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Generation of Spin 1 System

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Abstract

In this paper, we shall establish the connection between the group theory and quantum mechanics by showing how the group theory helps us to construct the spin operators. We look to the group generators SU(3). From these generators, new spin 1 operators will be constructed. These operators S_{-x}, S_{-y} and S_{-z} satisfy all the properties of Pauli spin operators S_x, S_y and S_z. We shall discuss the notion of spin squeezing and correlations for pure spin 1 system using our spin operators S_{-x}, S_{-y} and S_{-z}.

Keywords

Group Generators, Spin Operators

1. Introduction

The fundamental problem in quantum physics is to investigate the Hamiltonian operator *H* which is a Hermitian operator.

The operator H may correspond to physical observables such as spin operators…

The number of eigen-functions of H in the most quantum systems is in the fact infinite and the set of all eigen-functions of a Hermitian operator is a complete set. These eigen-functions define a Hilbert space on which the operator acts.

The group theory becomes almost inevitable when the system contains a large number of particles. The operator representation can be better understood in the light of the group theoretical interpretation such as spin or Isospin.

The spin operators possess a wide range of applications which plays a key role in both foundation of quantum physics and quantum information [1]-[6], and also makes the atomic clocks more precise [7] [8].

The concept of spin is fascinating in quantum theory which is defined through the commutation relations ($\hbar = 1$) [9],

$$\boldsymbol{j} \times \boldsymbol{j} = i\boldsymbol{j} \tag{1}$$

On the other hand, the concept of spin is understood through the introduction of a spin operator s consisting of three Hermitian components S_x, S_y and S_z which is called Pauli matrices. The square of the spin operators s defined as:

$$S^2 = \boldsymbol{S} \cdot \boldsymbol{S} = S_x^2 + S_y^2 + S_z^2 \tag{2}$$

and

$$\left[S_x, S_y \right] = iS_z \quad (x,\ y,\ z \text{ cyclic}), \tag{3}$$

is again a Hermitian operator. One can now construct simultaneous eigen-states of S^2 and S_z. These eigen-states labeled by $|sm\rangle$ satisfy:

$$S^2 |sm\rangle = s(s+1)|sm\rangle; s = 0, \frac{1}{2}, 1, \cdots \tag{4}$$

$$S_z |sm\rangle = m|sm\rangle; m = -s, -s+1, \cdots, +s. \tag{5}$$

In the eigen basis $\{|sm\rangle\}$, the operators S_x and S_y possess non-zero off diagonal matrix element by virtue of their non-commuting property. These matrix elements can be obtained by noting that the raising and lowering non-Hermitian operator S_+ and S_- is defined through:

$$S_\pm = S_x \pm iS_y, \tag{6}$$

which possess the matrix element:

$$\langle sm' | S_\pm | sm \rangle = \sqrt{(s \pm m)(s \pm m + 1)} \delta_{m', m_{\pm 1}}, \tag{7}$$

All the above equations [1] [2] [3] [4] [5] refer to the Pauli spin operators which are well known as S_x, S_y and S_z. Our aim in this paper is to construct a new spin operator which will give equivalent results to the Pauli spin operator by using the group generator.

The spin operators S_x, S_y and S_z possess a wide range of applications such as spin squeezing for sharp as well as for non-sharp spin value [10] [11] [12] [13].

This paper is organized as follows.

In Section 2, we deal with the definition of the group SU(2) and SU(3); from these groups, we will generate new spin operators. We shall show that new spin operators will be acting on a physical system with respect to a new frame. These operators will be satisfying all the physical properties such as commutation relations.

In Section 3, we shall look to the squeezing spin system using a new spin operator with respect to the new frame.

In Section 4, the spin-spin correlations will be discussed and we will show that the new operators give an equivalent result of squeezing and correlation which had been calculated with respect to Pauli-operators [12].

2. Spin Operators and Group Generators

The group of all unitary matrices of ordered n is known as U(n), whereas the group of all unitary matrices of ordered n with determinant +1 is denoted by SU(n). It is clear that SU(n) is a subgroup of U(n), since a unitary matrix of order n has $n^2 - 1$ independent parameters.

The unitary unimodular group SU(2) in two complex dimensions is the simplest non-trivial example for a non-Abelian group which describe the two level quantum system, at the same time SU(2) is hemimorphic of SU(3).

Thus the Lie algebra of SU(2) in three dimensional representation can be chosen as the set of all real linear combination of σ_x, σ_y and σ_z, such that:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

The next group after SU(2) is SU(3) which possesses eight-dimensional unimodular matrices in three complex dimension [14].

$$\begin{aligned} &\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{9}$$

It is clear from Equation (2) that a spin function for an electron or nucleon has a representation of SU(2) ($s = \frac{1}{2}$).

Now for two electrons or two nucleons ($s = 1$), the generated representation of SU(3) which possesses eight generators that have become convention for $s = 1$.

From Equation (9) we construct three-subgroup spin operators in terms of group generators:

$$S_{\pm ij} = \frac{1}{\sqrt{2}}\left(\lambda_i \pm \lambda_j\right); \left(i, j = 1, 6; i \neq j\right) \tag{10}$$

$$S_{\pm kl} = \frac{1}{\sqrt{2}}\left(\lambda_k \pm \lambda_l\right); \left(k, l = 2, 7; k \neq l\right) \tag{11}$$

$$S_{-z} = \frac{1}{2}\left(\lambda_3 + \sqrt{3}\lambda_8\right) \tag{12}$$

It is clear from Equations (10), (11) and (12) that the first representation S_{+ij}, S_{+kl} and S_z are Pauli spin operators S_x, S_y, S_z while the second representation S_{-ij}, S_{-kl} and S_{-z} give other spin operators, which can be written in the matrix form:

$$\begin{aligned} S_{-x} &= S_{-ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\ S_{-y} &= S_{-kl} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} \\ S_{-z} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = S_z \end{aligned} \tag{13}$$

This latter Equation (13) provides new spin operators. We call these operators left hand operators, which satisfy the relation:

$$
\left.
\begin{aligned}
\left[S_{-x}, S_{-y} \right] &= iS_{-z} \\
\left[S^2, S_{\pm} \right] &= 0 \\
\left[S_{-z}, S_+ \right] &= S_+ \\
\left[S_{-z}, S_- \right] &= -S_- \\
S^2 = \boldsymbol{S} \cdot \boldsymbol{S} &= S_{-x}^2 + S_{-y}^2 + S_{-z}^2
\end{aligned}
\right\}
\tag{14}
$$

from Equation (13), it is clear that the two groups of spin operators coincide with the *z*-axis.

To illustrate this situation as shown in **Figure 1**, let us consider in the simplest case of $s = 1$. The spin ($s = 1$) state $|\psi\rangle$ makes an angle γ with respect to *z*-axis in the frame *xyz* and the corresponding operators are Pauli operators S_x, S_y and S_z. Now, in the second frame *x′y′z′* coincides with the *z*-axis in frame *xyz*. We see from **Figure 1**, that the polar angle with respect to *xyz* frame is γ, while with respect to *x′y′z′* it will be $\pi + \gamma$. The operator corresponding to the frame *x′y′z′* will be S_{-x}, S_{-y} and S_{-z}, which is called left-handed operators.
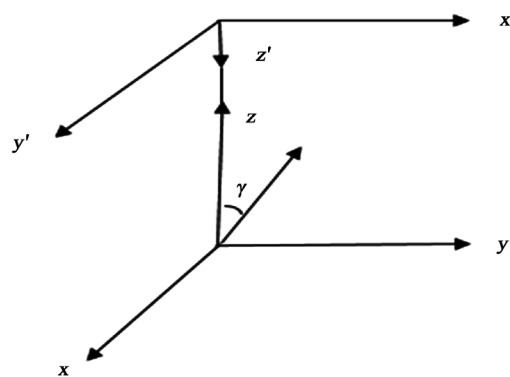

Figure 1. The *z′*-axis in the frame *x′y′z′* coincide with the *z*-axis in frame *xyz*.

3. Squeezing Criterion for Pure Spin State

In the last section, we construct left-handed spin operators. In this section we deal with the squeezing aspects for pure spin 1-system by using the left-handed operators.

The notion of squeezing had been discussed for spin system with arbitrary sharp spin value by several authors [10] [11] [12] [13].

In order to discuss the squeezing condition, we restrict ourselves to Heisenberg's relationship that a spins-state is squeezed in the spin component normal to the mean spin direction of

$$
\Delta S_{-x}^2 < \frac{\left| \langle S_z \rangle \right|}{2}
\tag{15}
$$

or

$$\Delta S^2_{-y} < \frac{\left| \langle S_z \rangle \right|}{2} \tag{16}$$

A normalized pure state of a spin 1 system can be expressed in terms of angular momentum state: $\left| 11 \right\rangle, \left| 00 \right\rangle, \left| 1-1 \right\rangle$ with respect to the frame $x'y'z'$ with a non-zero mean spin direction $\langle S_z \rangle$ can be written in the form

$$\left| \psi \right\rangle = \cos(\pi + \gamma) \left| 11 \right\rangle + \sin(\pi + \gamma) \left| 1-1 \right\rangle \tag{17}$$

It is clear from the above equation that:

$$\langle S_{-x} \rangle = \langle S_{-y} \rangle = 0 \tag{18}$$

For such non-oriented state, the relevant quantities indeed for studying the squeezing turn out to be

$$\Delta S^2_{-x} = \frac{1}{2}\left[1 - \sin 2\gamma\right] \tag{19}$$

$$\Delta S^2_{-y} = \frac{1}{2}\left[1 + \sin 2\gamma\right] \tag{20}$$

$$\Delta S_z = \cos 2\gamma \tag{21}$$

And the squeezing condition's

$$\frac{1}{2}\left[1 - \sin 2\gamma\right] < \frac{1}{2}\left| \cos 2\gamma \right| \tag{22}$$

or

$$\frac{1}{2}\left[1 + \sin 2\gamma\right] < \frac{1}{2}\left| \cos 2\gamma \right| \tag{23}$$

It is clear from Equation's (22), (23) that the spin state $\left| \psi \right\rangle$ is squeezed state with respect to frame $x'y'z'$ for a wide range of γ as shown in **Figure 2**.

Also, we shall mention here that for: $\gamma = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \pi, \cdots$ the spin state is not squeezed state which implies that a non-oriented state $\left| \psi \right\rangle$ indeed squeezed state, and the maximum value of squeezing in x' or y' components when $\gamma = \tan^{-1}(1)$.
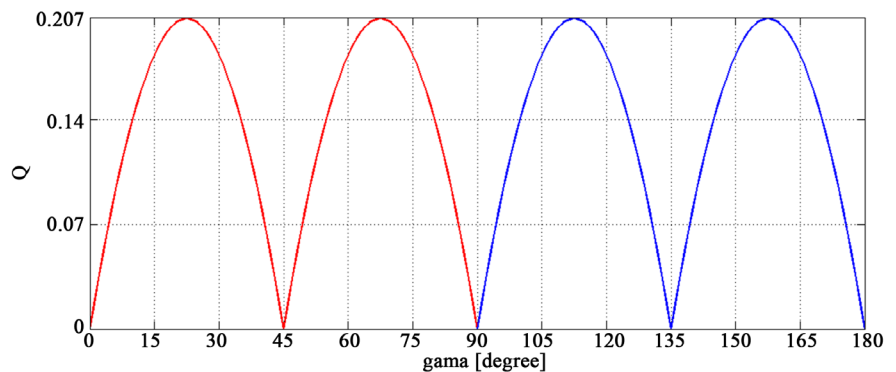

Figure 2. Variation of $Q_x = \dfrac{\left| \langle S_z \rangle \right|}{2} - \Delta S^2_{-x}$ (red carve) & $Q_y = \dfrac{\left| \langle S_z \rangle \right|}{2} - \Delta S^2_{-y}$ (blue carve) with respect to γ. Positive values of these functions imply the presence of squeezing.

The study in this section shows that the left-handed operators give the equivalent result done by Mallesh with respect to the frame *xyz* and Pauli operators [12].

4. Spin-Spin Correlations

In the last section, we have looked into squeezing aspect using left-handed operators of pure spin 1 system with respect to the frame *x'y'z'* as shown in **Figure 1**.

The squeezing of a spin system arises from the existence of quantum correlations between the spin components.

This can be done by assuming the model in which the spin 1 state is constructed by 2s spin $\frac{1}{2}$ system, this work had been done by Mallesh [12] with respect to the frame *xyz*.

In this section we will calculate the correlations with respect to the frame *x'y'z'* as shown in **Figure 3** using left-handed operators.

Now the two spinors are specified with respect to $Q_1(\pi+\theta,\varphi_1)$ and $Q_2(\pi-\theta,\varphi_2)$ in the frame *x'y'z'* so that the spin 1 state is given by:

$$|\psi\rangle = N_1 \sum_{m_1 m_2} D^{\frac{1}{2}}_{m_1 \frac{1}{2}}(\varphi_1,\pi+\theta,0) D^{\frac{1}{2}}_{m_2 \frac{1}{2}}(\varphi_2,\pi-\theta,0) C\left(\frac{1}{2}\frac{1}{2}1; m_1 m_2 m\right)|1m\rangle \quad (24)$$

Here the basis state is referred to the *z*-axis in frame *x'y'z'*. The state $|\psi\rangle$ expressed in terms of the spinor state with respect to *z* in frame *x'y'z'* has a form:

$$|\psi\rangle = \frac{\sqrt{2}i}{\sqrt{1+\cos^2\theta}}\left[\cos^2\left(\frac{\theta}{2}\right)\left|\frac{1}{2}\frac{1}{2}\right\rangle - \sin^2\left(\frac{\theta}{2}\right)\left|-\frac{1}{2}-\frac{1}{2}\right\rangle\right] \quad (25)$$

In terms of the constituent spinor states

$$|\psi\rangle = \frac{\sqrt{2}i}{\sqrt{1+\cos^2\theta}}\left[\cos^2\left(\frac{\theta}{2}\right)|11\rangle_z - \sin^2\left(\frac{\theta}{2}\right)|1-1\rangle_z\right] \quad (26)$$

It is clear from the above equations that:

$$\langle S_{-x}\rangle = \langle S_{-y}\rangle = 0$$

which happen to be a Lakin frame in *x'y'z'* frame.
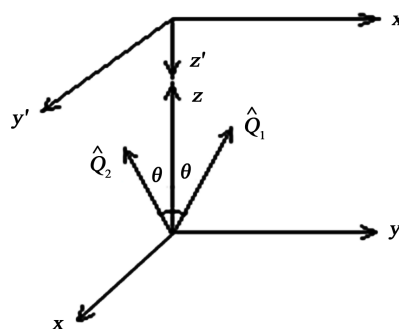
For such state, the relevant quantities:

Figure 3. The two frames *xyz* and *x'y'z'* with mean spin direction *z* along the bisector of two spin half quantization axes $\hat{Q}_1$ and $\hat{Q}_2$.

$$\Delta S_{-x}^2 = \frac{1}{1+\cos^2\theta} \tag{27}$$

$$\Delta S_{-y}^2 = \frac{1+\cos 2\theta}{2\left(1+\cos^2\theta\right)} \tag{28}$$

$$\Delta S_{-z} = \frac{2\cos\theta}{1+\cos^2\theta} \tag{29}$$

The squeezing conditions for S_{-x} and S_{-y} are respectively:

$$\frac{1}{1+\cos^2\theta} < \frac{1}{2}\left|\frac{2\cos\theta}{1+\cos^2\theta}\right| \tag{30}$$

or

$$\frac{1+\cos 2\theta}{2\left(1+\cos^2\theta\right)} < \frac{1}{2}\left|\frac{2\cos\theta}{1+\cos^2\theta}\right| \tag{31}$$

It is clear from Equations (30), (31) that the state $|\psi\rangle$ is squeezed for wide range of θ except for: $\theta = 0, \frac{\pi}{2}, \pi$ as shown in **Figure 4**.

Now let us look to the squeezing condition itself where the existence of spin-spin correlation is necessary and sufficient to the state to be squeezed.

The matrix element of correlation D^{12} is defined through its elements:

$$D_{12}^{\mu\nu} = \left\langle S_1^\mu S_2^\nu \right\rangle - \left\langle S_1^\mu \right\rangle \left\langle S_2^\nu \right\rangle \tag{32}$$

where S_1^μ and S_2^ν are the spin component associated with the two spinors. For such state the correlations matrix is diagonal in the frame $x'y'z'$ and the elements given by:

$$D_{-x-x}^{12} = \frac{\sin^2\theta}{4\left(1+\cos^2\theta\right)} \tag{34}$$

$$D_{-y-y}^{12} = \frac{-\sin^2\theta}{4\left(1+\cos^2\theta\right)} \tag{34}$$

$$D_{-z-z}^{12} = \left[\frac{\sin^2\theta}{2\left(1+\cos^2\theta\right)}\right]^2 \tag{35}$$

From Equations (33), (34) and (35) it's clear that the trace correlations matrix is:

$$Tr\left(D^{12}\right) = \left[\frac{\sin^2\theta}{2\left(1+\cos^2\theta\right)}\right]^2 \tag{36}$$

In addition, the maximum value of correlations as shown in **Figure 5** is 0.25. Thus, the trace condition Equation (36) is the necessary and sufficient condition for pure spin 1 state to be squeezed.

At the end of this section, we have shown that by using the left-handed operator in a calculation for squeezing or correlations we have got the same result which is done by Mallesh [12].
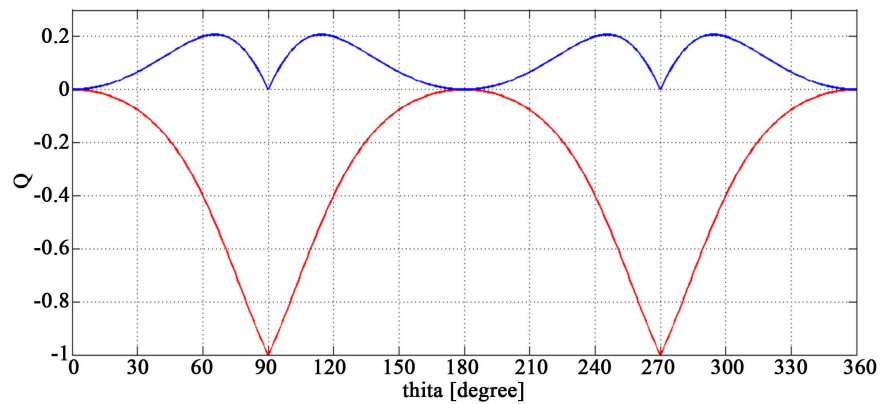
Figure 4. Variation of $Q_x = \dfrac{\left|\langle S_z \rangle\right|}{2} - \Delta S_{-x}^2$ (red carve) & $Q_y = \dfrac{\left|\langle S_z \rangle\right|}{2} - \Delta S_{-y}^2$ (blue carve)

with respect to θ. Positive values of these functions imply the presence of squeezing.
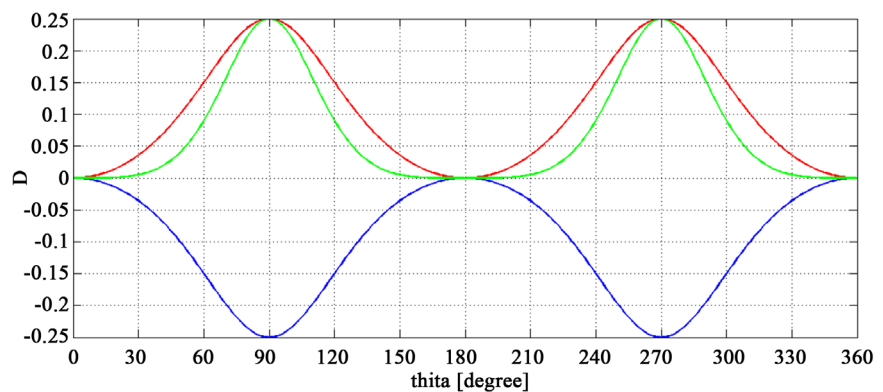

Figure 5. Variation spin-spin correlation D_{-x-x}^{12} (red carve), D_{-y-y}^{12} (blue carve) and D_{-z-z}^{12} (green carve) with respect to θ.

5. Conclusions

The quantum mechanics and group theory are playing an important role in theoretical physics especially in spin operators.

These spin operators have a wide range of applications such as squeezing [10] [11] [12] [13]. In this paper we have constructed new spin 1 operators from the SU(3) generators and we called them left handed spin operators.

These two groups of spin operators S_x, S_y, S_z and S_{-x}, S_{-y}, S_{-z} are acting into two frames. These frames coincide with the same z-axis.

A detailed study of squeezing and correlations of a pure spin 1 system is done by using left-handed operators (S_{-x}, S_{-y}, S_{-z}). Also, it is clear from our analysis that new operators give the equivalent result and the same values which are done by Mallesh [12].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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