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Counting the Microstates at the Black Hole Horizon and the Immirzi Parameter of Loop Quantum Gravity

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Abstract

Motivated by Bekenstein's original thought that led him to his famous area-entropy formula for a black hole and by our recent study regarding the black hole dynamics, we identify the appropriate microscopic degrees of freedom in loop quantum gravity that are responsible for the black hole entropy. We achieve consistent results by taking the $j = 1/2$ edges as dominant and by subjecting these edges to experience quantum fluctuations at the horizon. This also leads to a modification of the value of the Immirzi parameter in the $SU(2)$ framework.

Keywords

Immirzi Parameter, Spin Networks, Quantum Gravity, Loop Quantum Gravity, Isotropic Oscillator, $SU(2)$, Quasinormal Modes, Black Hole

1. Introduction

Amongst various approaches to quantum gravity, loop quantum gravity (LQG) seems to be the sole theory that has produced results regarding geometrical spectra (and other results based on these spectra) from first principle. LQG uses spin networks as basis for its Hilbert space. Spin networks are graphs with edges that carry labels $\{j \in 0, 1/2, 1, \cdots\}$ as the representations of $SU(2)$ that serves as the gauge group of the theory. The area of a given region of space is quantized in such a way that if a surface is punctured by an edge of the spin network carrying a label j the surface acquires an element of Planck size area [1] [2] [3]

$$A_j = 8\pi G\hbar\gamma\sqrt{j(j+1)}. \tag{1}$$

Here γ is the unfixed free Immirzi parameter [4] in the theory. LQG cannot

produce predictions unless this unknown parameter is fixed. It was proposed [5] [6] that γ can be fixed by comparing the loop quantum gravity results for the black hole entropy with the Bekenstein-Hawking semi-classical formula [7] [8],

$$S = \frac{A}{4G\hbar}. \tag{2}$$

In LQG, the entropy of a black hole is determined from the quantum properties of those microstates that contribute non-vanishing values to the physical area and is found from the number of spin network edges puncturing the surface, whereas each edge with label j_i contributes an area element given by (1). The Immirzi parameter was fixed as

$$\gamma = \frac{\ln\left(2j_{\min}+1\right)}{2\pi\sqrt{j_{\min}\left(j_{\min}+1\right)}}, \tag{3}$$

where $j = j_{\min}$ denotes the lowest possible non-zero spin label for the representation of the gauge group of the theory. It is not hard to see that statistically preferable punctures are the ones with $j_{\min} = 1/2$.

Dreyer [9] revealed yet another novel way of fixing γ by taking into account the asymptotic quasinormal mode (*QNM*) spectrum of a Schwarzschild black hole [10] [11] and the Bohr correspondence principle, an idea originally conceived by Hode [12] in the context of black hole dynamics. Based on numerical calculations Hod conjectured that the real part of the highly damped (*QNM*) frequencies ω_{QNM} asymptotically tends to a fixed quantity.

$$\omega_{QNM} = \frac{\ln 3}{8\pi GM}. \tag{4}$$

This conjecture was later proved analytically by Motl and Neitzke [13]. The argument due to Hod and Dreyer goes as follows. If one assumes that the Bohr's correspondence principle is applicable to black holes, the radiation or absorption of such an asymptotic frequency of *QNM* should be consistent with the corresponding change ΔM in the mass of the black hole, *i.e.*

$$\Delta M = \hbar\omega_{QNM} = \frac{\hbar\ln 3}{8\pi GM}. \tag{5}$$

Combining (5) with the area-mass relation for a Schwarzschild black hole,

$$A = 16\pi\left(GM\right)^2, \tag{6}$$

One can obtain the corresponding variation in the area of the horizon as

$$\Delta A = 4G\hbar\ln 3. \tag{7}$$

Dreyer argued that the most natural candidate for a transition of the quantum black hole, as described above, is the appearance or disappearance of a puncture with spin $j_{\min}$. The area of the black hole would then change by an amount,

$$A_{j_{\min}} = 8\pi G\hbar\gamma\sqrt{j_{\min}\left(j_{\min}+1\right)}. \tag{8}$$

Comparison of (7) and (8) yields the value of the Immirzi parameter as

$$\gamma = \frac{\ln 3}{2\pi\sqrt{j_{\min}\left(j_{\min}+1\right)}}. \tag{9}$$

Taking the dimensions of the Hilbert space on the boundary of area A as $\left(2j_{\min}+1\right)^N$, where N is the total number of punctures, the black hole entropy is obtained as

$$S = \frac{A}{4G\hbar}\frac{\ln\left(2j_{\min}+1\right)}{\ln 3}. \tag{10}$$

In order to cancel the ln3 factor in (10) and hence to comply with the Bekenstein-Hawking entropy formula (2), Dreyer was forced to choose $j_{\min}=1$ as the dominant edges as opposed to the expected $j_{\min}=1/2$ processes. Consequently, the Immirzi parameter assumed the value

$$\gamma = \frac{\ln 3}{2\pi\sqrt{2}}. \tag{11}$$

Unable to obtain the desired result with $j_{\min}=1/2$ punctures, Dreyer suggested that one has to adopt $SO(3)$ as the gauge group of loop quantum gravity in place of $SU(2)$. It was also remarked that there might be other physical reasons which might explain why consistent results cannot be obtained from $j_{\min}=1/2$ transitions. In this paper we will point out that it is due to the neglect of the appropriate degrees of freedom at the boundary that consistent results do not follow with $j_{\min}=1/2$ punctures.

Various attempts were made to save $SU(2)$ as the relevant gauge group of the theory and to formulate convincing explanation as to why $j=1$ processes contribute dominantly to the black hole entropy, but these came with additional considerations and assumptions. Corichi [14] argued that changing the group structure of the theory from $SU(2)$ to $SO(3)$ was not that appealing an idea if fermions were to be included in the theory. Keeping $SU(2)$ as the gauge group and invoking the local fermion number conservation, Corichi reasoned why were the processes with $j=1$ dominant, and reported the same value of γ as given by (11) for $j=1$ processes. However, it remained unclear why the $j=1/2$ processes, though kinamatically allowed, had to be highly suppressed. Remarkably, agreement with the correct entropy formula was achieved for $j=1/2$ as the dominant contributing processes by considering supersymmetric extension of spin networks [15].

In order to get a consistent picture of matter-geometry transitions at the horizon we modeled a *QNM* frequency at the horizon as a two-dimensional isotropic oscillator with frequency ω_{QNM}, a dynamical system that carry the same $SU(2)$ symbols as those of the edges of LQG [16]. That work led to a systematic modification in the value of γ as twice the value reported in [14] for $j=1$ processes. However, no explanation was provided as to why the edges with $j=1/2$ were suppressed. In the present note, guided by our approach in [16] together with Bekenstein's seminal idea [7] regarding the black hole entropy, we show that consistent results can be obtained with $j=1/2$ edges as dominant by

identifying the appropriate mechanism responsible for the thermal properties of a black hole.

In Section 2 we identify the relevant microstates in the $SU(2)$ framework responsible for the black hole thermodynamics. Section III is devoted to results and discussion.

2. Counting the Degrees of Freedom at the Horizon

Generally, one would think of the thermal properties of a black hole as to originate from fluctuating horizon geometry rather than a spherically symmetric static horizon. A quantum theory of gravity would therefore relate such fluctuations to some microscopic degrees of freedom at the horizon. In the LQG approach, the microstates that determine the thermal behavior of a black hole are the ones belonging to the irreducible representations of edges, with symbols $j > 0$, puncturing the horizon. It is of particular interest to note that edges with $j = 0$ are often disregarded in the black hole entropy calculations, mainly because they do not contribute to the area spectrum. But here we point out that $j = 0$ states do of course contribute to the entropy even though they yield zero area. In the following, we will find out that it is due to neglect of the contribution from the $j = 0$ edges that consistent results are difficult to achieve with $j = 1/2$ edges as dominant in the $SU(2)$ framework. Our argument in favor of selecting the true degrees of freedom, that includes the $j = 0$ state, is motivated by the Bekenstein's original idea [7] that led him to his area law of entropy and by our recent work [16].

Bekenstein [7] brought in quantum uncertainty to rescue the second law of thermodynamics at black hole horizon which would otherwise be violated if the black holes were purely classical objects. Bekenstein argued that when a particle (which he identified with a bit of information) is one Compton wavelength away from the horizon it should be considered as part of the black hole; it increases the mass as well as the area of the black hole by a small amount. At the same time it also increases the entropy of the horizon by ln2, which is the maximum entropy associated with a bit of information when nothing whatsoever is known if the bit is either present or absent at the horizon. This uncertainty or the inaccessibility of information (to an exterior agent) associated with a bit was argued to be at the root of the black hole thermodynamics.

In order to strengthen further our argument that will follow, we wish to recall our recent work [16] in which we argued that geometry-matter transitions at the horizon could be conceived in a consistent way if *QNM* frequencies at the horizon were emulated as 2D isotropic oscillators that carried the same $SU(2)$ quantum numbers as those of the edges of LQG. We observed that the ground state ($j = 0$) of a *QNM* with energy $\hbar\omega_{QNM}$ corresponds to the zero eigenvalue ($j = 0$) of the area operator of LQG. In addition, absorption and emission of edges at the horizon could be viewed as excitation and de-excitation of *QNM* frequencies, respectively. A quantum jump of a *QNM* from an excited state j to

its ground state would be equivalent to the detachment of an edge j from the surface, leaving no puncture ($j=0$) behind. Reversibly, the excitation of a *QNM* from its ground state to a higher j state would correspond to the attachment of an edge j to the horizon.

Our argument about the correct dynamics at the horizon goes as follows: If our assessment of representing a *QNM* frequency by a 2D oscillator is accurate, then it is certain that the ground state (edge with $j=0$), regardless of being unable "alone" to contribute to the area spectrum, must contribute significantly to the thermal properties of a black hole because of having energy $\hbar\omega_{QNM}$. We want to remind the reader that we are dealing with a quantum black hole and that the ground state is as respectable a state as in any complete quantum theory; a quantum black hole is built up from the ground state. So how are we to consider the effects of the ground state on the black hole entropy? The only plausible way to appreciate the role of the $j=0$ state seems to think of an edge j at the horizon as to be in a superposition state of being either attached or detached, analogously to Bekenstein's bit. In our scheme, this would correspond to a *QNM* mode being in a superposition of an excited state ($j>0$) and the ground state ($j=0$). This uncertainty in the state of an edge can be thought of as correctly describing a quantum fluctuating horizon. Thus, by including the $j=0$ state in the superposition one readily observes that the actual number of states accessible to an edge j at the horizon are in fact $\left[(2j+1)+1\right]$ rather than $(2j+1)$. This implies that a fluctuating edge with $j=1/2$ is in fact worth ln3 of entropy. It immediately follows that the ln3 term present in the definition (5) of *QNM* frequency can as well be cancelled by considering a superposition state containing the $j=0$ and $j_{\min}=1/2$ punctures.

Now, we are compelled to find the expectation value of the area element that the superposition state will contribute to the horizon. To maximize entropy, we take an equal superposition that yields the expectation value of the area element as

$$\bar{A}(0, j_{\min}) = 4\pi G\hbar\gamma\sqrt{j_{\min}(j_{\min}+1)}. \tag{12}$$

This is half the value given by (8). Using (12), the entropy of a black hole with horizon area Aacquires the form

$$S = \frac{A\ln(2j_{\min}+2)}{4\pi G\hbar\gamma\sqrt{j_{\min}(j_{\min}+1)}}. \tag{13}$$

Since we are using the expectation value of the area element, formula (13) is more suitable than a purely quantum to be compared to the semi-classical formula (2) for the entropy. Equating (13) for $j_{\min}=1/2$ to the formula (2) one can fix the value of γ as

$$\gamma = \frac{2\ln 3}{\pi\sqrt{3}}. \tag{14}$$

This value of γ is different than all the previously reported values.

The observation made here that an edge with $j_{\min} = 1/2$ is worth ln3 of entropy is consistent with Equation (5) for the mass increment ΔM expressed in terms of the *QNM* frequency. It is straightforward to show that formula (5) follows from the area mass relation (6) provided one assumes that the edges are uniformly distributed over the horizon and that each edge contributes ln3 of entropy.

3. Discussion

In the LQG approach, thermal properties of the horizon are assumed to arise from the punctures of those edges $j > 0$ that wander only on their internal spaces. The $j = 0$ states are usually ignored because they do not contribute to the horizon area and are therefore not worthy of contributing to the entropy. However, to much of one's surprise, we observed that null punctures ($j = 0$) do play significant role in the thermodynamics of a black hole when *QNM* frequencies are viewed as $SU(2)$ oscillating systems and edges are subjected to quantum uncertainty at the horizon. This uncertainty provides degrees of freedom that best describe the dynamics at the horizon. The superposition state of $j = 0$ and $j = 1/2$ edges appears to account for the cancellation of the ln3 term inherent in (5) and leads to consistent results while preserving $SU(2)$ as the working group of LQG. The value of the Immirzi parameter thus gets modified.

One would be tempted to have a full understanding of the underlying quantum reality at the horizon. Projecting *QNM* frequencies as $SU(2)$ oscillators may prove as a step forward to a complete quantum picture. This realization is in conformity with the black hole spectroscopy initiated by Bekenstein [17] [18]. It clearly allows for the mass (and hence the area) of a Schwarzschild black hole to have an equally spaced discrete spectrum, with each level being $(2j+1)$-fold degenerate.

Dreyer's results are valid for large black hole where the macroscopic area is proportional to entropy and for which the area-mass relation (6) holds. It is worthwhile noting that in Dreyer's work the area element (7) was obtained by incorporating a semi-classical argument (5) and the classical result (6). And the Immirzi parameter in (9) was fixed by comparing a purely quantum result (8) to the semi-classical result (7). Similarly, the result (3) for the Immirzi parameter was obtained [5] [6] [19] by comparing a purely quantum result for the entropy with the semi-classical result (2). In contrast, in our approach, we were forced to equate result (13) based on the "expectation value" of the minimal area element (13) to the semi-classical result (2), which looks more legitimate.

Notably, the minimal area element in (7) can be obtained directly from the Bakenstein-Hawking formula (2) provided that one assumes that edges are evenly spread over the surface and that each edge contributes ln3 of entropy. Expression (5) for the *QNM* frequency combined with the Bohr's correspondence principle is implicit in (7) and can be derived with the help of (6). Thus, for the black hole calculation to be consistent with Hod's conjecture one must

consider the dominant edges as those which contribute ln3 of entropy each. This was the reason why Dreyer chose $j=1$ as the dominant edges. Here, we reached at consistent results by assuming superposition states of edges with $j=0$ and $j=1/2$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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