Journal of Applied Mathematics and Physics, 2020, 8, 2020-2030
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Some Connections in Almost Hermitian Manifold

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How to cite this paper: Kankarej, M. (2020) Some Connections in Almost Hermitian Manifold. *Journal of Applied Mathematics and Physics*, **8**, 2020-2030.
https://doi.org/10.4236/jamp.2020.89150

Received: August 12, 2020
Accepted: September 25, 2020
Published: September 28, 2020

Abstract

The idea of this research is to study different types of connections in an almost Hermite manifold. The connection has been established between linear connection and Riemannian connection. Three new linear connections $\nabla^1, \nabla^2, \nabla^3$ are introduced. The necessary and sufficient condition for $\nabla^1, \nabla^2, \nabla^3$ to be metric is discussed. A new metric $s^*(X,Y)$ has been defined for (M^n, F, g^*) and additional properties are discussed. It is also proved that for the quarter symmetric connection ∇ is unique in given manifold. The hessian operator with respect to all connections defined above has also been discussed.

Keywords

Almost Hermitian Manifold, Hessian Operator, Quarter Symmetric Metric Connection, Quarter Symmetric Non-Metric Connection

1. Introduction

The study of connection has been the field of interest for most of the mathematicians. The study of connections, semi symmetric connection was done in detail by Yano [1] followed by Konar and Chaki [2], De and Biswas [3], Pandey and Dubey [4], Pandey and Chaturvedi [5], Andonie [6] and many more, Quarter symmetric connection by Golab [7], Rastogi [8], Mishra and Pandey [9] [10], Biswas and De [11], De and Sengupta [12]. Quarter symmetric non-metric connection was studied in Riemannian, Kaehlerian and Sasakian manifolds. Quarter symmetric non-metric connection was studied in detail by Bhowmik [13], Mondal and De [14], Haseeb, Prakash and Siddiqi [15]. Kankarej [16] has studied the quarter symmetric non-metric connection in almost Hermitian Manifold. In this research three new types of connection have been discussed in almost Hermitian Manifold and the necessary and sufficient condition for it to be a metric has been discussed. A new metric has been defined and some additional properties with respect to the new metric is discussed.

This research has been divided in different sections. First section introduces some elementary definitions. Second section shows the relation between linear connection and Riemannian connection. Section three introduces some new connections in almost Hermitian Manifold and also the necessary conditions for new connections to be metric are established. Section four introduces a new metric ∇^*. Later in the same section some properties of new connections are proved with respect to new metric. In section five, Hessian operator is defined on all the connections established in section three.

Definition 1.1.

An even dimensional C^∞ differentiable manifold M^n is said to be an almost complex manifold (of class C^∞) if there exists a vector valued real linear function *f* of differentiable class C^∞ satisfying

$$f^2(X)+X=0$$

for any vector field *X*, *f* is said to give an almost complex structure on M^n.

Definition 1.2.

A metric g on an almost complex manifold M^n is said to be a Hermite metric if

$$g(\bar{X},\bar{Y})=g(FX,FY)=g(X,Y).$$

It is always possible to introduce a Hermite metric on an almost complex manifold.

An almost complex manifold M^n endowed with an almost complex structure F and a Hermite metric g is called an almost Hermite manifold with structure $\{F,g\}$ if

$$\bar{\bar{X}}+X=0 \tag{1.1}$$

and

$$g(\bar{X},\bar{Y})=g(X,Y), \tag{1.2}$$

where $\bar{X}=FX$, F is a tensor of type (1,1), g is a metric tensor and X,Y are arbitrary vector fields.

Definition 1.3.

In an almost Hermitian manifold there exists a unique torsion free metric connection D which is called Riemannian connection.

Riemannian connection D on an *n*-dimensional C^∞ Riemannian Manifold (M^n,g) is said to be a quarter symmetric connection if the torsion tensor S of D satisfies

$$S(X,Y)=\omega(Y)\Phi X-\omega(X)\Phi Y \tag{1.3}$$

where Φ is a tensor field of type $(1,1)$ and ω is a 1-form associated with vector field ρ

$$\omega(X)=g(X,\rho) \tag{1.4}$$

If the quarter symmetric connection D satisfies:

$$(Dg)(X,Y,Z)=0 \tag{1.5}$$

where $(Dg)(X,Y,Z)=D_X g(Y,Z)-g(D_X Y,Z)-g(Y,D_X Z)$.
then the connection D is said to be quarter symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection.

Definition 1.4.

A necessary and sufficient condition that vector field X on a Riemannian Manifold (M^n,g) is a Killing vector is that

$$g(D_Y X,Z)+g(Y,D_Z X)=0 \tag{1.6}$$

for any vector fields Y and Z. The connection D is unique in Riemannian manifold and is also called Levi-Civita connection on M^n.

2. Relation between the Riemannian Connection and a Linear Connection

The set of connections in M^n defines a unique (2,1) tensor B such that

$$\nabla_X Y=D_X Y+B(X,Y), \tag{2.1}$$

the tensor B is a subject to the requirement and ∇ is any linear connection

The torsion tensor of ∇ is

$$T(X,Y)=B(X,Y)-B(Y,X) \tag{2.2}$$

∇ is symmetric if $T(X,Y)=0$;
∇ is semi-symmetric if $T(X,Y)=\omega(X)Y-\omega(Y)X$;
∇ is quarter-symmetric if $T(X,Y)=\omega(X)\phi Y-\omega(Y)\phi X$;
where Φ is a tensor field of type $(1,1)$ and ω is a differential 1-form.

Theorem 2.1. ω being a differential 1-form is De Rham closed.

As ω is a differential 1-form it can be represented as

$$\omega=f_1\wedge \mathrm{d}x^1$$

$$\mathrm{d}\omega=\mathrm{d}f_1\wedge \mathrm{d}x^1+f_1\wedge \mathrm{dd}x^1=0+0=0$$

$$\mathrm{d}\omega=0 \quad \text{and} \quad \mathrm{d}^2\omega=0$$

ω is closed and also it is De Rham closed.

3. Some Connections on an Almost Hermite Manifold

Theorem 3.1. Let ∇^1 be the linear connection and D be a Riemannian con-

nection of a Hermite manifold $\{F, g\}$ such that

$$\nabla^1_X Y = D_X Y + \omega(X)Y + \omega(Y)X \tag{3.1}$$

where is a 1-form associated with vector field ρ and X and Y are vector fields. Then ∇^1 is a symmetric connection.

Proof: From (2.1) and (3.1) we have

$$B(X,Y) = \omega(X)Y + \omega(Y)X \tag{3.2}$$

Interchanging X and Y we have

$$B(Y,X) = \omega(Y)X + \omega(X)Y \tag{3.3}$$

From (3.2) and (3.3) we have

$$B(X,Y) - B(Y,X) = 0 \tag{3.4}$$

(2.2) and (3.4) show $T(X,Y) = 0$, which means ∇^1 is a symmetric connection.

Thus it is proved that ∇^1 is a symmetric connection.

Theorem 3.2. The necessary and sufficient condition for ∇^1 to be a metric connection is

$$\omega(Y)g(X,Z) + \omega(Z)g(Y,X) = 0 \tag{3.5}$$

Proof: We know

$$\begin{aligned}
&\left(\nabla^1 g\right)(X,Y,Z) = \nabla^1_X\left(g(Y,Z)\right) - g\left(\nabla^1_X Y, Z\right) - g\left(Y, \nabla^1_X Z\right) \\
&= D_X\left(g(Y,Z)\right) + \omega(X)g(Y,Z) + \omega\left(g(Y,Z)\right)X \\
&\quad - g\left(D_X Y + \omega(X)Y + \omega(Y)X, Z\right) - g\left(Y, D_X Z + \omega(X)Z + \omega(Z)X\right) \\
&= D_X\left(g(Y,Z)\right) + \omega(X)g(Y,Z) + \omega\left(g(Y,Z)\right)X - g\left(D_X Y, Z\right) - \omega(X)g(Y,Z) \\
&\quad - \omega(Y)g(X,Z) - g\left(Y, D_X Z\right) - \omega(X)g(Y,Z) - \omega(Z)g(Y,X) \\
&= D_X\left(g(Y,Z)\right) - g\left(D_X Y, Z\right) - g\left(Y, D_X Z\right) + \omega(X)g(Y,Z) + \omega\left(g(Y,Z)\right)X \\
&\quad - \omega(X)g(Y,Z) - \omega(Y)g(X,Z) - \omega(X)g(Y,Z) - \omega(Z)g(Y,X) \\
&= D_X\left(g(Y,Z)\right) - g\left(D_X Y, Z\right) - g\left(Y, D_X Z\right) + \omega(X)g(Y,Z) + \omega(X)\left(g(Y,Z)\right) \\
&\quad - \omega(X)g(Y,Z) - \omega(Y)g(X,Z) - \omega(X)g(Y,Z) - \omega(Z)g(Y,X) \\
&= -\omega(Y)g(X,Z) - \omega(Z)g(Y,X) + \left(D_X g\right)(Y,Z) - g\left(D_X Y, Z\right) - g\left(Y, D_X Z\right)
\end{aligned}$$

D being a Riemannian metric $\left(D_X g\right)(Y,Z) = g\left(D_X Y, Z\right) + g\left(Y, D_X Z\right)$

$$\left(\nabla^1 g\right)(X,Y,Z) = -\omega(Y)g(X,Z) - \omega(Z)g(Y,X) \tag{3.6}$$

Above equation proves that the given connection ∇^1 is non metric.

Necessary and sufficient condition for connection ∇^1 to be metric:

$$\left(\nabla^1 g\right)(X,Y,Z) = 0$$

implies $\omega(Y)g(X,Z) + \omega(Z)g(Y,X) = 0$ from (3.6).

Theorem 3.3. Let ∇^2 be the linear connection and D be a Riemannian con-

nection of a Hermite manifold $\{F,g\}$ such that

$$\nabla_X^2 Y = D_X Y + \omega(X)Y + s^*(X,Y) \tag{3.7}$$

Then ∇^2 is a semi symmetric connection.

From (2.1) and (3.7) we have

$$B(X,Y) = \omega(X)Y + s^*(X,Y) \tag{3.8}$$

assuming $s^*(X,Y)$ being symmetric.

Interchanging X and Y we have

$$B(Y,X) = \omega(Y)X + s^*(Y,X) \tag{3.9}$$

From (3.8) and (3.9)

$$B(X,Y) - B(Y,X) = \omega(X)Y - \omega(Y)X + s^*(X,Y) - s^*(Y,X)$$

$$B(X,Y) - B(Y,X) = \omega(X)Y - \omega(Y)X$$

$$T(X,Y) = \omega(X)Y - \omega(Y)X$$

which proves ∇^2 is a semi symmetric connection.

Theorem 3.4. The necessary and sufficient condition for ∇^2 to be a metric connection is

$$s^*(X, g(Y,Z)) = \omega(X)g(Y,Z) + g(Y, s^*(X,Z)) + g(s^*(X,Y), Z) \tag{3.10}$$

Proof: We know $(\nabla g)(X,Y,Z) = \nabla_X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$

$$\begin{aligned}
&(\nabla_X g)(Y,Z) \\
&= D_X g(Y,Z) + \omega(X)g(Y,Z) + s^*(X, g(Y,Z)) \\
&\quad - g(D_X Y + \omega(X)Y + s^*(X,Y), Z) - g(Y, D_X Z + \omega(X)Z + s^*(X,Z)) \\
&= D_X g(Y,Z) + \omega(X)g(Y,Z) + s^*(X, g(Y,Z)) - g(D_X Y, Z) \\
&\quad - \omega(X)g(Y,Z) - g(s^*(X,Y), Z) - g(Y, D_X Z) \\
&\quad - \omega(X)g(Y,Z) - g(Y, s^*(X,Z)) \\
&= D_X g(Y,Z) - g(Y, D_X Z) - g(D_X Y, Z) + \omega(X)g(Y,Z) - \omega(X)g(Y,Z) \\
&\quad + s^*(X, g(Y,Z)) - g(s^*(X,Y), Z) - \omega(X)g(Y,Z) - g(Y, s^*(X,Z))
\end{aligned}$$

As D is a Riemannian connection $D_X g(Y,Z) = g(Y, D_X Z) + g(D_X Y, Z)$

$$\begin{aligned}
&(\nabla^2 g)(X,Y,Z) \\
&= -\omega(X)g(Y,Z) + s^*(X, g(Y,Z)) - g(Y, s^*(X,Z)) - g(s^*(X,Y), Z)
\end{aligned} \tag{3.11}$$

It proves that ∇^2 defined in (3.7) is non metric.

Necessary and sufficient condition for ∇^2 to be metric:

$$(\nabla^2 g)(X,Y,Z) = 0$$

$$s^*(X, g(Y,Z)) = \omega(X)g(Y,Z) + g(Y, s^*(X,Z)) + g(s^*(X,Y), Z)$$

Theorem 3.5. Let ∇^3 be the linear connection and D be a Riemannian con-

nection of a Hermite manifold $\{F, g\}$ such that

$$\nabla_X Y = D_X Y + \omega(X)\phi Y + s^*(X,Y) \tag{3.12}$$

Then ∇^3 is a quarter symmetric connection.

From (2.1) and (3.12) we have

$$B(X,Y) = \omega(X)\phi Y + s^*(X,Y) \tag{3.13}$$

assuming $s^*(X,Y)$ being symmetric.

Interchanging X and Y we have

$$B(Y,X) = \omega(Y)\phi X + s^*(Y,X) \tag{3.14}$$

From (3.13) and (3.14)

$$B(X,Y) - B(Y,X) = \omega(X)\phi Y - \omega(Y)\phi X + s^*(X,Y) - s^*(Y,X)$$

$$B(X,Y) - B(Y,X) = \omega(X)\phi Y - \omega(Y)\phi X$$

$$T(X,Y) = \omega(X)\phi Y - \omega(Y)\phi X$$

which proves ∇^3 is a quarter symmetric connection.

Theorem 3.6. The necessary and sufficient condition for ∇^3 to be a metric connection is

$$s^*(X, g(Y,Z)) = g(Y, s^*(X,Z)) + g(s^*(X,Y), Z) \tag{3.15}$$

Proof: We know $(\nabla g)(X,Y,Z) = \nabla_X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$

$$\begin{aligned}
&(\nabla g)(X,Y,Z) \\
&= D_X g(Y,Z) + \omega(X)\phi(g(Y,Z)) + s^*(X, g(Y,Z)) \\
&\quad - g(D_X Y + \omega(X)\phi Y + s^*(X,Y), Z) - g(Y, D_X Z + \omega(X)\phi Z + s^*(X,Z)) \\
&= D_X g(Y,Z) + \omega(X)\phi g(Y,Z) + s^*(X, g(Y,Z)) - g(D_X Y, Z) \\
&\quad - \omega(X) g(\phi Y, Z) - g(s^*(X,Y), Z) - g(Y, D_X Z) \\
&\quad - \omega(X) g(Y, \phi Z) - g(Y, s^*(X,Z)) \\
&= D_X g(Y,Z) - g(Y, D_X Z) - g(D_X Y, Z) + \omega(X)\phi(g(Y,Z)) \\
&\quad - \omega(X) g(\phi Y, Z) - g(s^*(X,Y), Z) - \omega(X) g(Y, \phi Z) \\
&\quad - g(Y, s^*(X,Z)) + s^*(X, g(Y,Z))
\end{aligned}$$

As D is a Riemannian connection $D_X g(Y,Z) = g(Y, D_X Z) + g(D_X Y, Z)$

$$\begin{aligned}
(\nabla g)(X,Y,Z) &= \omega(X)\phi(g(Y,Z)) - \omega(X)\{g(\phi Y, Z) + g(Y, \phi Z)\} \\
&\quad + s^*(X, g(Y,Z)) - g(Y, s^*(X,Z)) - g(s^*(X,Y), Z)
\end{aligned} \tag{3.16}$$

By the property of ϕ we have $\phi(g(Y,Z)) = g(\phi Y, Z) + g(Y, \phi Z)$

$$(\nabla g)(X,Y,Z) = s^*(X, g(Y,Z)) - g(Y, s^*(X,Z)) - g(s^*(X,Y), Z)$$

It proves that ∇^3 defined by (3.12) is non metric.

Necessary and sufficient condition for ∇^3 to be metric:

$$\left(\nabla^3 g\right)(X,Y,Z)=0$$

$$s^\star\left(X,g(Y,Z)\right)=g\left(Y,s^\star(X,Z)\right)+g\left(s^\star(X,Y),Z\right)$$

4. Existence of a New Metric $g^\star$

Theorem 4.1. If there exist a quarter symmetric non metric connection in $\{M^n,F,g\}$ then there exist another metric $g^\star$ such that the connection is metric in $\{M^n,F,g^\star\}$.

Proof: Let us define a metric $g^\star$ in the Hermitian metric such that:

$$s^\star\left(X,g^\star(Y,Z)\right)=g^\star\left(s^\star(X,Y),Z\right)+g^\star\left(Y,s^\star(X,Z)\right) \tag{4.1}$$

Then using ∇^3, $\nabla^3_X Y=D_X Y+\omega(X)Y+s^\star(X,Y)$

We know

$$\left(\nabla^3 g^\star\right)(X,Y,Z)=\nabla^3_X\left(g^\star(Y,Z)\right)-g^\star\left(\nabla_X Y,Z\right)-g^\star\left(Y,\nabla_X Z\right)$$

$$\begin{aligned}
&\left(\nabla^3 g^\star\right)(X,Y,Z)\\
&=D_X g^\star(Y,Z)+\omega(X)\phi\left(g^\star(Y,Z)\right)+s^\star\left(X,g^\star(Y,Z)\right)\\
&\quad-g^\star\left(D_X Y+\omega(X)\phi Y+s^\star(X,Y),Z\right)-g^\star\left(Y,D_X Z+\omega(X)\phi Z+s^\star(X,Z)\right)\\
&=D_X g^\star(Y,Z)+\omega(X)\phi g^\star(Y,Z)+s^\star\left(X,g^\star(Y,Z)\right)-g^\star\left(D_X Y,Z\right)\\
&\quad-\omega(X)g^\star(\phi Y,Z)-g^\star\left(s^\star(X,Y),Z\right)-g^\star\left(Y,D_X Z\right)\\
&\quad-\omega(X)g^\star(Y,\phi Z)-g^\star\left(Y,s^\star(X,Z)\right)\\
&=D_X g^\star(Y,Z)-g^\star\left(Y,D_X Z\right)-g^\star\left(D_X Y,Z\right)+\omega(X)\phi\left(g^\star(Y,Z)\right)\\
&\quad-\omega(X)g^\star(\phi Y,Z)-g^\star\left(s^\star(X,Y),Z\right)-\omega(X)g^\star(Y,\phi Z)\\
&\quad-g^\star\left(Y,s^\star(X,Z)\right)+s^\star\left(X,g^\star(Y,Z)\right)
\end{aligned}$$

As D is a Riemannian connection $D_X g(Y,Z)=g\left(Y,D_X Z\right)+g\left(D_X Y,Z\right)$

Using (3.16)

$$\begin{aligned}
\left(\nabla^3 g^\star\right)(X,Y,Z)&=\omega(X)\phi\left(g^\star(Y,Z)\right)-\omega(X)\left\{g^\star(\phi Y,Z)+g^\star(Y,\phi Z)\right\}\\
&\quad+s^\star\left(X,g^\star(Y,Z)\right)-g^\star\left(Y,s^\star(X,Z)\right)-g^\star\left(s^\star(X,Y),Z\right)
\end{aligned}$$

By the property of ϕ we have $\phi\left(g^\star(Y,Z)\right)=g^\star(\phi Y,Z)+g^\star(Y,\phi Z)$

$$\left(\nabla^3 g^\star\right)(X,Y,Z)=s^\star\left(X,g^\star(Y,Z)\right)-g^\star\left(Y,s^\star(X,Z)\right)-g^\star\left(s^\star(X,Y),Z\right)$$

By theorem 3.5, we have $\left(\nabla^3 g^\star\right)(X,Y,Z)=0$.

Thus we have proved that if there is a connection, ∇^3 in $\{M^n,F,g\}$ which is not metric then it is metric in $\{M^n,F,g^\star\}$.

Necessary and sufficient condition for $\left(\nabla^3 g^\star\right)(X,Y,Z)$ to be metric is:

$$s^\star\left(X,g^\star(Y,Z)\right)-g^\star\left(Y,s^\star(X,Z)\right)-g^\star\left(s^\star(X,Y),Z\right)=0$$

$$s^\star\left(X,g^\star(Y,Z)\right)=g^\star\left(Y,s^\star(X,Z)\right)+g^\star\left(s^\star(X,Y),Z\right)$$

Theorem 4.2. For every quarter symmetric metric connection ∇ in

$\{M^n, F, g\}$ there exist a unique connection $\nabla^\star$ such that it is quarter symmetric also.

Proof: Let us assume that ∇ is quarter symmetric and metric in $\{M^n, F, g\}$ and define a connection $\nabla^\star$ such that

$$g(\nabla_X^\star Y, Z) = Xg(Y,Z) - g(Y, \nabla_X Z) \tag{4.2}$$

Let us check if $\nabla^\star$ is quarter symmetric.

$$\begin{aligned}
(\nabla^\star g)(X,Y,Z) &= \nabla_X^\star g(Y,Z) - g(\nabla_X^\star Y, Z) - g(Y, \nabla_X^\star Z) \\
&= \nabla_X^\star g(Y,Z) - \{Xg(Y,Z) - g(Y,\nabla_X Z)\} - \{Xg(Y,Z) - g(\nabla_X Y, Z)\} \\
&= \nabla_X^\star g(Y,Z) - \nabla_X g(Y,Z) + g(Y,\nabla_X Z) - \nabla_X g(Y,Z) + g(\nabla_X Y, Z)
\end{aligned}$$

$$\begin{aligned}
&= \nabla_X^\star g(Y,Z) - \{\nabla_X g(Y,Z) - g(Y,\nabla_X Z) - g(\nabla_X Y, Z)\} - \nabla_X g(Y,Z) \\
&= \nabla_X^\star g(Y,Z) - \{(\nabla g)(X,Y,Z)\} - \nabla_X g(Y,Z) \\
&= \nabla_X^\star g(Y,Z) - \{(\nabla g)(X,Y,Z)\} - \nabla_X g(Y,Z)
\end{aligned}$$

By the assumption made in connection 4 implies that $(\nabla g)(X,Y,Z) = 0$

$$(\nabla^\star g)(X,Y,Z) = \nabla_X^\star g(Y,Z) - \nabla_X g(Y,Z)$$

As g is a metric so, $(\nabla^\star g)(X,Y,Z) = 0$.

So, $\nabla_X^\star g(Y,Z) = \nabla_X g(Y,Z)$.

This proves that $\nabla = \nabla^\star$.

Hence there exist a unique connection on $\{M^n, F, g\}$ which is metric and quarter symmetric.

5. Hessian Operator on Different Connections

In this section we introduce the notion of hessian operator on different connections.

Definition 5.1. The Hessian of a smooth function $f : M^n \to R$ on a smooth manifold with a connection ∇ is the covariant derivative of the function *f*, such that

$$Hess(f) = \nabla\nabla f \tag{5.1}$$

so that $Hess(f) \in \Gamma(T^\star M^n \otimes T^\star M^n)$, it is a (0,2) tensor field on M^n.

For any two vector fields X and Y on M^n, we have a smooth real valued function.

$Hess(f)(X,Y) = \nabla\nabla f(X,Y)$ on manifold M^n, which is a bilinear function.

It is defined as

$$Hess(f)(X,Y) = X(Yf) - (\nabla_X Y)f \tag{5.2}$$

Theorem 5.1. For every connection in an almost hermitian manifold there exist a unique hessian operator.

$$Hess(f')(X,Y) = Hess(f)(X,Y)$$

Proof: Let us define connection ∇^1, ∇^2 and ∇^3 as mentioned above in (3.1), (3.7) and (3.12). Then we have

$$\nabla^1_X Y = D_X Y + \omega(X)Y + \omega(Y)X$$

$$\nabla^2_X Y = D_X Y + \omega(X)Y + s^\star(X,Y)$$

$$\nabla^3_X Y = D_X Y + \omega(X)\phi Y + s^\star(X,Y)$$

By (5.2) we can say that for a Riemannian connection D in an almost hermitian manifold we have

$$Hess(f)(X,Y) = D_X(Yf) - (D_X Y)f \tag{5.3}$$

The hessian operator for connection (3.1) corresponds

$$\begin{aligned} Hess(f^1)(X,Y) &= \nabla^1_X(Yf) - (\nabla^1_X Y)f \\ &= \{D_X Yf + \omega(X)Yf + \omega(Yf)X\} - \{D_X Y + \omega(X)Y + \omega(Y)X\}f \\ &= D_X(Yf) - (D_X Y)f + \omega(X)Yf - \omega(X)Yf + \omega(Yf)X - \omega(Y)Xf \end{aligned}$$

$Hess(f^1)(X,Y) = Hess(f)(X,Y)$ using (5.3) and property of 1-form ω.

Thus for connection in (3.1) above theorem holds true.

The hessian operator for connection (3.7) corresponds

$$\begin{aligned} Hess(f^2)(X,Y) &= \nabla^2_X(Yf) - (\nabla^2_X Y)f \\ &= \{D_X Yf + \omega(X)Yf + s^\star(X,Yf)\} - \{D_X Y + \omega(X)Y + s^\star(X,Y)\}f \\ &= D_X(Yf) - (D_X Y)f + \omega(X)Yf - \omega(X)Yf + s^\star(X,Yf) - s^\star(X,Y)f \end{aligned}$$

$Hess(f^2)(X,Y) = Hess(f)(X,Y)$ using (5.3) and property of 1-form ω.

Thus for connection in (3.7) above theorem holds true.

The hessian operator for connection (3.12) corresponds

$$\begin{aligned} Hess(f^3)(X,Y) &= \nabla^3_X(Yf) - (\nabla^3_X Y)f \\ &= \{D_X Yf + \omega(X)\phi Yf + s^\star(X,Yf)\} - \{D_X Y + \omega(X)\phi Y + s^\star(X,Y)\}f \\ &= D_X(Yf) - (D_X Y)f + \omega(X)\phi Yf - \omega(X)\phi(Y)f + s^\star(X,Yf) - s^\star(X,Y)f \end{aligned}$$

$Hess(f^3)(X,Y) = Hess(f)(X,Y)$ using (5.3) and property of 1-form ω and the tensor field ϕ of type (1,1).

Thus for connection in (3.12) above theorem holds true.

6. Conclusion

Thus in this paper I have introduced three new kind of connections $\nabla^1_X Y = D_X Y + \omega(X)Y + \omega(Y)X$, $\nabla^2_X Y = D_X Y + \omega(X)Y + s^\star(X,Y)$, $\nabla^3_X Y = D_X Y + \omega(X)\phi Y + s^\star(X,Y)$ and proved the necessary conditions for all to be metric are discussed. Also I have proved that in almost Hermitian manifold linear connection is unique by introducing a new metric $g^\star$. It is also discussed that Hessian operator in almost hermitian manifold is unique.

Acknowledgements

I am thankful to Dr. Michel Nguiff Boyom for his valuable suggestions while writing this article.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Notations and Symbols

C^{∞} = Smooth manifold

M^{n} = An even dimensional smooth differentiable manifold

X, Y = Vector fields

f= A vector valued real linear function F= A tensor of type (1,1)

g= Metric tensor

$g^{\star}$ = New kind of metric

$\{F, g\}$ = A Hermite manifold with structure g

D= Riemannian connection

$S(X, Y)$ = Tensor

$s^{\star}(X, Y)$ = Tensor

$T(X, Y)$ = Torsion tensor

$B(X, Y)$ = Tensor

ω = Differential 1-form

ρ = Vector field

Φ = Tensor field of type (1,1)

∇ = Linear connection

∇^{1} = 1st kind of linear connection

∇^{2} = 2nd kind of linear connection

∇^{3} = 3rd kind of linear connection

$\nabla^{\star}$ = New kind of connection