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Solution of Partial Derivative Equations of Poisson and Klein-Gordon with Neumann Conditions as a Generalized Problem of Two-Dimensional Moments

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Abstract

It will be shown that finding solutions from the Poisson and Klein-Gordon equations under Neumann conditions are equivalent to solving an integral equation, which can be treated as a generalized two-dimensional moment problem over a domain that is considered rectangular. The method consists to solve the integral equation numerically using the two-dimensional inverse moments problem techniques. We illustrate the different cases with examples.

Keywords

Equation in Poisson Partial Derivatives, Klein-Gordon Equation, Integral Equations, Generalized Moment Problem

1. Introduction

You want to find $w(x,t)$ such that

$$w_{xx} + w_{tt} = R(x,t) \quad \text{or} \quad w_{xx} - w_{tt} = R(x,t) \tag{1}$$

with $R(x,t)$ known about a D domain where

- $D = \{(x,t); a_1 < x < b_1, a_2 < t < b_2\}$
- $D = \{(x,t); a_1 < x < b_1, t > a_2\}$

The underlying space is $L^2(D)$. Under the conditions

$$\begin{aligned} w_x(a_1,t) &= k_1(t) \qquad & w_x(b_1,t) &= k_2(t) \\ w_t(x,a_2) &= h_1(t) \qquad & w_t(x,b_2) &= h_2(t) \\ w(x,a_2) &= s_1(t) \qquad & w(x,b_2) &= s_2(t) \end{aligned}$$

The problem has been largely studied and solved with different methods such as the method of finite differences [1–4] to name a few.

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation.

We want to present a an alternative method to solve a Poisson equation under Neumann conditions using techniques of generalized inverse moment problem, independently of other commonly used existing methods: finite difference method, Galerkin method, among many others. The interest is not to compare with the existing methods, but to present a different method to my novel criteria, and the one that I have already applied in other cases of partial differential equations under other conditions, for example the Poisson equation under Cauchy conditions or from Dirichlet. It turns out that a change in conditions implies a different approach. This is a significant change in the problem statement for its resolution.

The generalized moments problem [5–7] is to find a function $f(x)$ about a domain $\Omega \subset R^d$ that satisfies the sequence of equations

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i \epsilon N \tag{2}$$

where N is the set of the natural numbers, $(g_i(x))$ is a given sequence of functions in $L^2(\Omega)$ linearly independent known and the succession of real numbers $\{\mu_i\}_{i\epsilon N}$ is known data. The problem of Hausdorff moments [5–7], is to find a function $f(x)$ en (a,b) such that

$$\mu_i = \int_a^b x^i f(x) dx \quad i \epsilon N \tag{3}$$

In this case $g_i(x) = x^i$ with i belonging to the set N.

If the integration interval is $(0,\infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty,\infty)$ we have the problem of Hamburger moments [5–7]. The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [5–7]. There are several methods to build regularized solutions. One of them is the truncated expansion method [5]. This method is to approximate (2) with the finite moments problem

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i = 1,2,...,n \tag{4}$$

where it is considered as approximate solution of $f(x)$ to $p_n(x) = \sum_{i=0}^{n} \lambda_i \phi_i(x)$, and the functions $\{\phi_i(x)\}_{i=1,...,n}$ result of orthonormalize $\langle g_1, g_2, ..., g_n \rangle$ being λ_i the coefficients based on the data μ_i. In the subspace generated by $\langle g_1, g_2, ..., g_n \rangle$ the solution is stable. If $n \epsilon N$ is chosen in an appropriate way then the solution of (6) it approaches the solution of the original problem (2).

In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution (p. 19 a 30 de [5]).

2. Resolution of the Poisson Equation

We consider

$$w_{xx} + w_{tt} = R(x,t) \tag{5}$$

We take as an auxiliary function

$$u(m,r,x,t) = cos\left(\frac{m\pi}{b_1} x\right) e^{-r(t+1)}$$

If the D domain is bounded the conditions are:

$$\begin{aligned} w_x(a_1,t) &= k_1(t) \qquad w_x(b_1,t) = k_2(t) \\ w_t(x,a_2) &= h_1(x) \qquad w_t(x,b_2) = h_2(x) \end{aligned} \tag{6}$$

If the region D is not bounded the conditions are:

$$w_x(a_1,t) = k_1(t) \qquad w_x(b_1,t) = k_2(t) \qquad w_t(x,a_2) = h_1(x) \tag{7}$$

We define the vector field

$$F^* = (F_1(w), F_2(w)) = (w_x, w_t)$$

Since $div(F^*) = R(x,t)$ we have to:

$$\iint_D udiv(F^*)dA = \iint_D uR(x,t)dA$$

In addition, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, so

$$\iint_D udiv(F^*)dA = \iint_D div(uF^*)dA - \iint_D F^*.\nabla udA$$

where $\nabla u = (u_x, u_t)$.

And

$$\iint_D F^*.\nabla udA = \iint_D (F_1 u_x + F_2 u_t)\, dA$$

Integrating by parts:

$$\begin{aligned} \iint_D F_1 u_x dA &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_1 u_x dxdt \\ &= \int_{a_2}^{b_2} (w(b_1,t)u_x(m,r,b_1,t) - w(a_1,t)u_x(m,r,a_1,t))\, dt \\ &- \iint_D wu_{xx}dA \end{aligned}$$

$$u_x(m,r,b_1,t) = -e^{-r(t+1)} sen\left(\frac{m\pi}{b_1}b_1\right)\frac{m\pi}{b_1} = 0$$

$$u_x(m,r,a_1,t) = -e^{-r(t+1)} sen\left(\frac{m\pi}{b_1}a_1\right)\frac{m\pi}{b_1} \underbrace{=}_{a_1=0} 0$$

Analogously

$$\begin{aligned} \iint_D F_2 u_t dA &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} F_2 u_t dxdt \\ &= \int_{a_1}^{b_1} (w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2))\, dx \\ &- \iint_D wu_{tt}dA \end{aligned}$$

If $m \in N$ y $a_1 = 0$ then

$$\begin{aligned} &\iint_D F^*.\nabla udA \\ &= \int_{a_1}^{b_1} (w(x,b_2)u_t(m,r,x,b_2) - w(x,a_2)u_t(m,r,x,a_2))\, dx \\ &- \iint_D w(u_{xx} + u_{tt})dA \end{aligned}$$

where

$$\iint_D w(u_{xx}+u_{tt})dA = \iint_D wu\left(-\left(\frac{m\pi}{b_1}\right)^2+(t+1)^2\right)dA$$

On the other hand,

$$\begin{aligned}&\int_C (uF^*).nds\\ &= -\int_{a_1}^{b_1} u(m,r,x,a_2)w_t(x,a_2)dx \int_{a_1}^{b_1} u(m,r,x,b_2)w_t(x,b_2)dx\\ &+\int_{a_2}^{b_2} u(m,r,b_1,t)w_x(b_1,t)dt\\ &-\int_{a_2}^{b_2} u(m,r,a_1,t)w_x(a_1,t)dt = G(m,r)\end{aligned}$$

$$\begin{aligned}&\therefore \iint uR(x,t)dA = G(m,r)\\ &-\int_{a_1}^{b_1}(w(x,b_2)u_t(m,r,x,b_2)-w(x,a_2)u_t(m,r,x,a_2))dx\\ &+\iint_D wu\left(-\left(\frac{m\pi}{b_1}\right)^2+(r+1)^2)dA\right.\end{aligned}$$

$$\begin{aligned}&\therefore \iint_D wu\left(-\left(\frac{m\pi}{b_1}\right)^2+(r+1)^2\right)dA\\ &=-\int_{a_1}^{b_1}(w(x,b_2)u_t(m,r,x,b_2)-w(x,a_2)u_t(m,r,x,a_2))\,dx\\ &-G(m,r)+\iint_D uR(x,t)dA=\varphi(m,r)\end{aligned}$$

$$\therefore \iint_D wudA = \frac{\varphi(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2+(r+1)^2\right)}$$

If the equation is from Klein-Gordon, it is taken

$$F^*=(F_1(w),F_2(w))=(w_x,-w_t)$$

$$\therefore \iint_D wudA = \frac{\varphi^*(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2-(t+1)^2\right)}$$

where, in $\varphi^*(m,r)$ we have to $G(m,r)$ is different.

To solve this integral equation we give integer values to m and r:

$$m=0,1,2,...n_1-1;\qquad r=1,2,...,n_2$$

then

$$\iint_D w(x,t)H_{mr}(x,t)dA = \frac{\varphi(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2+(t+1)^2\right)}=\mu_{mr} \tag{8}$$

We interpret (8) as a moments problem of two-dimensional generalized. $p_n(x,t)$ is the numerical approximation with the truncated expansion method for $w(x,t)$ with

$$n = n_1.n_2$$

$$H_{mr}(x,t) = u(m,r,x,t) \quad m = 0,1,2,...n_1-1; \quad r = 1,2,...,n_2$$

3. Solution of the Generalized Moments Problem

We can apply the detailed truncated expansion method in [7] and generalized in [8] and [9] to find an approximation $p_n(x,t)$ for the corresponding finite problem with $i = 0,1,2,...,n$, where n is the number of moments μ_i. We consider the basis $\phi_i(x,t)$ $i = 0,1,2,...,n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t)$ $i = 0,1,2,...,n$. We approximate the solution $w(x,t)$ with [7] and generalized in [8] y [9]:

$$p_n(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{donde} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\mu_j \quad i = 0,1,2,...,n$$

And the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) \mid \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) \cdot \|\phi_i(x,t)\|^{-1} \quad 1 < i \le n; \ 1 \le j < i$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0,1,...,n.$$

The proof of the following theorem is in [9,10]. In [10] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [11] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

3.1. Theorem

Let $\{\mu_i\}_{i=0}^{n}$ be a set of real numbers and suppose that $f(x,t) \in L^2((a_1,b_1)\times(a_2,b_2))$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n} \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} H_i(x,t) f(x,t) dxdt - \mu_i \right|^2 \le \varepsilon^2$$

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} ((b_1-a_1)^2 f_x^2 + (b_2-a_2)^2 f_t^2) dxdt \le M^2 \tag{9}$$

then

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} |f(x,t)|^2 dxdt \le \min_i \left\{ \|CC^T\| \varepsilon^2 + \frac{M^2}{8(i+1)^2}; \ i = 0,1,...,n \right\}$$

where C it is a triangular matrix with elements C_{ij} ($1 < i \leq n$; $1 \leq j < i$) and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|p_n(x,t)-f(x,t)\right|^2 dxdt \leq \left\|CC^T\right\|\varepsilon^2+\frac{M^2}{8\left(n+1\right)^2} \quad (10)$$

If b_2 it is not finite then (9) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}(xf_x^2+tf_t^2)dxdt \leq M^2 \quad (11)$$

And it must be fulfilled that

$$t^i f(x,t) \longrightarrow 0 \quad \text{if} \quad t \longrightarrow \infty \quad \forall i \epsilon N$$

3.2. Numerical Examples

3.2.1. Example 1

We consider the equation

$$w_{xx}+w_{tt}=-3cos(2x)Cosech(\pi)Senh(t) \qquad \text{en} \quad (0,2)\times(0,2)$$

whose solution is: $w(x,t)=cos(2x)\dfrac{senh(t)}{senh(\pi)}$.

We take $n=9$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^2\int_0^2(p_9(x,t)-w(x,t))^2dtdx=0.0415341$$

In Figure 1 the graphics of: $p_9(x,t)$ (color magenta) $yw(x,t)$ (color celeste) are superimposed.

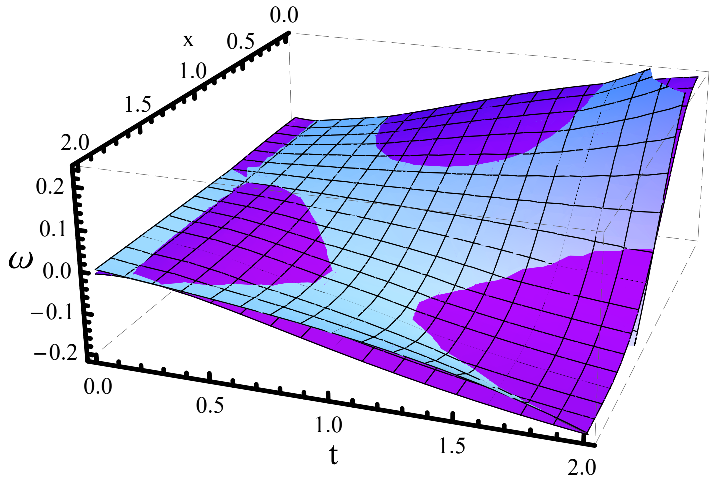

Figure 1. $p_9(x,t)$ and $w(x,t)$ for example 1.

3.2.2. Example 2

We consider the equation

$$w_{xx}+w_{tt}=0 \qquad \text{en} \quad (0,1)\times(0,\infty)$$

whose solution is: $sen(x)e^{-t}$.

We take $n = 9$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^1 \int_0^\infty (p_9(x,t) - w(x,t))^2 dtdx = 0.0193919$$

In Figure 2 the graphics of: $p_9(x,t)$ (color magenta) $yw(x,t)$ (color celeste) are superimposed.

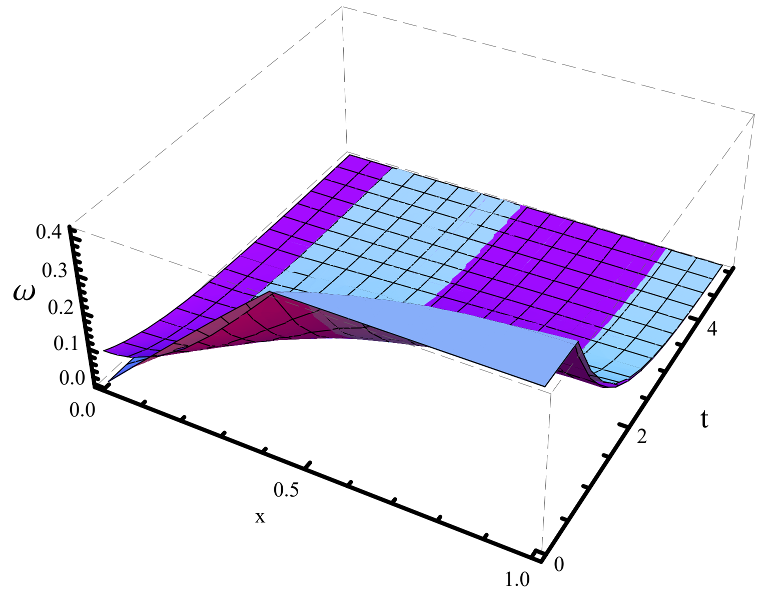

Figure 2. $p_9(x,t)$ and $w(x,t)$ for example 2.

3.2.3. Example 3

We consider the equation

$$w_{xx} - w_{tt} = -8e^{-3t+x} \qquad \text{en} \quad (0,1)\times(0,2)$$

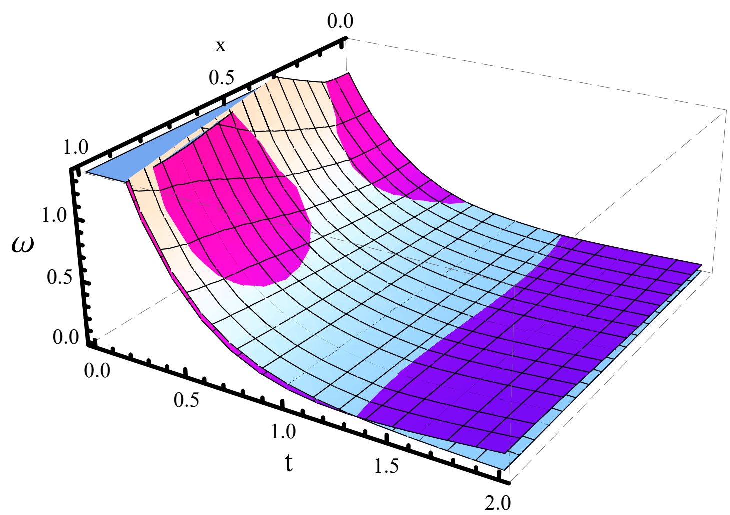

Figure 3. $p_9(x,t)$ and $w(x,t)$ for example 3.

whose solution is: e^{x-3t}.

We take $n = 9$ moments and is approaching $w(x,t)$ where the accuracy is

$$\int_0^1 \int_0^2 (p_9(x,t) - w(x,t))^2 dt dx = 0.0552525$$

In Figure 3: $p_9(x,t)$ (color magenta) $y w(x,t)$ (color celeste) are superimposed.

4. Conclusions

An equation in partial Poisson derivatives of the form $w_{xx} + w_{tt} = R(x,t)$ or from Klein-Gordon $w_{xx} - w_{tt} = R(x,t)$ where the unknown function $w(x,t)$ is defined in $D = (0,b_1) \times (a_2,b_2)$ or $D = (0,b_1) \times (a_2,\infty)$ under Neumann's conditions can be solved numerically by applying inverse moment problem techniques.

1. First the partial derivatives equation is written as an integral equation

$$\therefore \iint_D w u dA = \frac{\varphi(m,r)}{\left(-\left(\frac{m\pi}{b_1}\right)^2 + (r+1)^2\right)}$$

2. To solve this integral equation we give integer values to m and r:

$$m = 0,1,2,\ldots n_1 - 1; \qquad r = 1,2,\ldots,n_2$$

then

$$\iint_D w(x,t) H_{mr}(x,t) dA = \frac{\varphi(m,r)}{\left(-\left(\dfrac{m\pi}{b_1}\right)^2 + (t+1)^2\right)} = \mu_{mr}$$

We have a problem of two-dimensional generalized moments.
$p_n(x,t)$ is a numerical approximation with the truncated expansion method for $w(x,t)$, where $n = n_1.n_2$ $H_{mr}(x,t) = u(m,r,x,t)$ $m = 0,1,2,\ldots n_1$; $r = 1,2,\ldots,n_2$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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