 < T < -\frac{2\left(\lambda + u_{0x}(x_0)\right)}{\left(\lambda + u_{0x}(x_0)\right)^2 - \left(\lambda^2 + C_1^2\right)}.$$

The proof of the theorem 3.4 is complete.

Next we will show the blow-up rate of solutions and the result shows: the blow-up rate is not affected by the weakly dissipation.

Theorem 3.5 (blow-up rate) Let $\sigma > 0$, and (u,ρ) be the solution of Equation (1.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s \ge 2$, and T is the maximal time of existence of the solution. If $T < \infty$, then

$$\lim_{t \to T^-}\left(\inf_{x \in S} u_x(t,x)(T-t)\right) = -2$$

Proof: Without loss of generality, assume $s = 3$.

Set

$$M = \frac{(3\sigma+2)(e+1)}{4(e-1)} + \left(\frac{7(e+1)}{4(e-1)} + \frac{1+k^2}{4}\right)\left(\|u_0\|_{H^1}^2 + \sigma\|\rho - 1\|_{L^2}^2\right) \tag{3.37}$$

From (3.30), we have

$$-\frac{1}{2}\left(m(t)+\lambda\right)^2 - \frac{1}{2}\lambda^2 - M \le m'(t) \le -\frac{1}{2}\left(m(t)+\lambda\right)^2 + \frac{1}{2}\lambda^2 + M \tag{3.38}$$

Because of $\lim_{t \to T^-}\left(m(t)+\lambda\right) = -\infty$, there exists $t_0 \in (0,T)$ satisfies $m(t_0)+\lambda < 0$ and $\left(m(t_0)+\lambda\right)^2 > \frac{1}{\varepsilon}\left(\lambda^2 + M\right), \varepsilon \in \left(0, \frac{1}{2}\right)$. Since m is locally Lipschitz, m is absolutely continuous. We deduce that m is decreasing in $[t_0, T)$ and

$$\left(m(t)+\lambda\right)^2 > \frac{1}{\varepsilon}\left(\lambda^2 + M\right) \tag{3.39}$$

According to (3.38) and (3.39)

$$\frac{1}{2} - \varepsilon \le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m(t)+\lambda}\right) \le \frac{1}{2} + \varepsilon, t \in (t_0, T)$$

Integrating (3.39) over (t,T) with respect to $t \in [t_0, T)$, notice that $\lim_{t \to T^-}\left(m(t)+\lambda\right) = -\infty$, then

$$\left(\frac{1}{2} - \varepsilon\right)(T-t) \le -\left(\frac{1}{m(t)+\lambda}\right) \le \left(\frac{1}{2} + \varepsilon\right)(T-t), t \in (t_0, T)$$

Since ε is arbitrary, so

$$\lim_{t\to T^-}\left\{m(t)(T-t)+\lambda(T-t)\right\}=-2$$

That is $\lim_{t\to T^-} m(t)(T-t)=-2$, the blow-up rate of solutions of Equation (1.1) is not effected by the weakly dissipation.

4. Global Existence

In this section, we provide a sufficient condition for the global solution of Equation (1.1) in the case $0<\sigma<2$.

Theorem 4.1 Let $0<\sigma<2$, $(u_0,\rho_0-1)\in H^s(S)\times H^{s-1}(S)$ with $s>\frac{3}{2}$, there exist a maximal time $T>0$ and a unique solution (u,ρ) of Equation (1.1) with initial data. Assume that $\inf_{x\in S}\rho_0(x)>0$, then

1) when $0<\sigma\le 1$,

$$\left|\inf_{x\in S}u_x(t,x)\right|\le\frac{1}{\inf_{x\in S}\rho_0(x)}C_4\mathrm{e}^{C_3t}$$

$$\left|\inf_{x\in S}u_x(t,x)\right|\le\frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_4^{\frac{1}{2-\sigma}}\mathrm{e}^{\frac{C_3t}{2-\sigma}}$$

2) when $1<\sigma<2$,

$$\left|\inf_{x\in S}u_x(t,x)\right|\le\frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_4^{\frac{1}{2-\sigma}}\mathrm{e}^{\frac{C_3t}{2-\sigma}}$$

$$\left|\inf_{x\in S}u_x(t,x)\right|\le\frac{1}{\inf_{x\in S}\rho_0(x)}C_4\mathrm{e}^{C_3t}$$

where

$$C_3=1+\frac{(3\sigma+2)(e+1)}{4(e-1)}+\left(\frac{7(e+1)}{4(e-1)}+\frac{1+k^2}{4}\right)\left(\|u_0\|_{H^1}^2+\sigma\|\rho-1\|_{L^2}^2\right)$$

$$C_4=1+\|u_{0x}\|_{L^\infty}^2+\|\rho_0\|_{L^\infty}^2$$

Proof: It is sufficient to prove the desired results for $s=3$.

1) We will estimate the $\left|\inf_{x\in S}u_x(t,x)\right|$.

From (3.22), we have

$$m(t)\le 0, t\in[0,T) \tag{4.1}$$

Let $\zeta(t)=\rho(t,\xi(t))$, thus we have

$$\begin{cases} m'(t)=-\frac{1}{2}m^2(t)-\lambda m(t)+\frac{\sigma}{2}\zeta^2(t)+f(t,q(t,x_2)) \\ \zeta'(t)=-\zeta(t)\overline{m}(t)\end{cases} \tag{4.2}$$

where *f* is defined as (3.15). The second Equation of (3.15) shows that $\zeta(t)$ and

$\zeta(0)$ have the same sign. Hence $\zeta(0)=\rho(0,\xi)(0)>0$.

Suppose $0<\sigma\le 1$, define the function

$$w_1(t)=\zeta(0)\zeta(t)+\frac{\zeta(0)}{\zeta(t)}\left(1+m^2(t)\right) \tag{4.3}$$

which is positive on $t\in[0,T)$.

Differentiate $w_1(t)$

$$\begin{aligned}
w_1'(t)&=\zeta(0)\zeta'(t)+\frac{\zeta(0)}{\zeta^2(t)}\left(1+m^2(t)\right)\zeta'(t)+\frac{2\zeta(0)}{\zeta(t)}m(t)m'(t)\\
&=-\zeta(0)\zeta(t)m(t)+\frac{\zeta(0)}{\zeta^2(t)}\left(1+m^2(t)\right)\zeta(t)m(t)\\
&\quad+\frac{2\zeta(0)}{\zeta(t)}m(t)\left(-\frac{1}{2}m^2(t)-\lambda m(t)+\frac{\sigma}{2}\zeta^2(t)+f\right)\\
&=(\sigma-1)\zeta(0)\zeta(t)m(t)+\frac{\zeta(0)}{\zeta(t)}m(t)-\frac{2\lambda\zeta(0)}{\zeta(t)}m^2(t)+\frac{2\zeta(0)}{\zeta(t)}m(t)f\\
&\le(\sigma-1)\zeta(0)\zeta(t)m(t)+\frac{\zeta(0)}{\zeta(t)}m(t)+\frac{2\zeta(0)}{\zeta(t)}m(t)f\\
&=\frac{2\zeta(0)}{\zeta(t)}m(t)\left(\frac{1}{2}+f+\frac{\sigma-1}{2}\zeta^2(t)\right)\\
&\le\frac{\zeta(0)}{\zeta(t)}\left(1+m^2(t)\right)\left(1+|f|\right)\\
&\le C_3w_1(t)
\end{aligned} \tag{4.4}$$

where $C_3=1+\frac{(3\sigma+2)(e+1)}{4(e-1)}+\left(\frac{7(e+1)}{4(e-1)}+\frac{1+k^2}{4}\right)\left(\|u_0\|_{H^1}^2+\sigma\|\rho-1\|_{L^2}^2\right)$.

Then

$$\begin{aligned}
w_1(t)&\le w_1'(0)\mathrm{e}^{C_3t}=\left(\zeta^2(0)+1+m^2(0)\right)\mathrm{e}^{C_3t}\\
&\le\left(1+\|u_{0x}\|_{L^\infty}^2+\|\rho_0\|_{L^\infty}^2\right)\mathrm{e}^{C_3t}=C_4\mathrm{e}^{C_3t}
\end{aligned} \tag{4.5}$$

where $C_4=1+\|u_{0x}\|_{L^\infty}^2+\|\rho_0\|_{L^\infty}^2$.

From (4.3), we have

$$\zeta(0)\zeta(t)\le w_1(t),|\zeta(0)|m(t)\le w_1(t) \tag{4.6}$$

then

$$\left|\inf_{x\in S}u_x(t,x)\right|=|m(t)|\le\frac{w_1(t)}{|\zeta(0)|}\le\frac{1}{\inf_{x\in S}\rho_0(x)}C_4\mathrm{e}^{C_3t},t\in[0,T)$$

Suppose $1<\sigma<2$, define the function

$$w_2(t)=\zeta^\sigma(0)\frac{\zeta^2(t)+1+m^2(t)}{\zeta^\sigma(t)} \tag{4.7}$$

Differentiate $w_2(t)$

$$
\begin{aligned}
w_2'(t) &= \frac{2\zeta^\sigma(0)}{\zeta^\sigma(t)} m(t)\left((\sigma-1)\zeta^2(t)+\frac{\sigma-1}{2}m^2(t)-\lambda m(t)+f+\frac{\sigma}{2}\right)\\
&\le \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}\left(1+m^2(t)\right)\left(|f|+\frac{\sigma}{2}\right)\\
&\le \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}\left(1+m^2(t)\right)\left(|f|+1\right)\\
&\le C_3 w_2(t)
\end{aligned}
\tag{4.8}
$$

then

$$
\begin{aligned}
w_2(t) &\le w_2(0)e^{C_3 t} = \left(\zeta^2(0)+1+m^2(0)\right)e^{C_3 t}\\
&\le \left(1+\|u_{0x}\|_{L^\infty}^2+\|\rho_0\|_{L^\infty}^2\right)e^{C_3 t}\\
&= C_4 e^{C_3 t}
\end{aligned}
\tag{4.9}
$$

Here we apply Young's inequality $ab \le \frac{a^p}{p}+\frac{b^q}{q}$, for $p=\frac{2}{\sigma}$, $q=\frac{2}{2-\sigma}$.

$$
\begin{aligned}
\frac{w_2(t)}{\zeta^\sigma(0)} &= \left(\zeta^{\frac{\sigma(2-\sigma)}{2}}\right)^{\frac{2}{\sigma}}+\left(\frac{\left(1+m^2\right)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\right)^{\frac{2}{2-\sigma}}\\
&\ge \frac{\sigma}{2}\left(\zeta^{\frac{\sigma(2-\sigma)}{2}}\right)^{\frac{2}{\sigma}}+\frac{2-\sigma}{2}\left(\frac{\left(1+m^2\right)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\right)^{\frac{2}{2-\sigma}}\\
&\ge \left(1+m^2\right)^{\frac{2-\sigma}{2}} \ge |m(t)|^{2-\sigma}
\end{aligned}
$$

Hence

$$
\left|\inf_{x\in S} u_x(t,x)\right| \le \left(\frac{w_2(t)}{\left|\zeta^\sigma(0)\right|}\right)^{\frac{1}{2-\sigma}} \le \frac{1}{\inf\limits_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}, t\in[0,T)
$$

2) Next we control $\left|\sup\limits_{x\in S} u_x(t,x)\right|$.

Similarly,

$$
\begin{cases}
\bar{m}'(t) = -\frac{1}{2}\bar{m}^2(t)-\lambda\bar{m}(t)+\frac{\sigma}{2}\bar{\zeta}^2(t)+f\left(t,q(t,x_1)\right)\\
\bar{\zeta}'(t) = -\bar{\zeta}(t)\bar{m}(t)
\end{cases}
$$

Suppose $0<\sigma\le 1$, define the function

$$
\bar{w}_1(t) = \bar{\zeta}^\sigma(0)\frac{\bar{\zeta}^2(t)+1+\bar{m}^2(t)}{\bar{\zeta}^\sigma(t)}
\tag{4.10}
$$

From (3.20) and (4.8), we obtain $\bar{w}_1'(t)\le C_3\bar{w}_1(t)$, then $\bar{w}_1(t)\le C_4 e^{C_3 t}$. Similarly, we get

$$
\frac{\bar{w}_1(t)}{\bar{\zeta}^\sigma(0)} \ge |\bar{m}(t)|^{2-\sigma}
$$

then

$$\left|\sup_{x\in S} u_x(t,x)\right| \le \left(\frac{\bar{w}_1(t)}{\left|\bar{\zeta}^{\sigma}(0)\right|}\right)^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in S} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} \mathrm{e}^{\frac{C_3 t}{2-\sigma}}, t\in[0,T)$$

Suppose $1<\sigma<2$, define the function

$$\bar{w}_2(t) = \bar{\zeta}(0)\bar{\zeta}(t) + \frac{\bar{\zeta}(0)}{\bar{\zeta}(t)}\left(1+\bar{m}^2(t)\right) \tag{4.11}$$

From (3.20) and (4.4), we have $\bar{w}_2'(t) \le C_3 \bar{w}_2(t)$, then $\bar{w}_2(t) \le C_4 \mathrm{e}^{C_3 t}$. Hence

$$\left|\sup_{x\in S} u_x(t,x)\right| = \left|\bar{m}(t)\right| \le \frac{\bar{w}_2(t)}{\left|\bar{\zeta}(0)\right|} \le \frac{1}{\inf_{x\in S} \rho_0(x)} C_4 \mathrm{e}^{C_3 t}, t\in[0,T)$$

Theorem 4.2 Let $0<\sigma<2$, $(u_0, \rho_0 - 1) \in H^s(S)\times H^{s-1}(S)$ with $s\ge 2$, there exist a maximal time $T>0$ and a unique solution (u,ρ) of Equation (1.1) with initial data. If $\inf_{x\in S} \rho_0(x) > 0$, then $T=\infty$ and the the solution (u,ρ) is global.

Proof: By contradictory arguement, assume $T<\infty$ and the solution blows up. The Theorem 3.1 shows

$$\int_0^T \left|u_x(t,x)\right|_{L^\infty} \mathrm{d}t = \infty \tag{4.12}$$

The assumptions and the Theorem 4.1 show

$$\left|u_x(t,x)\right| < \infty$$

For all $(t,x)\in[0,T)\times S$, that is a contradiction to (4.12).

The proof of Theorem 4.2 is complete.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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