# PNP Models with the Same Positively Charged Valence

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## Abstract

A steady-state Poisson-Nernst-Planck model with $n$ ion species is studied under the assumption that $n-1$ positively charged ion species have the same valence and there is only one negatively charged ion species. By re-scaling, this model can be viewed as a standard singularly perturbed system. Based on the explicit formulae for the solutions of its limit slow system and singular perturbation methods, the existence of the solutions is analyzed.

## Keywords

Poisson-Nernst-Planck Model, Singular Perturbation, Boundary Layer

## 1. Introduction

The Poisson-Nernst-Planck model is a well-known model of ion transport, which plays a crucial role in the study of many biological and physical problems, such as ion channels in cell membranes [1] [2] and semiconductor devices [3].

A steady-state Poisson-Nernst-Planck model [4] [5] [6] is

$$\frac{1}{h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon_r\varepsilon_0 h(x)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) = -e\left(\sum_{j=1}^{n} z_j c_j(x) + Q(x)\right),$$

$$\frac{\mathrm{d}\mathcal{J}_i}{\mathrm{d}x} = 0, \quad -\mathcal{J}_i = \frac{1}{kT}D_i h(x)c_i(x)\frac{\mathrm{d}\mu_i}{\mathrm{d}x}, \quad i = 1,2,\cdots,n, \tag{1.1}$$

where $\Phi$ is the electric potential; $c_i$ is the concentration for the $i$th ion species; $z_i$ is the valence; $Q(x)$ is the permanent charge of the channel; $\mu_i(x)$ is the electrochemical potential; $h(x)$ is the area of the cross-section of the channel; $\mathcal{J}_i$ is the flux density; $D_i$ is the diffusion coefficient; $\varepsilon_r$ is the relative dielectric coefficient; $\varepsilon_0$ is the vacuum permittivity; $k$ is the Boltzmann constant; $T$ is the absolute temperature; and $e$ is the elementary charge.

The boundary conditions are, for $i = 1,2,\cdots,n$,

$$\Phi(0) = V, \ c_i(0) = L_i; \quad \Phi(1) = 0, \ c_i(1) = R_i. \tag{1.2}$$

$\mu_i(x)$ in the classical Poisson-Nernst-Planck model takes the following form

$$\mu_i(x) = z_i e\phi(x) + kT \ln \frac{c_i(x)}{c_0} \tag{1.3}$$

with $c_0$ is a constant.

Migration of charges for ionic flow through ion channels is often described mathematically by the Poisson-Nernst-Planck model (1.1), which can be viewed as a simplified version of the Maxwell-Boltzmann equations [7] [8] and the Langevin-Poisson equations [9] [10] by focusing on the key features of biological functions. Recently, the model (1.1) has been greatly studied [11]-[17]. In [18], the author obtained the existence and uniqueness of solutions for systems (1.1) and (1.2) under the assumption that $Q(x) = 0$ and $n = 2$. In [19], the authors completely solved the existence and uniqueness of solutions for the boundary value problems (1.1) and (1.2) under the assumption that $Q(x) = 0$ and $n$ ions with the different valences are involved. However, the analysis of the dynamics of Poisson-Nernst-Planck model with nonzero permanent charges $Q(x)$ is much more difficult. In [20], the authors justified the existence of multiple solutions for the boundary value problems (1.1) and (1.2) under the assumption that $Q(x)$ is a piecewise constant function and $n = 2$. For the case that $n$ ions with the arbitrary valences are involved and $Q(x)$ is a piecewise constant function, the general geometric framework for analyzing the dynamics of systems (1.1) and (1.2) is provided in [21] based on the geometric singular perturbation theory [22] [23] [24]. In this paper, we intend to study the dynamics of the classical Poisson-Nernst-Planck model under the following hypotheses.

(H1). $z_1 = z_2 = \cdots = z_{n-1} = z > 0$ and $z_n < 0$.

(H2). $Q(x) = 0$ for $0 < x < a$, $Q(x) = Q$ for $a < x < b$ and $Q(x) = 0$ for $b < x < 1$, where $Q$ is a constant.

By re-scaling,

$$\phi = \frac{e}{kT}\Phi, \ \bar{V} = \frac{e}{kT}V, \ \varepsilon^2 = \frac{\varepsilon_r \varepsilon_0 kT}{e^2}, \ J_i = \frac{\mathcal{J}_i}{D_i}.$$

The model (1) is reduced to a standard singularly perturbed system of the following

$$\frac{\varepsilon^2}{h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi\right) = -\left[zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x)\right],$$

$$h(x)\left(\frac{dc_1}{dx} + zc_1\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_1,$$

$$\vdots$$

$$h(x)\left(\frac{\mathrm{d}c_{n-1}}{\mathrm{d}x} + zc_{n-1}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_{n-1}, \tag{1.4}$$

$$h(x)\left(\frac{\mathrm{d}c_n}{\mathrm{d}x} + z_n c_n\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_n,$$

$$\frac{\mathrm{d}J_1}{\mathrm{d}x} = \cdots = \frac{\mathrm{d}J_n}{\mathrm{d}x} = 0,$$

with the boundary condition, for $j = 1, \cdots, n$.

$$\phi(0) = \overline{V}, \quad c_j(0) = L_j, \quad \phi(1) = 0, \quad c_j(1) = R_j. \tag{1.5}$$

Actually, for the case $n = 2$, system (1.4) and (1.5) corresponds to the equations studied in [20]. Additionally, due to the above hypothesis (H1), system (1.4) and (1.5) is a special case of the equations studied in [21], but in this special case, the explicit formulae for the solutions of its limit slow system can be obtained, which is crucial for the analysis of the existence of solutions for systems (1.4) and (1.5) in this paper by combining the technique of the geometric singular perturbation theory.

## 2. Limiting Fast Orbits and Limiting Slow Orbits on [0, 1]

Let $u = \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}x} \phi$, $\tau = x$. System (1.4) becomes

$$
\begin{aligned}
&\varepsilon \dot{\phi} = u, \quad \varepsilon \dot{u} = -\left[ zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x) \right] - \varepsilon h^{-1}(\tau) h_\tau(\tau) u, \\
&\varepsilon \dot{c}_1 = -zc_1 u - \varepsilon h_\tau(\tau) J_1, \\
&\qquad \vdots \\
&\varepsilon \dot{c}_{n-1} = -zc_{n-1} u - \varepsilon h_\tau(\tau) J_{n-1}, \\
&\varepsilon \dot{c}_n = -z_n c_n u - \varepsilon h_\tau(\tau) J_n, \\
&\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.
\end{aligned}
\tag{2.6}
$$

By using the rescaling $x = \varepsilon \xi$, one has

$$
\begin{aligned}
&\phi' = u, \quad u' = -\left[ zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x) \right] - \varepsilon h^{-1}(\tau) h_\tau(\tau) u, \\
&c_1' = -zc_1 u - \varepsilon h_\tau(\tau) J_1, \\
&\qquad \vdots \\
&c_{n-1}' = -zc_{n-1} u - \varepsilon h_\tau(\tau) J_{n-1}, \\
&c_n' = -z_n c_n u - \varepsilon h_\tau(\tau) J_n, \\
&J_1' = 0, \cdots, J_n' = 0, \quad \tau' = \varepsilon.
\end{aligned}
\tag{2.7}
$$

Define

$$
\begin{aligned}
B_L &= \left\{ \left( \overline{V}, u, L_1, \cdots, L_n, J_1, \cdots, J_n, 0 \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}, \\
B_R &= \left\{ \left( 0, u, R_1, \cdots, R_n, J_1, \cdots, J_n, 1 \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}.
\end{aligned}
\tag{2.8}
$$

Due to the fact that $Q(x)$ is a piecewise constant function, therefore, we identify the limiting fast and slow orbits on three intervals $[0, a]$, $[a, b]$ and $[b, 1]$ respectively.

### 2.1. Limiting Fast Orbits on [0, a]

Let $\phi(a) = \phi^a$, $c_1(a) = c_1^a$, $\cdots$, $c_n(a) = c_n^a$, where $\phi^a, c_1^a, \cdots, c_n^a$ are unknowns to be determined later. Let

$$B_a = \left\{ \left( \phi^a, u, c_1^a, \cdots, c_n^a, J_1, \cdots, J_n, a \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}.$$

We will identify the limiting fast and limiting slow orbits connecting $B_L$ to

$B_a$ on the interval $[0, a]$, where $Q(x) = 0$. Letting $\varepsilon = 0$ in (2.7), one gets the limiting fast system

$$
\begin{aligned}
&\phi' = u, \quad u' = -zc_1 - \cdots - zc_{n-1} - z_n c_n, \\
&c_1' = -zc_1 u, \\
&\quad\vdots \\
&c_{n-1}' = -zc_{n-1} u, \\
&c_n' = -z_n c_n u, \\
&J_1' = 0, \cdots, J_n' = 0, \quad \tau' = 0.
\end{aligned}
\tag{2.9}
$$

By letting $\varepsilon = 0$ in (2.6), we obtain the critical manifold

$$
\mathcal{Z}_l = \{ u = 0,\ zc_1 + \cdots + zc_{n-1} + z_n c_n = 0 \},
\tag{2.10}
$$

which is normally hyperbolic.

The flow of $B_L$ under system (2.9) in forward time is denoted by $M_L$, and the flow of $B_a$ under system (2.9) in backward time is denoted by $M_l^a$. Then the following results can be established [20] [21].

**Lemma 2.1.** *System* (2.9) *has the following $n+1$ nontrivial first integrals:*

$$
H_1 = \mathrm{e}^{z\phi} c_1, \cdots, H_{n-1} = \mathrm{e}^{z\phi} c_{n-1}, H_n = \mathrm{e}^{z_n \phi} c_n, H_{n+1} = c_1 + \cdots + c_n - \frac{1}{2} u^2.
$$

**Proposition 2.2.** (i) *The stable manifold $W^s(\mathcal{Z}_l)$ intersects $B_L$ transversally at points*

$$
(\overline{V}, u_0, L_1, \cdots, L_n, J_1, \cdots, J_n, 0),
$$

*and the $\omega$-limit set of $N_L = M_L \cap W^s(\mathcal{Z}_l)$ is*

$$
\omega(N_L) = \left\{ \left( \phi^L, 0, c_1^L, \cdots, c_n^L, J_1, \cdots, J_n, 0 \right) \right\},
$$

*where $J_i$ for $i = 1, \cdots, n$ are arbitrary, and*

$$
\phi^L = \overline{V} - \frac{1}{z - z_n} \ln \frac{-z_n L_n}{z(L_1 + \cdots + L_{n-1})},
$$

$$
c_1^L = L_1 \left[ \frac{-z_n L_n}{z(L_1 + \cdots + L_{n-1})} \right]^{\frac{z}{z - z_n}}, \cdots, c_n^L = L_n \left[ \frac{-z_n L_n}{z(L_1 + \cdots + L_{n-1})} \right]^{\frac{z_n}{z - z_n}},
$$

$$
u_0 = \mathrm{sgn}\left( \phi^L - \overline{V} \right) \sqrt{2 \left[ L_1 + \cdots + L_n - \left( c_1^L + \cdots + c_n^L \right) \right]}.
$$

(ii) *The unstable manifold $W^u(\mathcal{Z}_l)$ intersects $B_a$ transversally at points*

$$
\left( \phi^a, u_l(a), c_1^a, \cdots, c_n^a, J_1, \cdots, J_n, a \right),
$$

*and the $\alpha$-limit set of $N_l^a = M_l^a \cap W^u(\mathcal{Z}_l)$ is*

$$
\alpha(N_l^a) = \left\{ \left( \phi^{a,l}, 0, c_1^{a,l}, \cdots, c_n^{a,l}, J_1, \cdots, J_n, a \right) \right\},
$$

*where $J_i$ for $i = 1, \cdots, n$ are arbitrary, and*

$$
\phi^{a,l} = \phi^a - \frac{1}{z - z_n} \ln \frac{-z_n c_n^a}{z(c_1^a + \cdots + c_{n-1}^a)},
$$

$$c_1^{a,l} = c_1^a \left[ \frac{-z_n c_n^a}{z \left( c_1^a + \cdots + c_{n-1}^a \right)} \right]^{\frac{z}{z-z_n}}, \cdots, c_n^{a,l} = c_n^a \left[ \frac{-z_n c_n^a}{z \left( c_1^a + \cdots + c_{n-1}^a \right)} \right]^{\frac{z_n}{z-z_n}},$$

$$u_l(a) = \mathrm{sgn}\left( \phi^a - \phi^{a,l} \right) \sqrt{2 \left[ c_1^a + \cdots + c_n^a - \left( c_1^{a,l} + \cdots + c_n^{a,l} \right) \right]}.$$

**Remark 2.3.** At $x = 0$, the limiting fast orbits $\Gamma^0 \subset N_L$ are a segment connecting $\left( \overline{V}, u_0, L_1, \cdots, L_n, J_1, \cdots, J_n, 0 \right)$ to $\omega(N_L)$, and at $x = a$ the limiting fast orbits $\Gamma_l^a \subset N_l^a$ are a segment connecting $\left( \phi^a, u_l(a), c_1^a, \cdots, c_n^a, J_1, \cdots, J_n, a \right)$ to $\alpha\left( N_l^a \right)$.

## 2.2. Limiting Slow Orbits on [0, a]

Now we identify the limiting slow orbits $\Lambda_l$ on the critical manifold $\mathcal{Z}_l$. By using a rescaling

$$u = \varepsilon p, \; z c_1 + \cdots + z c_{n-1} + z_n c_n = -\varepsilon q.$$

System (2.6) becomes

$$\dot{\phi} = p, \quad \varepsilon \dot{p} = q - \varepsilon h^{-1}(\tau) h_\tau(\tau) p,$$
$$\varepsilon \dot{q} = \left[ z(z - z_n)(c_1 + \cdots + c_{n-1}) - \varepsilon z_n q \right] p + \left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right] h^{-1}(\tau),$$
$$\dot{c}_1 = -z c_1 p - J_1 h^{-1}(\tau), \cdots, \dot{c}_{n-1} = -z c_{n-1} p - J_{n-1} h^{-1}(\tau),$$
$$\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1,$$

(2.11)

where $c_n = \dfrac{-z c_1 - \cdots - z c_{n-1} - \varepsilon q}{z_n}$, and its limiting slow system is

$$\dot{\phi} = p, \quad q = 0, \quad p = -\frac{h^{-1}(\tau)\left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right]}{z(z - z_n)(c_1 + \cdots + c_{n-1})},$$
$$\dot{c}_1 = -z c_1 p - J_1 h^{-1}(\tau), \cdots, \dot{c}_{n-1} = -z c_{n-1} p - J_{n-1} h^{-1}(\tau),$$
$$\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.$$

(2.12)

For system (2.12), the critical manifold is

$$\mathcal{S}_l = \left\{ q = 0, \; p = -\frac{h^{-1}(\tau)\left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right]}{z(z - z_n)(c_1 + \cdots + c_{n-1})} \right\},$$

where $c_3 = \dfrac{-z c_1 - \cdots - z c_{n-1}}{z_n}$.

It follows that the limiting slow system on $\mathcal{S}_l$ is

$$\dot{\phi} = -\frac{h^{-1}(\tau)\left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right]}{z(z - z_n)(c_1 + \cdots + c_{n-1})},$$
$$\dot{c}_1 = \frac{h^{-1}(\tau)\left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right] c_1}{(z - z_n)(c_1 + \cdots + c_{n-1})} - J_1 h^{-1}(\tau),$$
$$\vdots$$
$$\dot{c}_{n-1} = \frac{h^{-1}(\tau)\left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right] c_{n-1}}{(z - z_n)(c_1 + \cdots + c_{n-1})} - J_{n-1} h^{-1}(\tau),$$
$$\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.$$

(2.13)

For convenience, we denote

$$H(x) = \int_0^x h^{-1}(s)\,\mathrm{d}s, \quad P_l(x) = 1 + \frac{z_n(J_1 + \cdots + J_n)}{(z - z_n)(c_1^L + \cdots + c_{n-1}^L)} H(x)$$

and

$$\lambda_l^i = \frac{c_i^{a,l} - c_i^L e^{z(\phi^L - \phi^{a,l})}}{c_1^{a,l} + \cdots + c_{n-1}^{a,l} - (c_1^L + \cdots + c_{n-1}^L) e^{z(\phi^L - \phi^{a,l})}}, \quad i = 1, \cdots, n-1.$$

**Lemma 2.4.** *There is a unique solution*
$(\phi(x), c_1(x), \cdots, c_{n-1}(x), J_1, \cdots, J_n, \tau(x))$ *of* (2.13) *such that*

$$(\phi(0), c_1(0), \cdots, c_{n-1}(0), \tau(0)) = (\phi^L, c_1^L, \cdots, c_{n-1}^L, 0) \in \omega(N_L)$$

*and* $(\phi(a), c_1(a), \cdots, c_{n-1}(a), \tau(a)) = (\phi^{a,l}, c_1^{a,l}, \cdots, c_{n-1}^{a,l}, a) \in \alpha(N_l^a),$

*where* $\phi^L, c_1^L, \cdots, c_{n-1}^L, \phi^{a,l}, c_1^{a,l}, \cdots, c_{n-1}^{a,l}$ *are given in Proposition 2.2. It is given by*

$$\phi(x) = \phi^L - \frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{zz_n(J_1 + \cdots + J_n)} \ln P_l(x),$$

$$c_1(x) = \frac{c_1^L(J_2 + \cdots + J_{n-1}) - (c_2^L + \cdots + c_{n-1}^L)J_1}{J_1 + \cdots + J_{n-1}} \left[P_l(x)\right]^{\frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z_n(J_1 + \cdots + J_n)}}$$

$$+ \frac{(c_1^L + \cdots + c_{n-1}^L)J_1}{J_1 + \cdots + J_{n-1}} P_l(x),$$

$$\vdots$$

$$c_{n-1}(x) = \frac{c_{n-1}^L(J_1 + \cdots + J_{n-2}) - (c_1^L \cdots + c_{n-2}^L)J_{n-1}}{J_1 + \cdots + J_{n-1}} \left[P_l(x)\right]^{\frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z_n(J_1 + \cdots + J_n)}}$$

$$+ \frac{(c_1^L + \cdots + c_{n-1}^L)J_{n-1}}{J_1 + \cdots + J_{n-1}} P_l(x),$$

$$J_1 = \frac{c_1^L + \cdots + c_{n-1}^L - (c_1^{a,l} + \cdots + c_{n-1}^{a,l})}{H(a)} \left[1 - \frac{z(\phi^L - \phi^{a,l})}{\ln \dfrac{c_1^{a,l} + \cdots + c_{n-1}^{a,l}}{c_1^L + \cdots + c_{n-1}^L}}\right] \lambda_l^1,$$

$$\vdots$$

$$J_{n-1} = \frac{c_1^L + \cdots + c_{n-1}^L - (c_1^{a,l} + \cdots + c_{n-1}^{a,l})}{H(a)} \left[1 - \frac{z(\phi^L - \phi^{a,l})}{\ln \dfrac{c_1^{a,l} + \cdots + c_{n-1}^{a,l}}{c_1^L + \cdots + c_{n-1}^L}}\right] \lambda_l^{n-1}, \quad (2.14)$$

$$J_n = \frac{z\left[c_1^{a,l} + \cdots + c_{n-1}^{a,l} - (c_1^L + \cdots + c_{n-}^L)\right]}{z_n H(a)} \left[1 - \frac{z_n(\phi^L - \phi^{a,l})}{\ln \dfrac{c_1^{a,l} + \cdots + c_{n-1}^{a,l}}{c_1^L + \cdots + c_{n-1}^L}}\right],$$

$$\tau(x) = x.$$

*Proof.* By system (2.13), it follows that

$$\dot{c}_1(x) + \dot{c}_2(x) + \cdots + \dot{c}_{n-1}(x) = \frac{z_n}{z - z_n}(J_1 + J_1 + \cdots + J_n)h^{-1}(x).$$

Therefore,

$$c_1(x) + c_2(x) + \cdots + c_{n-1}(x)$$
$$= c_1^L + c_2^L + \cdots + c_{n-1}^L + \frac{z_n}{z - z_n}(J_1 + J_1 + \cdots + J_n)H(x).$$

By inserting the above formula for $c_1(x) + c_2(x) + \cdots + c_{n-1}(x)$ into system (2.13) and using the variation of constants formula, the formulas for

$$c_1(x), \cdots, c_{n-1}(x), \phi(x), J_1, \cdots, J_n$$

in the statement can be obtained. □

### 2.3. Limiting Fast Orbits on [a, b]

Let $\phi(b) = \phi^b$, $c_1(b) = c_1^b$, $\cdots$, $c_n(b) = c_n^b$, where $\phi^b, c_1^b, \cdots, c_n^b$ are unknowns to be determined later. Let

$$B_b = \left\{ \left( \phi^b, u, c_1^b, \cdots, c_n^b, J_1, \cdots, J_n, b \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}.$$

We will identify the limiting fast and limiting slow orbits connecting $B_a$ to $B_b$ on the interval $[a, b]$, where $Q(x) = Q$. The limiting fast system is obtained by letting $\varepsilon = 0$ in (2.7):

$$\begin{aligned}
&\phi' = u, \quad u' = -zc_1 - \cdots - zc_{n-1} - z_n c_n - Q, \\
&c_1' = -zc_1 u, \\
&\qquad \vdots \\
&c_{n-1}' = -zc_{n-1}u, \\
&c_n' = -z_n c_n u, \\
&J_{1'} = 0, \cdots, J_{n'} = 0, \quad \tau' = 0.
\end{aligned} \qquad (2.15)$$

By letting $\varepsilon = 0$ in (2.6), we obtain the critical manifold

$$\mathbb{Z}_m = \left\{ u = 0, \ zc_1 + \cdots + zc_{n-1} + z_n c_n + Q = 0 \right\}, \qquad (2.16)$$

which is normally hyperbolic.

The flow of $B_a$ under system (2.15) in forward time is denoted by $M_m^a$, and the flow of $B_b$ under system (2.15) in backward time is denoted by $M_m^b$. Then the following results can be established [20] [21].

**Lemma 2.5.** *System* (2.15) *has the following* $n+1$ *nontrivial first integrals:*

$$H_1 = e^{z\phi}c_1, \cdots, H_{n-1} = e^{z\phi}c_{n-1}, H_n = e^{z_n\phi}c_n, H_{n+1} = c_1 + \cdots + c_n - \frac{1}{2}u^2 - Q\phi.$$

**Proposition 2.6.** (i) *Letting* $\phi = \phi^{a,m}$ *be the unique solution of*

$$zc_1^a e^{z(\phi^a - \phi)} + \cdots + zc_{n-1}^a e^{z(\phi^a - \phi)} + z_n c_n^a e^{z_n(\phi^a - \phi)} + Q = 0,$$

*and letting*

$$c_1^{a,m} = c_1^a e^{z(\phi^a - \phi^{a,m})}, \cdots, c_{n-1}^{a,m} = c_{n-1}^a e^{z(\phi^a - \phi^{a,m})}, c_n^{a,m} = c_n^a e^{z_n(\phi^a - \phi^{a,m})},$$

$$u_m(a) = \text{sgn}\left(\phi^{a,m} - \phi^a\right)\sqrt{2\left[c_1^a + \cdots + c_n^a - \left(c_1^{a,m} + \cdots + c_n^{a,m}\right) - Q\left(\phi^a - \phi^{a,m}\right)\right]}.$$

The stable manifold $W^s\left(\mathcal{Z}_m\right)$ intersects $B_a$ transversally at points

$$\left(\phi^a, u_m(a), c_1^a, \cdots, c_n^a, J_1, \cdots, J_n, a\right),$$

and the $\omega$-limit set of $N_m^a = M_m^a \bigcap W^s\left(\mathcal{Z}_m\right)$ is

$$\omega\left(N_m^a\right) = \left\{\left(\phi^{a,m}, 0, c_1^{a,m}, \cdots, c_n^{a,m}, J_1, \cdots, J_n, a\right)\right\},$$

where $J_i$ for $i = 1, \cdots, n$ are arbitrary.

(ii) Letting $\phi = \phi^{b,m}$ be the unique solution of

$$zc_1^b e^{z\left(\phi^b - \phi\right)} + \cdots + zc_{n-1}^b e^{z\left(\phi^b - \phi\right)} + z_n c_n^b e^{z_n\left(\phi^b - \phi\right)} + Q = 0,$$

and letting

$$c_1^{b,m} = c_1^b e^{z\left(\phi^b - \phi^{b,m}\right)}, \cdots, c_{n-1}^{b,m} = c_{n-1}^b e^{z\left(\phi^b - \phi^{b,m}\right)}, c_n^{b,m} = c_n^b e^{z_n\left(\phi^b - \phi^{b,m}\right)},$$

$$u_m(b) = \text{sgn}\left(\phi^b - \phi^{b,m}\right)\sqrt{2\left[c_1^b + \cdots + c_n^b - \left(c_1^{b,m} + \cdots + c_n^{b,m}\right) - Q\left(\phi^b - \phi^{b,m}\right)\right]}.$$

The unstable manifold $W^u\left(\mathcal{Z}_m\right)$ intersects $B_b$ transversally at points

$$\left(\phi^b, u_m(b), c_1^b, \cdots, c_n^b, J_1, \cdots, J_n, b\right),$$

and the $\alpha$-limit set of $N_m^b = M_m^b \bigcap W^u\left(\mathcal{Z}_m\right)$ is

$$\alpha\left(N_m^b\right) = \left\{\left(\phi^{b,m}, 0, c_1^{b,m}, \cdots, c_n^{b,m}, J_1, \cdots, J_n, b\right)\right\},$$

where $J_i$ for $i = 1, \cdots, n$ are arbitrary.

**Remark 2.7.** At $x = a$, the limiting fast orbits $\Gamma_m^a \subset N_m^a$ are a segment connecting $\left(\phi^a, u_m(a), c_1^a, \cdots, c_n^a, J_1, \cdots, J_n, a\right)$ to $\omega\left(N_m^a\right)$, and at $x = b$ the limiting fast orbits $\Gamma_m^b \subset N_m^b$ are a segment connecting $\left(\phi^b, u_m(b), c_1^b, \cdots, c_n^b, J_1, \cdots, J_n, b\right)$ to $\alpha\left(N_m^b\right)$.

## 2.4. Limiting Slow Orbits on [a, b]

Now we identify the limiting slow orbits $\Lambda_m$ on the critical manifold $\mathcal{Z}_m$. By using a rescaling

$$u = \varepsilon p, \quad zc_1 + \cdots + zc_{n-1} + z_n c_n + Q = -\varepsilon q.$$

System (2.6) becomes

$$\begin{aligned}
&\dot{\phi} = p, \quad \varepsilon\dot{p} = q - \varepsilon h^{-1}(\tau)h_\tau(\tau)p, \\
&\varepsilon\dot{q} = \left[z(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q - \varepsilon z_n q\right]p \\
&\qquad + \left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]h^{-1}(\tau), \\
&\dot{c}_1 = -zc_1 p - J_1 h^{-1}(\tau), \cdots, \dot{c}_{n-1} = -zc_{n-1} p - J_{n-1} h^{-1}(\tau), \\
&\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1,
\end{aligned} \tag{2.17}$$

where $c_n = \dfrac{-zc_1 - \cdots - zc_{n-1} - Q - \varepsilon q}{z_n}$, and its limiting slow system is

$$\dot{\phi} = p, \quad q = 0, \quad p = -\frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]}{z(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q},$$

$$\dot{c}_1 = -zc_1 p - J_1 h^{-1}(\tau), \cdots, \dot{c}_{n-1} = -zc_{n-1}p - J_{n-1}h^{-1}(\tau), \qquad (2.18)$$

$$\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.$$

For system (2.18), the critical manifold is

$$S_m = \left\{ q = 0, \ p = -\frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]}{z(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q} \right\},$$

where $c_n = \dfrac{-zc_1 - \cdots - zc_{n-1} - Q}{z_n}$.

It follows that the limiting slow system on $S_m$ is

$$\dot{\phi} = -\frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]}{z(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q},$$

$$\dot{c}_1 = \frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]c_1}{(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q} - J_1 h^{-1}(\tau),$$

$$\vdots \qquad\qquad (2.19)$$

$$\dot{c}_{n-1} = \frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]c_{n-1}}{(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q} - J_{n-1}h^{-1}(\tau),$$

$$\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.$$

Following the idea in [20] [21], system (2.19) can be transformed to

$$\frac{d}{dy}\phi = -\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right],$$

$$\frac{d}{dy}c_1 = \left[z(J_1 + \cdots + J_n) + z_n J_n\right]zc_1 - z(z - z_n)J_1(c_1 + \cdots + c_{n-1}) + z_n Q J_1,$$

$$\vdots \qquad\qquad (2.20)$$

$$\frac{d}{dy}c_{n-1} = \left[z(J_1 + \cdots + J_n) + z_n J_n\right]zc_{n-1} - z(z - z_n)J_{n-1}(c_1 + \cdots + c_{n-1})$$

$$+ z_n Q J_{n-1},$$

$$\frac{d}{dy}J_1 = 0, \cdots, \frac{d}{dy}J_n = 0, \quad \frac{d}{dy}\tau = \left[z(z - z_n)(c_1 + \cdots + c_{n-1}) - z_n Q\right]h(\tau).$$

The solution of (2.20) is

$$\phi(y) = \phi^{a,m} - \left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]y,$$

$$c_1(y) = \frac{c_1^{a,m}(J_2 + \cdots + J_{n-1}) - \left(c_2^{a,m} + \cdots + c_{n-1}^{a,m}\right)J_1}{J_1 + \cdots + J_{n-1}}D_1(y) - \frac{QJ_1}{z(J_1 + \cdots + J_n)}$$

$$+ \frac{J_1}{J_1 + \cdots + J_{n-1}}\left[c_1^{a,m} + \cdots + c_{n-1}^{a,m} + \frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)}\right]D_2(y),$$

$$\vdots$$

$$c_{n-1}(y) = \frac{c_{n-1}^{a,m}(J_1 + \cdots + J_{n-2}) - (c_1^{a,m} + \cdots + c_{n-2}^{a,m})J_{n-1}}{J_1 + \cdots + J_{n-1}} D_1(y) - \frac{QJ_{n-1}}{z(J_1 + \cdots + J_n)}$$

$$+ \frac{J_{n-1}}{J_1 + \cdots + J_{n-1}} \left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} + \frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)} \right] D_2(y),$$

$$\int_a^\tau h^{-1}(s)\,\mathrm{d}s = -\frac{(z - z_n)(c_1^{a,m} + \cdots + c_{n-1}^{a,m})}{z_n(J_1 + \cdots + J_n)} \left[ 1 - D_2(y) \right] \qquad (2.21)$$

$$- \frac{(z - z_n)Q(J_1 + \cdots + J_{n-1})}{J_1 + \cdots + J_n} y$$

$$- \frac{(z - z_n)Q(J_1 + \cdots + J_{n-1})}{zz_n(J_1 + \cdots + J_n)^2} \left[ 1 - D_2(y) \right] - z_n Q y,$$

where

$$D_1(y) = \mathrm{e}^{z[z(J_1 + \cdots + J_{n-1}) + z_n J_n]y}, \quad D_2(y) = \mathrm{e}^{zz_n(J_1 + \cdots + J_n)y}.$$

It can be seen that there exists a $y_0 > 0$ such that $\tau(y_0) = b$, which implies that $\phi(y_0) = \phi^{b,m}$ and $c_1(y_0) = c_1^{b,m}, \cdots, c_{n-1}(y_0) = c_{n-1}^{b,m}$, then it follows that

$$\phi^{b,m} = \phi^{a,m} - \left[ z(J_1 + \cdots + J_{n-1}) + z_n J_n \right] y_0,$$

$$c_1^{b,m} = \frac{c_1^{a,m}(J_2 + \cdots + J_{n-1}) - (c_2^{a,m} + \cdots + c_{n-1}^{a,m})J_1}{J_1 + \cdots + J_{n-1}} D_1(y_0) - \frac{QJ_1}{z(J_1 + \cdots + J_n)}$$

$$+ \frac{J_1}{J_1 + \cdots + J_{n-1}} \left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} + \frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)} \right] D_2(y_0),$$

$$\vdots$$

$$c_{n-1}^{b,m} = \frac{c_{n-1}^{a,m}(J_1 + \cdots + J_{n-2}) - (c_1^{a,m} + \cdots + c_{n-2}^{a,m})J_{n-1}}{J_1 + \cdots + J_{n-1}} D_1(y_0) - \frac{QJ_{n-1}}{z(J_1 + \cdots + J_n)}$$

$$+ \frac{J_{n-1}}{J_1 + \cdots + J_{n-1}} \left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} + \frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)} \right] D_2(y_0), \qquad (2.22)$$

$$J_1 + \cdots + J_n = \frac{(z - z_n)\left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} - (c_1^{b,m} + \cdots + c_{n-1}^{b,m}) \right]}{-z_n(H(b) - H(a))} - \frac{Q(\phi^{a,m} - \phi^{b,m})}{H(b) - H(a)}.$$

## 2.5. Limiting Fast Orbits on [b, 1]

In this section, we will identify the limiting fast and limiting slow orbits connecting $B_b$ to $B_R$ on the interval $[b,1]$, where $Q(x) = 0$. The limiting fast system is obtained by letting $\varepsilon = 0$ in (2.7):

$$\phi' = u, \quad u' = -zc_1 - \cdots - zc_{n-1} - z_n c_n,$$

$$c_1' = -zc_1 u,$$

$$\vdots$$

$$c_{n-1}' = -zc_{n-1}u, \qquad (2.23)$$

$$c_n' = -z_n c_n u,$$

$$J_1' = 0, \cdots, J_n' = 0, \quad \tau' = 0.$$

By letting $\varepsilon = 0$ in (2.6), we obtain the critical manifold

$$\mathcal{Z}_r = \left\{ u = 0, zc_1 + \cdots + zc_{n-1} + z_n c_n = 0 \right\}, \tag{2.24}$$

which is normally hyperbolic.

The flow of $B_b$ under system (2.23) in forward time is denoted by $M_r^b$, and the flow of $B_R$ under system (2.23) in backward time is denoted by $M_R$. Then the following results can be established [20] [21].

**Lemma 2.8.** *System* (2.23) *has the following* $n+1$ *nontrivial first integrals:*

$$H_1 = e^{z\phi} c_1, \cdots, H_{n-1} = e^{z\phi} c_{n-1}, H_n = e^{z_n \phi} c_n, H_{n+1} = c_1 + \cdots + c_n - \frac{1}{2} u^2.$$

**Proposition 2.9.** (i) *The stable manifold* $W^s(\mathcal{Z}_r)$ *intersects* $B_b$ *transversally at points*

$$\left( \phi^b, u_r(b), c_1^b, \cdots, c_n^b, J_1, \cdots, J_n, b \right),$$

*and the* $\omega$*-limit set of* $N_r^b = M_r^b \cap W^s(\mathcal{Z}_r)$ *is*

$$\omega\left( N_r^b \right) = \left\{ \left( \phi^{b,r}, 0, c_1^{b,r}, \cdots, c_n^{b,r}, J_1, \cdots, J_n, b \right) \right\},$$

*where* $J_i$ *for* $i = 1, \cdots, n$ *are arbitrary, and*

$$\phi^{b,r} = \phi^b - \frac{1}{z - z_n} \ln \frac{-z_n c_n^b}{z\left( c_1^b + \cdots + c_{n-1}^b \right)},$$

$$c_1^{b,r} = c_1^b \left[ \frac{-z_n c_n^b}{z\left( c_1^b + \cdots + c_{n-1}^b \right)} \right]^{\frac{z}{z - z_n}}, \cdots, c_n^{b,r} = c_n^b \left[ \frac{-z_n c_n^b}{z\left( c_1^b + \cdots + c_{n-1}^b \right)} \right]^{\frac{z_n}{z - z_n}},$$

$$u_r(b) = \operatorname{sgn}\left( \phi^{b,r} - \phi^b \right) \sqrt{2 \left[ c_1^b + \cdots + c_n^b - \left( c_1^{b,r} + \cdots + c_n^{b,r} \right) \right]}.$$

(ii) *The unstable manifold* $W^u(\mathcal{Z}_r)$ *intersects* $B_R$ *transversally at points*

$$\left( 0, u_1, R_1, \cdots, R_n, J_1, \cdots, J_n, 1 \right),$$

*and the* $\alpha$*-limit set of* $N_R = M_R \cap W^u(\mathcal{Z}_r)$ *is*

$$\alpha\left( N_R \right) = \left\{ \left( \phi^R, 0, c_1^R, \cdots, c_n^R, J_1, \cdots, J_n, 1 \right) \right\},$$

*where* $J_i$ *for* $i = 1, \cdots, n$ *are arbitrary, and*

$$\phi^R = -\frac{1}{z - z_n} \ln \frac{-z_n R_n}{z\left( R_1 + \cdots + R_{n-1} \right)},$$

$$c_1^R = R_1 \left[ \frac{-z_n R_n}{z\left( R_1 + \cdots + R_{n-1} \right)} \right]^{\frac{z}{z - z_n}}, \cdots, c_n^R = R_n \left[ \frac{-z_n R_n}{z\left( R_1 + \cdots + R_{n-1} \right)} \right]^{\frac{z_n}{z - z_n}},$$

$$u_1 = \operatorname{sgn}\left( -\phi^R \right) \sqrt{2 \left[ R_1 + \cdots + R_n - \left( c_1^R + \cdots + c_n^R \right) \right]}.$$

**Remark 2.10.** *At* $x = b$, *the limiting fast orbits* $\Gamma_r^b \subset N_r^b$ *are a segment connecting* $\left( \phi^b, u_r(b), c_1^b, \cdots, c_n^b, J_1, \cdots, J_n, b \right)$ *to* $\omega\left( N_r^b \right)$, *and at* $x = 1$ *the limiting fast orbits* $\Gamma^1 \subset N_R$ *are a segment connecting* $\left( 0, u_1, R_1, \cdots, R_n, J_1, \cdots, J_n, 1 \right)$ *to* $\alpha\left( N_R \right)$.

## 2.6. Limiting Slow Orbits on [b, 1]

Now we identify the limiting slow orbits $\Lambda_r$ on the critical manifold $\mathcal{Z}_r$. Just as in sections 2.1 and 2.2, it can be shown that the limiting slow system is

$$
\dot{\phi} = -\frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]}{z(z - z_n)(c_1 + \cdots + c_{n-1})},
$$

$$
\dot{c}_1 = \frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]c_1}{(z - z_n)(c_1 + \cdots + c_{n-1})} - J_1 h^{-1}(\tau),
$$

$$
\vdots
$$

$$
\dot{c}_{n-1} = \frac{h^{-1}(\tau)\left[z(J_1 + \cdots + J_{n-1}) + z_n J_n\right]c_{n-1}}{(z - z_n)(c_1 + \cdots + c_{n-1})} - J_{n-1} h^{-1}(\tau),
$$

$$
\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.
$$

(2.25)

For convenience, we denote

$$
P_r(x) = 1 + \frac{z_n(J_1 + \cdots + J_n)\left[H(x) - H(b)\right]}{(z - z_n)(c_1^{b,r} + \cdots + c_{n-1}^{b,r})}
$$

and

$$
\lambda_r^i = \frac{c_i^R - c_i^{b,r} e^{z(\phi^{b,r} - \phi^R)}}{c_1^R + \cdots + c_{n-1}^R - (c_1^{b,r} + \cdots + c_{n-1}^{b,r}) e^{z(\phi^{b,r} - \phi^R)}}, i = 1, \cdots, n-1.
$$

**Lemma 2.11.** *There is a unique solution*
$\left(\phi(x), c_1(x), \cdots, c_{n-1}(x), J_1, \cdots, J_n, \tau(x)\right)$ *of* (2.25) *such that*

$$
\left(\phi(b), c_1(b), \cdots, c_{n-1}(b), \tau(b)\right) = \left(\phi^{b,r}, c_1^{b,r}, \cdots, c_{n-1}^{b,r}, b\right) \in \omega\left(N_r^b\right)
$$

*and* $\left(\phi(1), c_1(1), \cdots, c_{n-1}(1), \tau(1)\right) = \left(\phi^R, c_1^R, \cdots, c_{n-1}^R, 1\right) \in \alpha(N_R),$

*where* $\phi^{b,r}, c_1^{b,r}, \cdots, c_{n-1}^{b,r}, \phi^R, c_1^R, \cdots, c_{n-1}^R$ *are given in Proposition 2.9. It is given by*

$$
\phi(x) = \phi^{b,r} - \frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z z_n(J_1 + \cdots + J_n)} \ln P_r(x),
$$

$$
c_1(x) = \frac{c_1^{b,r}(J_2 + \cdots + J_{n-1}) - (c_2^{b,r} + \cdots + c_{n-1}^{b,r})J_1}{J_1 + \cdots + J_{n-1}} \left[P_r(x)\right]^{\frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z_n(J_1 + \cdots + J_n)}}
$$

$$
+ \frac{(c_1^{b,r} + \cdots + c_{n-1}^{b,r})J_1}{J_1 + \cdots + J_{n-1}} P_r(x),
$$

$$
\vdots
$$

$$
c_{n-1}(x) = \frac{c_{n-1}^{b,r}(J_1 + \cdots + J_{n-2}) - (c_1^{b,r} + \cdots + c_{n-2}^{b,r})J_{n-1}}{J_1 + \cdots + J_{n-1}} \left[P_r(x)\right]^{\frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z_n(J_1 + \cdots + J_n)}}
$$

$$
+ \frac{(c_1^{b,r} + \cdots + c_{n-1}^{b,r})J_{n-1}}{J_1 + \cdots + J_{n-1}} P_r(x),
$$

$$
J_1 = \frac{c_1^{b,r} + \cdots + c_{n-1}^{b,r} - (c_1^R + \cdots + c_{n-1}^R)}{H(1) - H(b)} \left[1 - \frac{z(\phi^{b,r} - \phi^R)}{\ln \frac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b,r} + \cdots + c_{n-1}^{b,r}}}\right] \lambda_r^1,
$$

$$\vdots$$

$$J_{n-1} = \frac{c_1^{b,r} + \cdots + c_{n-1}^{b,r} - \left(c_1^R + \cdots + c_{n-1}^R\right)}{H(1) - H(b)} \left[1 - \frac{z\left(\phi^{b,r} - \phi^R\right)}{\ln \dfrac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b,r} + \cdots + c_{n-1}^{b,r}}}\right] \lambda_r^{n-1},$$

$$J_n = \frac{z\left[c_1^R + \cdots + c_{n-1}^R - \left(c_1^{b,r} + \cdots + c_{n-1}^{b,r}\right)\right]}{z_n\left(H(1) - H(b)\right)} \left[1 - \frac{z_n\left(\phi^{b,r} - \phi^R\right)}{\ln \dfrac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b,r} + \cdots + c_{n-1}^{b,r}}}\right],$$

$$\tau(x) = x.$$

(2.26)

## 3. Results

Based on Propositions 2.2, 2.6, 2.9 and formulas (2.14), (2.22), (2.26), we get

$$zc_1^a e^{z\left(\phi^a - \phi\right)} + \cdots + zc_{n-1}^a e^{z\left(\phi^a - \phi\right)} + z_n c_n^a e^{z_n\left(\phi^a - \phi\right)} + Q = 0,$$

$$zc_1^b e^{z\left(\phi^b - \phi\right)} + \cdots + zc_{n-1}^b e^{z\left(\phi^b - \phi\right)} + z_n c_n^b e^{z_n\left(\phi^b - \phi\right)} + Q = 0,$$

$$\operatorname{sgn}\left(\phi^a - \phi^{a,l}\right)\sqrt{2\left[c_1^a + \cdots + c_n^a - \left(c_1^{a,l} + \cdots + c_n^{a,l}\right)\right]}$$

$$= \operatorname{sgn}\left(\phi^{a,m} - \phi^a\right)\sqrt{2\left[c_1^a + \cdots + c_n^a - \left(c_1^{a,m} + \cdots + c_n^{a,m}\right) - Q\left(\phi^a - \phi^{a,m}\right)\right]},$$

$$\operatorname{sgn}\left(\phi^b - \phi^{b,m}\right)\sqrt{2\left[c_1^b + \cdots + c_n^b - \left(c_1^{b,m} + \cdots + c_n^{b,m}\right) - Q\left(\phi^b - \phi^{b,m}\right)\right]}$$

$$= \operatorname{sgn}\left(\phi^{b,r} - \phi^b\right)\sqrt{2\left[c_1^b + \cdots + c_b^b - \left(c_1^{b,r} + \cdots + c_n^{b,r}\right)\right]},$$

$$J_1 + \cdots + J_{n-1}$$

$$= \frac{c_1^L + \cdots + c_{n-1}^L - \left(c_1^{a,l} + \cdots + c_{n-1}^{a,l}\right)}{H(a)} \left[1 - \frac{z\left(\phi^L - \phi^{a,l}\right)}{\ln \dfrac{c_1^{a,l} + \cdots + c_{n-1}^{a,l}}{c_1^L + \cdots + c_{n-1}^L}}\right]$$

$$= \frac{c_1^{b,r} + \cdots + c_{n-1}^{b,r} - \left(c_1^R + \cdots + c_{n-1}^R\right)}{H(1) - H(a)} \left[1 - \frac{z\left(\phi^{b,r} - \phi^R\right)}{\ln \dfrac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b,r} + \cdots + c_{n-1}^{b,r}}}\right],$$

$$J_n = \frac{z\left[c_1^{a,l} + \cdots + c_{n-1}^{a,l} - \left(c_1^L + \cdots + c_{n-}^L\right)\right]}{z_n H(a)} \left[1 - \frac{z_n\left(\phi^L - \phi^{a,l}\right)}{\ln \dfrac{c_1^{a,l} + \cdots + c_{n-1}^{a,l}}{c_1^L + \cdots + c_{n-1}^L}}\right]$$

$$= \frac{z\left[c_1^R + \cdots + c_{n-1}^R - \left(c_1^{b,r} + \cdots + c_{n-1}^{b,r}\right)\right]}{z_n\left(H(1) - H(a)\right)} \left[1 - \frac{z_n\left(\phi^{b,r} - \phi^R\right)}{\ln \dfrac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b,r} + \cdots + c_{n-1}^{b,r}}}\right],$$

$$\phi^{b,m} = \phi^{a,m} - \left[z\left(J_1 + \cdots + J_{n-1}\right) + z_n J_n\right] y_0,$$

$$c_1^{b,m} + \cdots + c_{n-1}^{b,m}$$

$$= -\frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)} + \left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} + \frac{Q(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)} \right] D_2(y_0),$$

$$J_1 + \cdots + J_n = \frac{(z - z_n)\left[ c_1^{a,m} + \cdots + c_{n-1}^{a,m} - \left( c_1^{b,m} + \cdots + c_{n-1}^{b,m} \right) \right]}{-z_n (H(b) - H(a))} - \frac{Q(\phi^{a,m} - \phi^{b,m})}{H(b) - H(a)}, \tag{3.27}$$

and

$$J_1 = (J_1 + \cdots + J_{n-1})\lambda_l^1 = (J_1 + \cdots + J_{n-1})\lambda_m^1 = (J_1 + \cdots + J_{n-1})\lambda_r^1,$$

$$\vdots \tag{3.28}$$

$$J_{n-1} = (J_1 + \cdots + J_{n-1})\lambda_l^{n-1} = (J_1 + \cdots + J_{n-1})\lambda_r^{n-1} = (J_1 + \cdots + J_{n-1})\lambda_m^{n-1},$$

where

$$\lambda_m^i = \frac{c_i^{b,m} - c_i^{a,m} e^{z(\phi^{a,m} - \phi^{b,m})}}{c_1^{b,m} + \cdots + c_{n-1}^{b,m} - \left( c_1^{a,m} + \cdots + c_{n-1}^{a,m} \right) e^{z(\phi^{a,m} - \phi^{b,m})}}, i = 1, \cdots, n-1.$$

Now, we consider a special case that $z = -z_n = 1$, and $a = 1/3, b = 2/3, h(x) = 1$. Following the idea in [20], let

$$\left( c_1^a + \cdots + c_{n-1}^a \right) c_n^a = A^2, \quad \left( c_1^b + \cdots + c_{n-1}^b \right) c_n^b = B^2$$

and

$$\left( L_1 + \cdots + L_{n-1} \right) L_n = L^2, \quad \left( R_1 + \cdots + R_{n-1} \right) R_n = R^2, \quad Q = 2Q_0.$$

Then system (3.27) reduces to

$$F(A) = e^{K(A)} \left( \sqrt{Q_0^2 + A^2} - \frac{Q_0 \left[ J_n - (J_1 + \cdots + J_{n-1}) \right]}{6(L - A)} \right)$$

$$+ \frac{Q_0 \left[ J_n - (J_1 + \cdots + J_{n-1}) \right]}{6(L - A)} - \sqrt{Q_0^2 + B^2} = 0, \tag{3.29}$$

where

$$K(A) = -6(L - A) \frac{\sqrt{Q_0^2 + B^2} - \sqrt{Q_0^2 + A^2} + L - A}{Q_0 \left[ J_n - (J_1 + \cdots + J_{n-1}) \right]},$$

and system (3.28) reduces to

$$\frac{J_1}{J_1 + \cdots + J_{n-1}}$$

$$= \frac{c_1^a \sqrt{\dfrac{c_n^a}{c_1^a + \cdots + c_{n-1}^a}} - L_1 \sqrt{\dfrac{L_n}{L_1 + \cdots + L_{n-1}}} e^{\bar{V} - \phi^a - \frac{1}{2} \ln \frac{L_n}{L_1 + \cdots + L_{n-1}} + \frac{1}{2} \ln \frac{c_n^a}{c_1^a + \cdots + c_{n-1}^a}}}{A - L e^{\bar{V} - \phi^a - \frac{1}{2} \ln \frac{L_n}{L_1 + \cdots + L_{n-1}} + \frac{1}{2} \ln \frac{c_n^a}{c_1^a + \cdots + c_{n-1}^a}}}$$

$$= \frac{(J_1 + \cdots + J_{n-1}) \dfrac{c_1^b c_n^b}{\sqrt{Q_0^2 + B^2} + Q_0}}{\sqrt{Q_0^2 + B^2} - Q_0 - \left( \sqrt{Q_0^2 + A^2} - Q_0 \right) e^{\phi^a - \phi^b + \ln \frac{\left( \sqrt{Q_0^2 + B^2} - Q_0 \right)\left( c_1^a + \cdots + c_{n-1}^a \right)}{\left( \sqrt{Q_0^2 + A^2} - Q_0 \right)\left( c_1^b + \cdots + c_{n-1}^b \right)}}}$$

$$\left(J_1 + \cdots + J_{n-1}\right) \frac{c_1^a c_n^a}{\sqrt{Q_0^2 + A^2} + Q_0} \mathrm{e}^{\phi^a - \phi^b + \ln \frac{\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left(c_1^a + \cdots + c_{n-1}^a\right)}{\left(\sqrt{Q_0^2 + A^2} - Q_0\right)\left(c_1^b + \cdots + c_{n-1}^b\right)}}$$

$$- \frac{}{\sqrt{Q_0^2 + B^2} - Q_0 - \left(\sqrt{Q_0^2 + A^2} - Q_0\right)\mathrm{e}^{\phi^a - \phi^b + \ln \frac{\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left(c_1^a + \cdots + c_{n-1}^a\right)}{\left(\sqrt{Q_0^2 + A^2} - Q_0\right)\left(c_1^b + \cdots + c_{n-1}^b\right)}}}$$

$$= \frac{R_1 \sqrt{\dfrac{R_n}{R_1 + \cdots + R_{n-1}}} - c_1^b \sqrt{\dfrac{c_n^b}{c_1^b + \cdots + c_{n-1}^b}} \, \mathrm{e}^{\phi^b - \frac{1}{2}\ln \frac{c_n^b}{c_1^b + \cdots + c_{n-1}^b} + \frac{1}{2}\ln \frac{R_n}{R_1 + \cdots + R_{n-1}}}}{R - B\mathrm{e}^{\phi^b - \frac{1}{2}\ln \frac{c_n^b}{c_1^b + \cdots + c_{n-1}^b} + \frac{1}{2}\ln \frac{R_n}{R_1 + \cdots + R_{n-1}}}},$$

$$\vdots$$

$$\frac{J_{n-1}}{J_1 + \cdots + J_{n-1}}$$

$$= \frac{c_{n-1}^a \sqrt{\dfrac{c_n^a}{c_1^a + \cdots + c_{n-1}^a}} - L_{n-1} \sqrt{\dfrac{L_n}{L_1 + \cdots + L_{n-1}}} \, \mathrm{e}^{\bar{V} - \phi^a - \frac{1}{2}\ln \frac{L_n\left(c_1^a + \cdots + c_{n-1}^a\right)}{c_n^a\left(L_1 + \cdots + L_{n-1}\right)}}}{A - L\mathrm{e}^{\bar{V} - \phi^a - \frac{1}{2}\ln \frac{L_n}{L_1 + \cdots + L_{n-1}} + \frac{1}{2}\ln \frac{c_n^a}{c_1^a + \cdots + c_{n-1}^a}}}$$

$$= \frac{\left(J_1 + \cdots + J_{n-1}\right) \dfrac{c_{n-1}^b c_n^b}{\sqrt{Q_0^2 + B^2} + Q_0}}{\sqrt{Q_0^2 + B^2} - Q_0 - \left(\sqrt{Q_0^2 + A^2} - Q_0\right)\mathrm{e}^{\phi^a - \phi^b + \ln \frac{\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left(c_1^a + \cdots + c_{n-1}^a\right)}{\left(\sqrt{Q_0^2 + A^2} - Q_0\right)\left(c_1^b + \cdots + c_{n-1}^b\right)}}}$$

$$- \frac{\left(J_1 + \cdots + J_{n-1}\right) \dfrac{c_{n-1}^a c_n^a}{\sqrt{Q_0^2 + A^2} + Q_0} \mathrm{e}^{\phi^a - \phi^b + \ln \frac{\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left(c_1^a + \cdots + c_{n-1}^a\right)}{\left(\sqrt{Q_0^2 + A^2} - Q_0\right)\left(c_1^b + \cdots + c_{n-1}^b\right)}}}{\sqrt{Q_0^2 + B^2} - Q_0 - \left(\sqrt{Q_0^2 + A^2} - Q_0\right)\mathrm{e}^{\phi^a - \phi^b + \ln \frac{\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left(c_1^a + \cdots + c_{n-1}^a\right)}{\left(\sqrt{Q_0^2 + A^2} - Q_0\right)\left(c_1^b + \cdots + c_{n-1}^b\right)}}} \tag{3.30}$$

$$= \frac{R_{n-1} \sqrt{\dfrac{R_n}{R_1 + \cdots + R_{n-1}}} - c_{n-1}^b \sqrt{\dfrac{c_n^b}{c_1^b + \cdots + c_{n-1}^b}} \, \mathrm{e}^{\phi^b - \frac{1}{2}\ln \frac{c_n^b}{c_1^b + \cdots + c_{n-1}^b} + \frac{1}{2}\ln \frac{R_n}{R_1 + \cdots + R_{n-1}}}}{R - B\mathrm{e}^{\phi^b - \frac{1}{2}\ln \frac{c_n^b}{c_1^b + \cdots + c_{n-1}^b} + \frac{1}{2}\ln \frac{R_n}{R_1 + \cdots + R_{n-1}}}}.$$

Additionally, it is demonstrated in [20] that $F(A) = 0$ has solutions, therefore, it follows that the following unknowns

$$\phi^a, c_1^a + \cdots + c_{n-1}^a, c_n^a, \phi^b, c_1^b + \cdots + c_{n-1}^b, c_n^b, J_1 + \cdots + J_{n-1}, J_n, y_0$$

can be determined. The remaining unknown

$$c_1^a, \cdots, c_{n-2}^a, c_1^b, \cdots, c_{n-2}^b, J_1, \cdots, J_{n-2}$$

will be determined by Equation (3.30).

By solving (3.30), we get

$$c_1^a = \frac{\left(c_1^a + \cdots + c_{n-1}^a\right)\left[\dfrac{L_1\left(R_1 + \cdots + R_{n-1}\right)}{L_1 + \cdots + L_{n-1}} - R_1\right]Le^{T_1}}{A\left[R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}\right]}$$
$$+ \frac{\left(c_1^a + \cdots + c_{n-1}^a\right)\left(R_1 - L_1 e^{\bar{V}}\right)}{R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}},$$

$$c_1^b = \frac{\left(c_1^b + \cdots + c_{n-1}^b\right)\left[\dfrac{L_1\left(R_1 + \cdots + R_{n-1}\right)}{L_1 + \cdots + L_{n-1}} - R_1\right]L\left(\sqrt{Q_0^2 + A^2} - Q_0\right)e^{T_2}}{A\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left[R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}\right]}$$
$$+ \frac{\left(c_1^b + \cdots + c_{n-1}^b\right)\left(R_1 - L_1 e^{\bar{V}}\right)}{R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}},$$

$$\vdots$$

$$c_{n-1}^a = \frac{\left(c_1^a + \cdots + c_{n-1}^a\right)\left[\dfrac{L_{n-1}\left(R_1 + \cdots + R_{n-1}\right)}{L_1 + \cdots + L_{n-1}} - R_{n-1}\right]Le^{T_1}}{A\left[R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}\right]}$$
$$+ \frac{\left(c_1^a + \cdots + c_{n-1}^a\right)\left(R_{n-1} - L_{n-1} e^{\bar{V}}\right)}{R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}},$$

$$c_{n-1}^b = \frac{\left(c_1^b + \cdots + c_{n-1}^b\right)\left[\dfrac{L_{n-1}\left(R_1 + \cdots + R_{n-1}\right)}{L_1 + \cdots + L_{n-1}} - R_{n-1}\right]L\left(\sqrt{Q_0^2 + A^2} - Q_0\right)e^{T_2}}{A\left(\sqrt{Q_0^2 + B^2} - Q_0\right)\left[R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}\right]}$$
$$+ \frac{\left(c_1^b + \cdots + c_{n-1}^b\right)\left(R_{n-1} - L_{n-1} e^{\bar{V}}\right)}{R_1 + \cdots + R_{n-1} - \left(L_1 + \cdots + L_{n-1}\right)e^{\bar{V}}}, \tag{3.31}$$

where

$$T_1 = \bar{V} - \phi^a - \frac{1}{2}\ln\frac{L_n}{L_1 + \cdots + L_{n-1}} + \frac{1}{2}\ln\frac{c_n^a}{c_1^a + \cdots + c_{n-1}^a},$$

and

$$T_2 = \bar{V} - \phi^b - \frac{1}{2}\ln\frac{L_n\left(c_1^a + \cdots + c_{n-1}^a\right)}{c_n^a\left(L_1 + \cdots + L_{n-1}\right)} + \ln\frac{\left(c_1^a + \cdots + c_{n-1}^a\right)\left(\sqrt{Q_0^2 + B^2} - Q_0\right)}{\left(c_1^b + \cdots + c_{n-1}^b\right)\left(\sqrt{Q_0^2 + A^2} - Q_0\right)}.$$

**Remark 3.1.** *Once* $c_1^a, \cdots, c_{n-1}^a, c_1^b, \cdots, c_{n-1}^b$ *are determined, then* $J_1, \cdots, J_{n-1}$ *are also determined. Therefore, all unknowns involved in Equations* (3.27) *and* (3.28) *are determined.*

Therefore, a limiting fast and limiting slow orbit is identified as follows, see **Figure 1** for an illustration.

$$\Gamma^0 \bigcup \Lambda_l \bigcup \Gamma_l^a \bigcup \Gamma_m^a \bigcup \Lambda_m \bigcup \Gamma_m^b \bigcup \Gamma_r^b \bigcup \Lambda_r \bigcup \Gamma^1.$$

By employing the Exchange Lemma [20] [21] [22] [23] [24], it can be verified that

**Theorem 3.2.** *For* $\varepsilon > 0$ *sufficiently small, there exists a unique solution of*
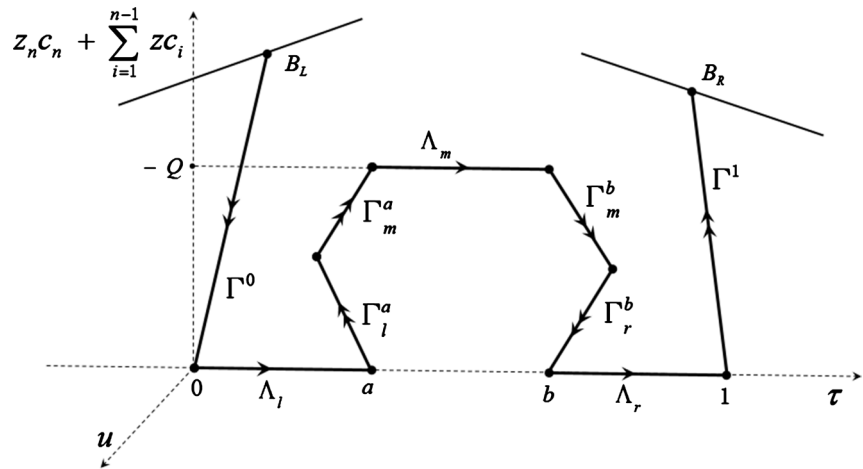
**Figure 1.** A limiting fast and limiting slow orbit connecting $B_L$ to $B_R$, where $\Gamma^0, \Gamma^1, \Gamma_l^a, \Gamma_m^a, \Gamma_m^b, \Gamma_r^b$ are limiting fast orbits and $\Lambda_l, \Lambda_m, \Lambda_r$ are limiting slow orbits.

*system* (1.4) *and* (1.5) *near the limiting fast and limiting slow orbits.*

## 4. Conclusion

In this paper, a steady-state Poisson-Nernst-Planck model with $n$ ion species is studied under the assumption that $n-1$ positively charged ion species have the same valence and there is only one negatively charged ion species. By using the geometric theory for singularly perturbed system, the existence of solutions for systems (1.4) and (1.5) is justified. As we know, it can be seen that the results in [20] correspond to those in this paper in the case that $n = 2$. Also, this paper shows that the number of solutions for systems (1.4) and (1.5) essentially is determined by the number of solutions for the algebraic Equation (3.29), that is, an increase in the number of positively charged ions with the same valence does not change the number of solutions for systems (1.4) and (1.5). Generally, for the case that there are more than two species ions involved in the Poisson-Nernst-Planck model, the dynamics become more subtle and complicated. Moreover, the mixture of multi-species ions, such as sodium ($Na^+$), potassium ($K^+$), calcium ($Ca^{2+}$), chloride ($Cl^-$), plays the very important role in many vital biological functions, for instance, opening and closing of ionic channels. In [25] [26], it was shown that the Poisson-Nernst-Planck model with three or more ionic species of different charge may admit multiple steady state solutions, and the existence of multiple steady state solutions is important to study transitions between such states which may be related to the gating (switching between open and closed states of ionic channels) and selectivity of ion channels.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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