Scientific
Research
Publishing

Runge-Kutta Method and Bolck by Block Method to Solve Nonlinear Fredholm-Volterra Integral Equation with Continuous Kernel

A. M. Al-Bugami, J. G. Al-Juaid

Department of Mathematics, Faculty of Sciences, Taif University, Taif, KSA
Email: abeer101aa@yahoo.com

Abstract

In this paper, the existence and uniqueness of the solution of Fredholm-Volterra integral equation is considered (NF-VIE) with continuous kernel; then we used a numerical method to reduce this type of equations to a system of nonlinear Volterra integral equations. Runge-Kutta method (RKM) and Bolck by block method (BBM) are used to solve the system of nonlinear Volterra integral equations of the second kind (SNVIEs) with continuous kernel. The error in each case is calculated.

Keywords

Nonlinear Fredholm-Volterra Integral Equation, System of Nonlinear Volterra Integral Equations, Runge-Kutta Method, Bolck by Block Method

1. Introduction

Integral equations of various types and kinds play an important role in many branches of linear and nonlinear function analysis and their applications in the theory of elasticity, engineering, mathematical physical and contact mixed problems. Therefor, many different methods are used to obtain the solution of the Volterra integral equation. In [1] Linz, studied analytical and numerical methods of Volterra equation. In [2], Mirzaee and Rafei used the BBM for the numerical solution of the nonlinear two-dimensional Volterra integral equations. In the references [3]-[8] the authors considered many different methods to solve linear and nonlinear system of Volterra integral equations numerically with continuous and singular kernels. In [9], Al-waqdani studied linear F-VIE with continuous kernel and solved the linear SVIEs numerically with continuous kernel.

$$\mu\phi(x,t) = f(x,t) + \lambda\int_a^b K(x,y)\gamma(y,t,\phi(y,t))\mathrm{d}y + \lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))\mathrm{d}\tau \quad (1)$$

Equation (1) is called the NF-VIE in the space $L_2[a,b] \times C[0,T], T < \infty$. Here the Fredholm integral term is considered in position with a positive continuous kernel $K(x,y)$ for all $x,y \in [a,b]$, while the Volterra integral term is considered in time with a positive continuous kernel $F(t,\tau)$ for all $t,\tau \in [0,T], T < \infty$. The free term $f(x,t)$ is known continuous function in the space $L_2[a,b] \times C[0,T]$, while $\phi(x,t)$ is unknown function representing the solution of the nonlinear integral Equation (1). The numerical coefficient λ is called the parameter of the integral equation, may be complex, and has physical meaning, while the constant parameter μ defines the kind of the integral Equation (1).

2. Existence of Solution of NF-VIE

To prove the existence of a unique solution of Equation (1) using fixed point theorem.

We write it in the integral operator form:

$$\overline{W}\phi = f + W\phi \qquad (2)$$

where

$$W\phi = K\phi + F\phi \qquad (3)$$

$$K\phi = \lambda\int_a^b K(x,y)\gamma(y,t,\phi(y,t))\mathrm{d}y, \quad F\phi = \lambda\int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))\mathrm{d}\tau \qquad (4)$$

Then, we assume the following conditions:

i) The kernel of Fredholm integral term satisfies:

$$|K(x,y)| \le A_1 \quad (A_1 \text{ is a constant}).$$

ii) The kernel of Volterra integral term satisfies:

$$|F(t,\tau)| \le A_2 \quad (A_2 \text{ is a constant}).$$

iii) The given function $f(x,t)$ with its partial derivatives is continuous in $L_2[a,b] \times C[0,T]$ where:

$$\|f(x,t)\| = \max \int_0^t \left(\int_a^b |f(x,\tau)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}\tau = A_3 \quad (A_3 \text{ is a constant}).$$

iv) The known continuous function $\gamma(x,t,\phi(x,t))$, for the constant $B > B_1, B > p$, the following conditions:

a) $\left(\int_0^t \int_a^b |\gamma(x,t,\phi(x,t))|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \le B_1 \|\phi(x,t)\|_{L_2[a,b] \times C[0,T]}$

b) $\|\gamma(x,t,\phi_1(x,t)) - \gamma(x,t,\phi_2(x,t))\| \le N(x,t)|\phi_1(x,t) - \phi_2(x,t)|$

where $\|N(x,t)\|_{L_2[a,b] \times C[0,T]} = p$.

Theorem 1:

If the condition i)-iv) are verified, then Equation (1) has unique solution in the Banach space $L_2[a,b] \times C[0,T], T < \infty$.

The provement of this theorem depends on the following two lemmas:

Lemma 1:

Under the conditions i)-iv-a), the operator $\overline{W}$ defined by (2), maps the space $L_2[a,b] \times C[0,T], T < \infty$ into itself.

Proof:

In view of Formula (2) and (3) we get:

$$\left\| \overline{W}\phi(x,t) \right\| \leq \left\| f(x,t) \right\| + |\lambda| \left\| \int_a^b |K(x,y)| |\gamma(y,t,\phi(y,t))| \, \mathrm{d}y \right\|$$
$$+ |\lambda| \left\| \int_0^t |F(t,\tau)| |\gamma(x,\tau,\phi(x,\tau))| \, \mathrm{d}\tau \right\|$$

Using the conditions (i)-(iii), then applying Cauchy-Schwarz inequality, we have:

$$\left\| \overline{W}\phi(x,t) \right\| \leq A_3 + |\lambda| A_1 \left(\int_a^b |\gamma(y,t,\phi(y,t))|^2 \, \mathrm{d}y \right)^{\frac{1}{2}}$$
$$+ |\lambda| A_2 \left(\int_0^t |\gamma(x,\tau,\phi(x,\tau))|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}}$$

In the light of the condition (iv-a), the above inequality take the form:

$$\left\| \overline{W}\phi(x,t) \right\| \leq A_3 + \alpha \left\| \phi(x,t) \right\|, \quad \alpha = |\lambda| B_1 (A_1 + A_2 + L) \tag{5}$$

The lost inequality (5) shows that, the operator $\overline{W}$ maps the ball S_r into itself, where

$$r = \frac{A_3}{1 - |\lambda| B_1 (A_1 + A_2 L)} \tag{6}$$

Since $r > 0, A_3 > 0$, therefore we have $\alpha < 1$. Moreover, the inequality (5) involves the boundedness of the operator W of Equation (2) where:

$$\left\| W\phi(x,t) \right\| \leq \alpha \left\| \phi(x,t) \right\| \tag{7}$$

Also, the inequalities (5) and (7) define the boundedness of the operator $\overline{W}$.

Lemma 2:

If the conditions (i),(ii) and (iv-b) are satisfied, then the operator $\overline{W}$ is contractive in space $L_2[a,b] \times C[0,T], T < \infty$.

Proof:

For two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in the space $L_2[a,b] \times C[0,T]$ Formula (2), (3) leads to:

$$\left\| \overline{W}\phi_1(x,t) - \overline{W}\phi_2(x,t) \right\| \leq |\lambda| \left\| \int_a^b |K(x,y)| |\gamma(y,t,\phi_1(y,t)) - \gamma(y,t,\phi_2(y,t))| \, \mathrm{d}y \right\|$$
$$+ |\lambda| \left\| \int_0^t |F(t,\tau)| |\gamma(x,\tau,\phi_1(x,\tau)) - \gamma(x,\tau,\phi_2(x,\tau))| \, \mathrm{d}\tau \right\|$$

Using the condition (iv-b), then apply Cauchy-Schwarz inequality we have:

$$\left\| \overline{W}\phi_1(x,t) - \overline{W}\phi_2(x,t) \right\| \leq |\lambda| |K(x,y)| \left(\int_a^b N^2(x,t) |\phi_1(y,t) - \phi_2(y,t)|^2 \, \mathrm{d}y \right)^{\frac{1}{2}}$$
$$+ |\lambda| |F(t,\tau)| \left(\int_a^b N^2(x,\tau) |\phi_1(x,\tau) - \phi_2(x,\tau)|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}}$$

Finally, with the aid of conditions (i), (ii), and (iv-b) we obtain:

$$\left\| \overline{W}\phi_1(x,t) - \overline{W}\phi_2(x,t) \right\| \leq \alpha \left\| \phi_1(x,t) - \phi_2(x,t) \right\| \tag{8}$$

In equality (8) shows that, the operator $\overline{W}$ is continuous in the space $L_2[a,b] \times C[0,T], T < \infty$, then $\overline{W}$ is a contraction operator under the condition $\alpha < 1$.

3. The SVIEs

Consider:

$$\phi(x,t) = f(x,t) + \lambda \int_a^b K(x,y)\gamma(y,t,\phi(y,t))\mathrm{d}y$$

$$+ \lambda \int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))\mathrm{d}\tau, \ (\mu = 1) \tag{9}$$

when $a = b = 0$, Equation (9) becomes:

$$\phi(0,t) = f(0,t) + \lambda \int_0^t F(t,\tau)\gamma(0,\tau,\phi(0,\tau))\mathrm{d}\tau$$

Then,

$$\phi_0(t) = f_0(t) + \lambda \int_0^t F(t,\tau)\gamma(\tau,\phi_0(\tau))\mathrm{d}\tau \tag{10}$$

where $u_0(t) = u(0,t), f_0(t) = f(0,t)$.

Formula (10) represents Volterra integral equation of the second kind at $a = b = 0$. For representing (9) as a VIEs, we use the numerical method. Divide the interval $[a,b]$ as $a = x_0 \leq x_1 \leq \cdots \leq x_N = b$. Using the quadrature formula, Equation (9) becomes:

$$\int_a^b K(x,y)\gamma(y,t,\phi(y,t))\mathrm{d}y$$

$$= \sum_{m=0}^n u_m K(x_n,x_m)\gamma(x_n,t,\phi(x_n,t)) \tag{11}$$

where $n = 1,2,\cdots,N$, and $u_0 = \frac{1}{2}h_0, u_n = \frac{1}{2}h_n, u_i = h_i, (i \neq 0,n)$.

Using (11) in (9), we have:

$$\phi(x,t) = f(x,t) + \lambda \int_0^t F(t,\tau)\gamma(x,\tau,\phi(x,\tau))\mathrm{d}\tau$$

$$+ \sum_{m=0}^n u_m K(x_n,x_m)\gamma(x_n,t,\phi(x_n,t)) \tag{12}$$

Then:

$$\phi_n = W_n(t) + \lambda \int_0^t F(t,\tau)\gamma(x_n,\tau,\phi_n(\tau))\mathrm{d}\tau \tag{13}$$

where $W_n(t) = f_n(t) + \lambda \sum_{m=0}^{n-1} u_m K(x_n,x_m)\gamma(x_m,t,\phi_m(t))$, $n = 0,1,\cdots,N$. Formula (13) represents a NSVIEs of the second kind, and we have N unknown functions $\phi_n(t)$ corresponding to time interval $[0,T]$.

4. Some Numerical Methods for Solving SVIEs

4.1. RKM

In this section, the RKM is used to solve NF-VIE of the second kind. By divide the interval $[a,b]$ as $a = x_0 \leq x_1 \leq \cdots \leq x_i \leq \cdots \leq x_N = b$, $i = 0,1,\cdots,N$ and using the quadrature formula, the integral Equation (1) represent a NSVIEs as:

$$\phi_n(t) = W_n(t) + \int_0^t F(t,\tau)\gamma(\tau,\phi_n(\tau))\,\mathrm{d}\tau$$

where $W_n(t) = f_i(t) + \sum_{i=0}^n u_i k_{ni} \gamma(t,\phi_i(t))$.

To solve the NSVIEs:

$$\varphi(t) = \int_0^t F(t,\tau)\gamma(\tau,\varphi(\tau))\,\mathrm{d}\tau + F(t) + \sum_{i=0}^n u_i k_{ni} \gamma(t,\varphi(t)), \quad n = 0,1,\cdots,N \quad (14)$$

where

$$\varphi(t) = \left(\phi_1(t),\phi_2(t),\cdots,\phi_l(t)\right)^{\mathrm{T}},$$
$$\gamma(\tau,\varphi(\tau)) = \left(\gamma(\tau,\varphi_1(\tau)),\gamma(\tau,\varphi_2(\tau)),\cdots,\gamma(\tau,\varphi_l(\tau))\right),$$
$$F(t) = \left(f_1(t),f_2(t),\cdots,f_l(t)\right)^{\mathrm{T}}$$

and

$$F(t,\tau)\gamma(\tau,\varphi(\tau))$$
$$= \begin{bmatrix} F_{1,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{1,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{1,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \\ F_{2,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{2,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{2,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \\ \vdots & \vdots & \ddots & \vdots \\ F_{s,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{s,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{s,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \end{bmatrix}$$

Then, we get

$$\phi_n(t) = W_n(t) + \int_0^t F(t,\tau)\gamma(\tau,\varphi(\tau))\,\mathrm{d}\tau, \quad n = 0,1,\cdots,N \quad (15)$$

Now, applying the RKM for solve (15):

Suppose that:

$$F(t,\tau) = \sum_s \psi_s(t)\chi_s(\tau) \quad (16)$$

Substituting from (16) into (15),

$$\phi_n(t) = W_n(t) + \int_0^t \sum_s \psi_s(t)\chi_s(\tau)\gamma(\tau,\varphi(\tau))\,\mathrm{d}\tau$$

$$\phi_n(t) = W_n(t) + \sum_s \psi_s(t)\int_0^t \chi_s(\tau)\gamma(\tau,\varphi(\tau))\,\mathrm{d}\tau$$

Then, we have,

$$\phi_n(t) = W_n(t) + \sum_s \psi_s(t)z_s(t), \quad t > 0 \quad (17)$$

where

$$z_s(t) = \int_0^t \chi_s(\tau)\gamma(\tau, \varphi(\tau))\,\mathrm{d}\tau \tag{18}$$

By derivative (18), we have,

$$z_s'(t) = \chi_s(\tau)\gamma(\tau, \varphi(\tau)), \quad z_s(0) = 0$$

Now, apply the RKM to this system of equations to give,

$$\tilde{z}(v_p h) = h\sum_{q=0}^{p-1} A_{pq}\chi_s(v_q h)\gamma(v_q h, \tilde{\varphi}(v_q h)), \quad p = 1, 2, \cdots, m$$

which lead to,

$$\tilde{\phi}_n(v_p h) = W_n(v_p h) + \sum_s \tilde{z}_s(v_p h)\psi_s(v_p h)$$

$$= W_n(v_p h) + \sum_s \psi_s(v_p h) h\sum_{q=0}^{p-1} A_{pq}\chi_s(v_q h)\gamma(v_q h, \tilde{\varphi}(v_q h))$$

$$= W_n(v_p h) + h\sum_{q=0}^{p-1} A_{pq}\gamma(v_q h, \tilde{\varphi}(v_q h))\sum_s \psi_s(v_p h)\chi_s(v_q h)$$

By using Equation (16) to give,

$$\tilde{\phi}_n(v_p h) = W_n(v_p h) + h\sum_{q=0}^{p-1} A_{pq}F(v_p h, v_q h)\gamma(v_q h, \tilde{\varphi}(v_q h)), \quad p = 1, 2, \cdots, m \tag{19}$$

which is approximate solution for Equation (15).

Now, if $m = 4$ consider the Pouzet's derivation, we define:

$$p_{i,j}(t) = \phi_{i,j-1}(t)$$

$$q_{i,j}(t) = G_{i,j}\left(t_{j+\frac{1}{2}}\right) + \frac{h}{2}F\left(t_{j+\frac{1}{2}}, t_j\right)p_{i,j}$$

$$r_{i,j}(t) = G_{i,j}\left(t_{j+\frac{1}{2}}\right) + \frac{h}{2}F\left(t_{j+\frac{1}{2}}, t_{j+\frac{1}{2}}\right)q_{i,j} \tag{20}$$

$$s_{i,j}(t) = G_{i,j}(t_{j+1}) + hF(t_{j+1}, t_{j+1})r_{i,j}$$

$$\phi_{i,j}(t) = G_{i,j}(t_{j+1}) + \frac{h}{6}\left[F(t_{j+1}, t_j)p_{i,j} + 2F\left(t_{j+1}, t_{j+\frac{1}{2}}\right)\left[q_{i,j} + r_{i,j}\right] + F(t_{j+1}, t_{j+1})s_{i,j}\right]$$

The function $\phi_{i,j}(t)$ is unknown function, such that

$$G_{i,j}(t) = W_i(t) + \frac{h}{6}\sum_{n=0}^{j-1}\left[F(t, t_n)p_{i,n} + 2F\left(t, t_{n+\frac{1}{2}}\right)\left[q_{i,n} + r_{i,n}\right] + F(t, t_{n+1})s_{i,n}\right] \tag{21}$$

where $G_{i,0}(t) = W_i(t)$, where $j = 2, 3, \cdots, N-1$.
when $j = 1$, Equation (10) becomes:

$$\phi_{i,1}(t) = G_{i,1}(t_2) + \frac{h}{6}\left[F(t_2, t_1)p_{i,1} + 2F\left(t_2, t_{1+\frac{1}{2}}\right)\left[q_{i,1} + r_{i,1}\right] + F(t_2, t_2)s_{i,1}\right]$$

such that $G_{i,1} = W(t_2)$ and

$$p_{i,1}(t) = W_i(t_1)$$

$$q_{i,1}(t) = G_{i,1}\left(t_{1+\frac{1}{2}}\right) + \frac{h}{2}F\left(t_{1+\frac{1}{2}}, t_1\right)p_{i,1}$$

$$r_{i,1}(t) = G_{i,1}\left(t_{1+\frac{1}{2}}\right) + \frac{h}{2}F\left(t_{1+\frac{1}{2}}, t_{1+\frac{1}{2}}\right)q_{i,1}$$

$$s_{i,1}(t) = G_{i,1}(t_2) + hF(t_2, t_2)r_{i,1}$$

Since the function $\phi_{i,j}(t) = \phi_{i,j}$ is the approximate solution at (x_i, t_j) for Equation (1).

4.2. BBM

In this section, we use the BBM for solving the NF-VIE of the second kind. The interval $[a,b]$ is divided into steps of width h, $t_j = jh, j = 0,1,\cdots,n$ and $h = (b-a)/n$. the approximate solution of $\phi_i(t)$ will be define at mesh-points t_j and denoted by $\phi_{ij}, j = 0,1,\cdots,n$ such as ϕ_{ij} is an approximation to $\phi_i(t_j)$.

To solve the NSVIEs:

$$\varphi(t) = F(t) + \lambda\sum_{j=0}^{i}u_j K_{nj}\gamma(t,\phi(t)) + \lambda\int_0^t F(t,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau, \ n = 0,1,\cdots,N \quad (22)$$

where

$$\varphi(t) = \left(\phi_1(t), \phi_2(t), \cdots, \phi_l(t)\right)^{\mathrm{T}},$$
$$\gamma(\tau,\varphi(\tau)) = \left(\gamma(\tau,\varphi_1(\tau)), \gamma(\tau,\varphi_2(\tau)), \cdots, \gamma(\tau,\varphi_l(\tau))\right),$$
$$F(t) = \left(f_1(t), f_2(t), \cdots, f_l(t)\right)^{\mathrm{T}}$$

and

$$F(t,\tau)\gamma(\tau,\varphi(\tau))$$
$$= \begin{bmatrix} F_{1,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{1,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{1,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \\ F_{2,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{2,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{2,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \\ \vdots & \vdots & \ddots & \vdots \\ F_{s,1}(t,\tau)\gamma(\tau,\varphi(\tau)) & F_{s,2}(t,\tau)\gamma(\tau,\varphi(\tau)) & \cdots & F_{s,s}(t,\tau)\gamma(\tau,\varphi(\tau)) \end{bmatrix}$$

Then, we get

$$\phi_n(t) = W_n(t) + \lambda\int_0^t F(t,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau, \ n = 0,1,\cdots,N \quad (23)$$

Rewrite Equation (23) as follows:

$$\phi_i(t_k) = W_i(t_k) + \lambda\int_0^{t_{pm}} F_{i,s}(t_k,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau + \lambda\int_{t_{pm}}^{t_n} F_{i,s}(t_k,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau$$

$$\phi_i(t_k) = f_i(t_k) + \lambda\sum_{n=0}^{i}u_n k_{in}\gamma(t_k,\phi_n(t_k)) + \lambda\int_0^{t_{pm}} F_{i,s}(t_k,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau$$
$$+ \lambda\int_{t_{pm}}^{t_n} F_{i,s}(t_k,\tau)\gamma(\tau,\varphi(\tau))\mathrm{d}\tau \quad (24)$$

where $s = 1, 2, \cdots$.

If the values $\phi_{i0}, \phi_{i1}, \cdots, \phi_{i,pm}$ are known, then the first integral can be approximated by standard quadrature methods, and the second integral is obtain by a quadrature rule using values of the integrand at $\tau = t_{pm}, t_{pm+1}, \cdots, t_{p(m+1)}$.

Since the values of ϕ_i at these points are unknown, we have a system of l_p nonlinear equations by applied the BBM:

$$
\begin{aligned}
\phi_{ik} = {} & f_i\left(t_k\right) + \lambda \sum_{n=0}^{i} u_n K_{in} \gamma\left(t_k, \phi_n\left(t_k\right)\right) \\
& + \lambda \left[h \sum_{j=0}^{mp} v_{kj} F_{i,s}\left(t_k, \tau_j\right)\left(\gamma\left(\tau_j, \phi_{ij}\right), \cdots, \gamma\left(\tau_j, \phi_{lj}\right)\right) \right] \\
& + \lambda \left[h \sum_{j=mp}^{(m+1)p} v_{kj}^* F_{i,s}\left(t_k, \tau_{mp+j}\right)\left(\gamma\left(\tau_{mp+j}, \phi_{1,mp+j}\right), \cdots, \gamma\left(\tau_{mp+j}, \phi_{l,mp+j}\right)\right) \right]
\end{aligned}
\tag{25}
$$

For $n = mp+1, mp+2, \cdots, (m+1)p$, $m = 0, 1, \cdots, (N-1)$, where v_{kj}, v_{kj}^* depend on the quadrature rule used.

Now, for the Modified method of two Blocks we take $p = 2$, this integration over $\left[0, t_{2m}\right]$ can be accomplished by Simpson's rule, and the integral over $\left[t_{2m}, t_n\right]$ by using a quadratic interpolation of the integrand at the point $t_{2m}, t_{2m+1}, t_{2m+2}$, then Equation (23) becomes:

$$
\phi_{i,2m+1} = W_i\left(t_{2m+1}\right) + \lambda \int_0^{(2m+1)h} F_{i,s}\left(t_{2m+1}, \tau\right) \gamma\left(\tau, \varphi\left(\tau\right)\right) \mathrm{d}\tau
\tag{26}
$$

and

$$
\begin{aligned}
& \phi_{i,2m+2} \\
& = W_i\left(t_{2m+2}\right) + \lambda \int_0^{(2m+2)h} F_{i,s}\left(t_{2m+2}, \tau\right) \gamma\left(\tau, \varphi\left(\tau\right)\right) \mathrm{d}\tau
\end{aligned}
\tag{27}
$$

where $i = 1, 2, \cdots, l; m = 0, 1, \cdots$.

Therefore, by Equation (25) the approximate solution is computed by:

$$
\begin{aligned}
\phi_{i,2m+1} = {} & W_i\left(t_{2m+1}\right) + \lambda \left[\frac{h}{3} \sum_{j=0}^{2m} v_j F_{i,s}\left(t_{2m+1}, \tau_j\right)\left(\gamma\left(\tau_j, \phi_{1j}\right), \cdots, \gamma\left(\tau_j, \phi_{lj}\right)\right) \right] \\
& + \lambda \left[\frac{h}{12} \left[5 F_{i,s}\left(t_{2m+1}, \tau_{2m}\right)\left(\gamma\left(\tau_{2m}, \phi_{1,2m}\right), \cdots, \gamma\left(\tau_{2m}, \phi_{l,2m}\right)\right) \right. \right. \\
& \left. \left. + 8 F_{i,s}\left(t_{2m+1}, \tau_{2m+1}\right)\left(\gamma\left(\tau_{2m+1}, \phi_{1,2m+1}\right), \cdots, \gamma\left(\tau_{2m+1}, \phi_{l,2m+1}\right)\right) \right. \right. \\
& \left. \left. - F\left(t_{2m+1}, \tau_{2m+2}\right)\left(\gamma\left(\tau_{2m+2}, \phi_{1,2m+2}\right), \cdots, \gamma\left(\tau_{2m+2}, \phi_{l,2m+2}\right)\right) \right] \right]
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
& \phi_{i,2m+2} \\
& = W_i\left(t_{2m+2}\right) + \lambda \frac{h}{3} \sum_{j=0}^{2m+2} v_j' F_{i,s}\left(t_{2m+2}, \tau_j\right)\left(\gamma\left(\tau_j, \phi_{1j}\right), \cdots, \gamma\left(\tau_j, \phi_{lj}\right)\right)
\end{aligned}
\tag{29}
$$

where $u_{i,0} = g_i\left(t_0\right)$.

Thus, replace the second term in Equation (28) by using integration formulas, then we get:

$$\phi_{i,2m+1} = W_i\left(t_{2m+1}\right) + \lambda\left[\frac{h}{3}\sum_{j=0}^{2m} v_j F_{i,s}\left(t_{2m+1},\tau_j\right)\left(\gamma\left(\tau_j,\phi_{1j}\right),\cdots,\gamma\left(\tau_j,\phi_{lj}\right)\right)\right]$$

$$+ \lambda\left[\frac{h}{6}\left[F_{i,s}\left(t_{2m+1},\tau_{2m}\right)\left(\gamma\left(\tau_{2m},\phi_{1,2m}\right),\cdots,\gamma\left(\tau_{2m},\phi_{l,2m}\right)\right)\right.\right.$$

$$+ 4F_{i,s}\left(t_{2m+1},\tau_{2m+\frac{1}{2}}\right)\left(\left(\frac{3}{8}\phi_{1,2m}+\frac{3}{4}\phi_{1,2m+1}-\frac{1}{8}\phi_{1,2m+2}\right),\cdots,\right. \tag{30}$$

$$\left.\left(\frac{3}{8}\phi_{l,2m}+\frac{3}{4}\phi_{l,2m+1}-\frac{1}{8}\phi_{l,2m+2}\right)\right)$$

$$\left.\left.+ F_{l,s}\left(t_{2m+1},\tau_{2m+1}\right)\left(\gamma\left(\tau_{2m+1},\phi_{1,2m+1}\right),\cdots,\gamma\left(\tau_{2m+1},\phi_{l,2m+1}\right)\right)\right]\right]$$

$$\phi_{i,2m+2} = W_i\left(t_{2m+2}\right) + \lambda\left[\frac{h}{3}F_{i,s}\left(t_{2m+2},\tau_0\right)\left(\gamma\left(\tau_0,\phi_{10}\right),\cdots,\gamma\left(\tau_0,\phi_{l0}\right)\right)\right.$$

$$+ 4F_{i,s}\left(t_{2m+2},\tau_1\right)\left(\gamma\left(\tau_1,\phi_{11}\right),\cdots,\gamma\left(\tau_1,\phi_{l1}\right)\right)+\cdots \tag{31}$$

$$\left.+ F_{i,s}\left(t_{2m+2},\tau_{2m+2}\right)\left(u_{1,2m+2},\cdots,u_{l,2m+2}\right)\right]$$

where

$$v_0 = v_{2m} = 1, \quad v_j = 3-(-1)^j, \quad j=1,2,\cdots,2m-1$$

$$v_0' = v_{2m+2}', \quad v_j' = 3-(-1)^j, \quad j=1,2,\cdots,2m+1$$

Finally, we construct $2l$ nonlinear equations from (30) and (31) to find the unknown functions $\phi_{i,2m+1}, \phi_{i,2m+2}$. The resulting system is solved by using modified Newton-Raphson method.

5. Numerical Examples

We solve two examples by RKM and BBMat $N = 20,50$, $T = 0.01,0.1,0.3$, $\lambda = 1$ and $\mu = 1$.

In Tables 1-6: $\phi_{Exact} \rightarrow$Exact solution, $\phi_{R.K.} \rightarrow$approximate solution of RKM, $E_{R.K.} \rightarrow$the absolute error of RKM, $\phi_{B.B.} \rightarrow$approximate solution of BBM, $E_{B.B.} \rightarrow$the absolute error of BBM.

Example 1
Consider: $\phi(x,t) = xt - \dfrac{x^2 t^2}{5} - \dfrac{x^2 t^6}{5} + \int_0^1 x^2 y^2 \phi^2(y,t)\,\mathrm{d}y + \int_0^t t\tau\phi^2(x,\tau)\,\mathrm{d}\tau$

Exact solution $\phi(x,t) = xt$

Case 1: $N = 20$

Table 1. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.01$, $N = 20$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0001000	0.0001499	4.9997×10^{-5}	0.0000999	2.4167×10^{-10}
0.2	0.0004000	0.0004999	9.9978×10^{-5}	0.0003999	3.853×10^{-9}
0.3	0.0009000	0.0010499	1.4992×10^{-4}	0.0008999	1.9247×10^{-8}

Continued

0.4	0.0016000	0.0017998	1.9980×10^{-4}	0.0015994	5.8590×10^{-8}
0.5	0.0025000	0.0027496	2.4960×10^{-4}	0.0024998	1.3151×10^{-7}
0.6	0.0036000	0.0038993	2.9932×10^{-4}	0.0035997	2.2922×10^{-7}
0.7	0.0049000	0.0052490	3.4901×10^{-4}	0.0048997	2.9202×10^{-7}
0.8	0.0064000	0.0067988	3.9884×10^{-4}	0.0063998	1.5103×10^{-7}
0.9	0.0081000	0.0085491	4.4918×10^{-4}	0.0081005	5.6689×10^{-7}
1	0.0100000	0.0104981	4.9815×10^{-4}	0.0100000	8.334×10^{-8}

Table 2. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.1$, $N = 20$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0010000	0.0014997	4.9997×10^{-4}	0.0009999	2.4161×10^{-8}
0.2	0.0040000	0.0049978	9.9978×10^{-4}	0.0039996	3.853×10^{-7}
0.3	0.0090000	0.0104922	1.4922×10^{-3}	0.0089980	1.9247×10^{-6}
0.4	0.0160000	0.0179805	1.9805×10^{-3}	0.0159941	5.8591×10^{-6}
0.5	0.0250000	0.0274608	2.4608×10^{-3}	0.0249868	1.3152×10^{-5}
0.6	0.0360000	0.0389329	2.9329×10^{-3}	0.0359770	2.2927×10^{-5}
0.7	0.0490000	0.0524020	3.4020×10^{-3}	0.0489707	2.9217×10^{-5}
0.8	0.0640000	0.0679851	3.8851×10^{-3}	0.0639848	1.5121×10^{-5}
0.9	0.0810000	0.0854199	4.4199×10^{-3}	0.0810568	5.682×10^{-5}
1	0.1000000	0.1048178	4.8178×10^{-3}	0.1000083	8.35×10^{-6}

Table 3. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.3$, $N = 20$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0030000	0.0049753	1.4975×10^{-3}	0.0029997	2.1745×10^{-7}
0.2	0.0120000	0.0014980	2.9804×10^{-3}	0.0119965	3.4685×10^{-6}
0.3	0.0270000	0.0314308	4.4308×10^{-3}	0.0269826	1.7322×10^{-5}
0.4	0.0480000	0.0538289	5.8289×10^{-3}	0.0479472	5.2734×10^{-5}
0.5	0.0750000	0.0821593	7.1593×10^{-3}	0.0748816	1.1838×10^{-4}
0.6	0.1080000	0.1164275	8.4275×10^{-3}	0.1077935	2.0644×10^{-4}
0.7	0.1470000	0.1566904	9.6904×10^{-3}	0.1467367	2.6328×10^{-4}
0.8	0.1920000	0.2031120	1.1112×10^{-2}	0.1918634	1.3654×10^{-4}
0.9	0.2430000	0.2560541	1.3054×10^{-2}	0.2435136	5.1366×10^{-4}
1	0.3000000	0.3138329	1.3832×10^{-2}	0.3000744	7.4446×10^{-5}

Case 2: $N = 50$

Table 4. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.01$, $N = 50$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0001000	0.0001199	1.9999×10^{-5}	0.0000999	9.867×10^{-11}
0.2	0.0004000	0.0004399	3.9991×10^{-5}	0.0003999	1.5735×10^{-9}
0.3	0.0009000	0.0009599	5.9969×10^{-5}	0.0008999	7.8606×10^{-9}
0.4	0.0016000	0.0016799	7.9923×10^{-5}	0.0015999	2.3949×10^{-8}
0.5	0.0025000	0.0025998	9.9844×10^{-5}	0.0024999	5.3855×10^{-8}
0.6	0.0036000	0.0037197	1.1973×10^{-4}	0.0035999	9.4282×10^{-8}
0.7	0.0049000	0.0050396	1.3960×10^{-4}	0.0048998	1.2160×10^{-7}
0.8	0.0064000	0.0065595	1.5952×10^{-4}	0.0063999	6.8595×10^{-8}
0.9	0.0081000	0.0082796	1.7964×10^{-4}	0.0081002	2.1360×10^{-7}
1	0.0100000	0.0101992	1.9922×10^{-4}	0.0100000	1.333×10^{-8}

Table 5. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.1$, $N = 50$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0010000	0.0011999	1.9990×10^{-4}	0.0009999	9.8647×10^{-9}
0.2	0.0040000	0.0043991	3.9917×10^{-4}	0.0039998	1.5735×10^{-7}
0.3	0.0090000	0.0095968	5.9698×10^{-4}	0.0089992	7.8607×10^{-7}
0.4	0.0160000	0.0167923	7.9237×10^{-4}	0.0159976	2.3948×10^{-6}
0.5	0.0250000	0.0259844	9.8448×10^{-4}	0.0249946	5.3855×10^{-6}
0.6	0.0360000	0.0371732	1.1732×10^{-3}	0.0359905	9.4288×10^{-6}
0.7	0.0490000	0.0503606	1.3606×10^{-3}	0.0489878	1.2163×10^{-5}
0.8	0.0640000	0.0655530	1.5530×10^{-3}	0.0639931	6.8630×10^{-6}
0.9	0.0810000	0.0827654	1.7654×10^{-3}	0.0810213	2.1380×10^{-5}
1	0.1000000	0.1019246	1.9246×10^{-3}	0.1000013	1.3344×10^{-6}

Table 6. The values of exact, approximate solutions, and errors by using RKM and BBM at $T = 0.3$, $N = 50$.

x	ϕ_{Exact}	$\phi_{R.K.}$	$E_{R.K.}$	$\phi_{B.B.}$	$E_{B.B.}$
0	0	0	0	0	0
0.1	0.0030000	0.0035991	5.9912×10^{-4}	0.0029999	8.8781×10^{-8}
0.2	0.0120000	0.0013192	1.1925×10^{-3}	0.0119985	1.4161×10^{-6}
0.3	0.0270000	0.0287733	1.7733×10^{-3}	0.0269929	7.0746×10^{-6}
0.4	0.0480000	0.0503338	2.3338×10^{-3}	0.0479784	2.1553×10^{-5}
0.5	0.0750000	0.0778692	2.8692×10^{-3}	0.0749515	4.8471×10^{-5}

Continued

0.6	0.1080000	0.1113838	3.3838×10^{-3}	0.1079151	8.4876×10^{-5}
0.7	0.1470000	0.1509044	3.9044×10^{-3}	0.1468904	1.0951×10^{-4}
0.8	0.1920000	0.1965019	4.5019×10^{-3}	0.1919381	6.1870×10^{-5}
0.9	0.2430000	0.2483267	5.3267×10^{-3}	0.2431926	1.9269×10^{-4}
1	0.3000000	0.3057405	5.7405×10^{-3}	0.3000111	1.1596×10^{-5}

6. Conclusions

From the previous discussions we conclude the following:

1) As N is increasing the errors are decreasing.

2) As x and t are increasing in $[0,1] \times [0,1]$, the errors due to RKM and BBM are also increasing.

3) The errors due to the BBM are less than the errors due to RKM (*i.e.* BBM the better than RKM to solve NF-VIE with continuous kernel).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Linz, P. (1985) Analytic and Numerical Methods for Volterra Equations. SIAM, Philadelphia. https://doi.org/10.1137/1.9781611970852

[2] Mirzaee, F. and Rafei, Z. (2011) The Block by Block Method for the Numerical Solution of the Nonlinear Two-Dimensional Volterra Integral Equations. *Journal of king Saud University-Science*, **23**, 191-195. https://doi.org/10.1016/j.jksus.2010.07.008

[3] Maleknejad, K. and Shahrezaee, M. (2004) Using Runge-Kutta Method for Numerical Solution of the System of Volterra Integral Equation. *Applied Mathematics and Computation*, **149**, 399-410. https://doi.org/10.1016/S0096-3003(03)00148-6

[4] Maleknejad, K. and Ostadi, A. (2017) Numerical Solution of System of Volterra Integral Equations with Weakly Singular Kernels and Its Convergence Analysis. *Applied Numerical Mathematics*, **115**, 82-98. https://doi.org/10.1016/j.apnum.2016.12.005

[5] Armand, A. and Gouyandch, Z. (2014) Numerical Solution of the System of Volterra Integral Equations of the First Kind. *IJIM*, **6**, 9.

[6] Biazar, J. and Babolian, E. (2003) Solution of a System of Volterra Integral Equations of the First Kind by Adomian Method. *Applied Mathematics and Computation*, **139**, 249-258. https://doi.org/10.1016/S0096-3003(02)00173-X

[7] Yaghouti, M.R. (2012) A Numerical Method for Solving a System of Volterra Integral Equations. *WAP*, **2**, 18-33.

[8] Katani, R. and Shahmorad, S. (2010) Block by Block Method for the Systems of Nonlinear Volterra Integral Equations. *Applied Mathematical Modelling*, **34**, 400-406. https://doi.org/10.1016/j.apm.2009.04.013

[9] Al-Wagdani, M.M. (2016) On Some Numerical Treatments for Solving Volterra Integral Equation and System of Volterra Integral Equations. Taif University, Taif.