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# New Exact Solutions for the Coupled Nonlinear Schrödinger Equations with Variable Coefficients

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## Abstract

In this paper, coupled nonlinear Schrödinger equations with variable coefficients are studied, which can be used to describe the interaction among the modes in nonlinear optics and Bose-Einstein condensation. Some novel bright-dark solitons and dark-dark solitons are obtained by modified Sine-Gordon equation method. Moreover, some figures are provided to illustrate how the soliton solutions propagation is determined by the different values of the variable group velocity dispersion terms, which can be used to model various phenomena.

## Keywords

## 1. Introduction

In nonlinear optics, the coupled nonlinear Schrödinger (CNLS) equations are often used to describe propagation of optical soliton in birefringence fibers, multimode fibers and optical fiber arrays. Many researchers have studied the CNLS equation with constant coefficient. In recent years, a number of methods are used to solve the coupled integrable nonlinear models, such as Hirota bilinear method [1] [2] [3], Painlev analysis method [4], Function expansion method [5] and direct perturbation method [6] and so on. However, the evolutions of vector solitons for CNLS equation with constant coefficients are not dependent on any controllable parameters. With the development of modern science, people need to manage and control soliton propagation, which will make solitons into practical information carriers. Therefore, variable coefficient equation has more prac-

tical significance.

In this paper, we will consider the following coupled nonlinear Schrödinger equation with variable coefficients (VCNLS) [7]:

$$
\begin{aligned}
& i\psi_{1t} + a(t)\psi_{1xx} + \left(b_1(t)|\psi_1|^2 + b_2(t)|\psi_2|^2\right)\psi_1 + v(t)\psi_1 = 0, \\
& i\psi_{2t} + a(t)\psi_{2xx} + \left(b_1(t)|\psi_1|^2 + b_2(t)|\psi_2|^2\right)\psi_2 + v(t)\psi_2 = 0.
\end{aligned}
\tag{1}
$$

where $\psi_1$ and $\psi_2$ are complex envelopes of the propagating beam of the two modes, and $x$, $t$ are the spatial coordinate and retarded time respectively. The coefficients $a(t)$ represent the group velocity dispersion; $b_1(t)$, $b_2(t)$ mean nonlinear interactions and $v(t)$ means external potential. Finding exact solutions of VCNLS, especially soliton solutions [8] [9] have been a subject of great interest to mathematicians and physicists. Han [7] constructed an explicit transformation, which maps VCNLS to the classical CNLS, and obtains Bright-Dark solitons for VCNLS. Exact traveling wave and soliton solutions of the VCNLS equation have been obtained by Zhong [10] using homogeneous balance principle and the F-expansion technique. Yu and Yang [11] presented the similarity transformations for this system. Because of the complexity of VCNLS form, it is difficult to solve it directly, so the study of this kind of soliton solutions is not so extensive.

The paper is organized as follows. Section 2 describes the modified Sine-Gordon equation method [12]. In Section 3, we have applied the method to the VCNLS, and derive some bright-dark solitons and dark-dark solitons. Section 4 is devoted to analysis of shape changing exhibited by these soliton solutions when variable coefficients are altered. In the last section, Section 5, conclusion is presented.

## 2. The Method

Let us consider a form of a nonlinear partial differential equation

$$
H_k\left(x, t, \frac{\partial \psi_k}{\partial x}, \frac{\partial \psi_k}{\partial t}, \frac{\partial^2 \psi_k}{\partial x^2}, \cdots, a(t), b_1(t), b_2(t), v(t)\right) = 0,
\tag{2}
$$

where $a(t)$, $b_k(t)$ and $v(t)$ are arbitrary functions in $t$ and $k = 1, 2$. In the following, we offer the main steps of this method:

**Step 1:** Use the following assumptions:

$$
\begin{aligned}
& \psi_1(x, t) = U_1(\zeta)e^{i(\alpha x - \theta(t))}, \\
& \psi_2(x, t) = U_2(\zeta)e^{i(\alpha x - \theta(t))}.
\end{aligned}
\tag{3}
$$

where $U_1(\zeta)$ and $U_2(\zeta)$ are the new dependant functions, $\zeta = \mu(x - \lambda(t))$ is the new independent variable, $\lambda(t)$ is an arbitrary function of $t$ and $\mu$ and $\alpha$ are the frequency and the width of the soliton respectively.

**Step 2:** Collect the coeffients of $U_1(\zeta)$ and $U_2(\zeta)$ and their derivatives, and then assume the imaginary part is equal to zero.

**Step 3:** Take the coefficient of the largest linear term as the normalization

coefficient.

**Step 4:** The derivatives and powers of $U_1(\zeta)$ and $U_2(\zeta)$ are equal to the term multiplied by a constant, so the arbitrary functions will be determined, and the Equation (2) is transformed into the following nonlinear ordinary differential system.

$$Q_i\left(U_i(\zeta), U_i'(\zeta), U_i''(\zeta), \cdots\right) = 0 \qquad (4)$$

**Step 5:** Use the solutions of the Sine-Gordon equation [13] [14] by assuming that

$$
\begin{aligned}
U_1(\xi) &= \sum_{i=1}^{n} \cos^{i-1}\left(w(\xi)\right) \times \left[B_i \sin\left(w(\xi)\right) + A_i \cos\left(w(\xi)\right)\right] + A_0, \\
U_2(\xi) &= \sum_{j=1}^{m} \cos^{j-1}\left(w(\xi)\right) \times \left[E_j \sin\left(w(\xi)\right) + D_j \cos\left(w(\xi)\right)\right] + D_0.
\end{aligned}
\qquad (5)
$$

where $A_i$, $B_i$, $D_i$, $E_i$, $A_0$ and $D_0$ are arbitrary constants and $n$ and $m$ are determined by balancing the most dispersive term and the greatest nonlinear term in Equation (4), and

$$w'(\xi) = \sin\left(w(\xi)\right) \qquad (6)$$

and

$$\sin\left(w(\xi)\right) = \operatorname{sech}(\xi) \ \text{ or } \ \cos\left(w(\xi)\right) = \tanh(\xi) \qquad (7)$$

**Step 6:** Equating the coefficients of $\sin^i\left(w(\xi)\right)$ and $\cos^i\left(w(\xi)\right)$ to zero and an algebraic system for the constant $A_i$, $B_i$, $D_i$, $E_i$, $A_0$ and $D_0$ are obtained, by solving them with a Maple program and back-substituting into Equation (5) and Equation (3) via Equation (7), novel soliton solutions are obtained for the system of Equation (2).

**Advantages of the method:** The Sine-Gordon equation method has limitations and is suitable for some constant coefficient systems, but modified Sine-Gordon equation method is applicable to systems with variable coefficients containing imaginary parts. As a result, some spanking new solutions might be originated via this method and this method can use computational software like Maple or Mathematica to reduce the amount of computation.

## 3. Exact Solutions for VCNLS

By substituting the assumptions in Equation (3) into Equation (1), we obtain

$$
\begin{aligned}
&a(t)\mu^2 U_1''(\xi) + i\mu\left(2\alpha a(t) - \lambda'\right)U_1'(\xi) + \left(\theta' - \alpha^2 a(t)\right)U_1(\xi) \\
&+ \left(b_1(t)U_1^2(\xi) + b_2(t)U_2^2(\xi)\right)U_1(\xi) + v(t)U_1(\xi) = 0, \\
&a(t)\mu^2 U_2''(\xi) + i\mu\left(2\alpha a(t) - \lambda'\right)U_2'(\xi) + \left(\theta' - \alpha^2 a(t)\right)U_2(\xi) \\
&+ \left(b_1(t)U_1^2(\xi) + b_2(t)U_2^2(\xi)\right)U_2(\xi) + v(t)U_2(\xi) = 0.
\end{aligned}
\qquad (8)
$$

to make Equation (8) real, the terms $U_1'(\xi)$ and $U_2'(\xi)$ must be eliminated, so according to Step 2, we get $\lambda(t) = 2\alpha \int a(t)\mathrm{d}t + \lambda_0$, and then follow Step 3, that is, take the coefficients of $U_1''(\xi)$ and $U_2''(\xi)$ as the normalized coeffi-

cients, and we get

$$\theta(t) = \int \left(\mu^2 c_1 + \alpha^2\right) a(t) \mathrm{d}t + \theta_0 \tag{9}$$

$$\lambda(t) = 2\alpha \int a(t) \mathrm{d}t + \lambda_0 \tag{10}$$

$$b_1(t) = c_2 \mu^2 a(t) \tag{11}$$

$$b_2(t) = c_3 \mu^2 a(t) \tag{12}$$

$$v(t) = c_4 \mu^2 a(t) \tag{13}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are constants and $\lambda_0$ and $\theta_0$ is an integration constant. Therefore,

$$\xi = \mu \left( x - 2\alpha \int a(t) \mathrm{d}t \right) \tag{14}$$

And, Equation (8) can be simplified as follows

$$U_1''(\xi) + c_1 U_1(\xi) + \left( c_2 U_1^2(\xi) + c_3 U_2^2(\xi) \right) U_1(\xi) + c_4 U_1(\xi) = 0,$$
$$U_2''(\xi) + c_1 U_2(\xi) + \left( c_2 U_1^2(\xi) + c_3 U_2^2(\xi) \right) U_2(\xi) + c_4 U_2(\xi) = 0. \tag{15}$$

By balancing the dispersive and nonlinear terms in Equation (15) we get $m+2 = 2m+m$ and $n+2 = 2n+n$, *i.e.* $m = n = 1$, so according to Step 5, we assume that

$$U_1(\xi) = B_1 \sin\left(w(\xi)\right) + A_1 \cos\left(w(\xi)\right) + A_0,$$
$$U_2(\xi) = E_1 \sin\left(w(\xi)\right) + D_1 \cos\left(w(\xi)\right) + D_0. \tag{16}$$

Substituting Equation (16) and the necessary derivatives into Equation (15) using Equation (6), applying trigonometric identities and collecting the coefficients of $\sin\left(w(\xi)\right)$ and $\cos\left(w(\xi)\right)$ that are containing independent combinations to zero, and we obtain the following independent parametric equations:

$\sin\left(w(\xi)\right)$:

$$c_1 B_1 + c_4 B_1 + 3c_2 B_1 A_0^2 + c_3 B_1 D_0^2 + 2c_3 E_1 D_0 A_0 = 0 \tag{17}$$

$\cos\left(w(\xi)\right)$:

$$c_1 A_1 + c_4 A_1 + 3c_2 A_1 A_0^2 + c_3 A_1 D_0^2 + 2c_3 D_1 D_0 A_0 + c_2 A_1^3 + c_3 A_1 D_1^2 = 0 \tag{18}$$

$\sin^3\left(w(\xi)\right)$:

$$-B_1 + c_2 B_1^3 + c_3 B_1 E_1^2 = 0 \tag{19}$$

$\sin\left(w(\xi)\right)\cos^2\left(w(\xi)\right)$:

$$B_1 + 3c_2 B_1 A_1^2 + c_3 B_1 D_1^2 + 2c_3 A_1 E_1 D_1 = 0 \tag{20}$$

$\sin^2\left(w(\xi)\right)\cos\left(w(\xi)\right)$:

$$-2A_1 + 3c_2 A_1 B_1^2 + c_3 A_1 E_1^2 + 2c_3 E_1 D_1 B_1 - c_2 A_1^3 - c_3 A_1 D_1^2 = 0 \tag{21}$$

$\sin^2\left(w(\xi)\right)$:

$$3c_2 A_0 B_1^2 + c_3 A_0 E_1^2 + 2c_3 E_1 D_0 B_1 = 0 \tag{22}$$

$\sin\left(w(\xi)\right)\cos\left(w(\xi)\right)$:

$$6c_2 B_1 A_1 A_0 + 2c_3 D_1 D_0 B_1 + 2c_3 A_1 E_1 D_0 + 2c_3 A_0 E_1 D_1 = 0 \tag{23}$$

$\cos^2\left(w(\xi)\right):$

$$3c_2 A_0 A_1^2 + 2c_3 A_1 D_1 D_0 + c_3 A_0 D_1^2 = 0 \tag{24}$$

constants:

$$c_1 A_0 + c_4 A_0 + c_2 A_0^3 + c_3 A_0 D_0^2 = 0 \tag{25}$$

Solving Equations (17)-(25), we obtain the following cases and solutions using Equation (7).

**Case 1:** When $A_0 = B_1 = 0, c_2 = -\dfrac{2}{A_1^2}, c_3 = 0, c_4 = -c_1 + 2$, we get the following bright-dark solitons:

$$
\begin{aligned}
\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right), \\
\psi_2(x,t) &= \left[E_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right) + D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right) + D_0\right] \quad (26) \\
&\quad \times \exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right).
\end{aligned}
$$

where $A_1$, $D_1$, $E_1$ and $c_1$ are arbitrary constants.

**Case 2:** When $A_0 = B_1 = D_0 = E_1 = 0, c_3 = -\dfrac{2}{D_1^2}, c_4 = -c_1$, we get the following dark-dark solitons:

$$
\begin{aligned}
\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right), \\
\psi_2(x,t) &= D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right).
\end{aligned} \tag{27}
$$

where $A_1$, $D_1$, $c_1$ and $c_2$ are arbitrary constants.

**Case 3:** When $A_0 = A_1 = D_0 = E_1 = 0, c_2 = -\dfrac{2}{B_1^2}, c_3 = -\dfrac{2}{D_1^2}, c_4 = -c_1$, we get the following bright-dark solitons:

$$
\begin{aligned}
\psi_1(x,t) &= B_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right), \\
\psi_2(x,t) &= D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right).
\end{aligned} \tag{28}
$$

where $B_1$, $D_1$, and $c_1$ are arbitrary constants.

**Case 4:** When $A_0 = B_1 = D_0 = D_1 = 0, c_2 = -\dfrac{c_4 + c_1}{A_1^2}, c_3 = -\dfrac{c_4 + c_1 - 2}{E_1^2}$, we get the following dark-bright solitons:

$$
\begin{aligned}
\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right)\exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right), \\
\psi_2(x,t) &= \left[E_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)\,\mathrm{d}t\right)\right) + D_0\right] \quad (29) \\
&\quad \times \exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)\,\mathrm{d}t + \theta_0\right)\right).
\end{aligned}
$$

where $A_1$, $E_1$, $c_1$ and $c_4$ are arbitrary constants.

## 4. Physical Application

In this segment, we will illustrate the figure and designate the acquired solutions to the VCNLS equations. The solutions (26)-(29) come in terms of hyperbolic function. Next, we study the evolution behavior of the dark-bright soliton solutions given by Equation (26), the bright-dark soliton solutions given by Equation (28), and interaction of the two solutions given by Equation (28), illustrated in the figures.

In **Figure 1**, we shows the soliton solutions evolution of Equation (26) with different variable coefficients 1, $t$, $t^2$, sin($t$). **Figure 1(a)** and **Figure 1(e)** depict the result with $a(t)=1$, which shows a dark bell-shaped soliton and a singular soliton. When $a(t)=t$, we obtain a parabolic cubic soliton, as shown in (b) and (f). When $a(t)=t^2$, we obtain a cubic soliton, as shown in (c) and (g). Periodical-oscillating soliton is obtained when we choose $a(t)=\sin(t)$ as depicted in (d) and (h).

In **Figure 2**, we shows the soliton solution evolutionis of Equation (28) with different variable coefficients 1, $t$, $t^2$, sin($t$). **Figure 2(a)** and **Figure 2(e)** depict the result with $a(t)=1$, which shows a bright bell-shaped solition and a dark bell-shaped soliton. When $a(t)=t$, we obtain a parabolic cubic soliton, as shown in (b) and (f). When $a(t)=t^2$, we obtain a cubic soliton, as shown in (c) and (g). Periodical-oscillating soliton is obtained when we choose $a(t)=\sin(t)$ as depicted in (d) and (h).
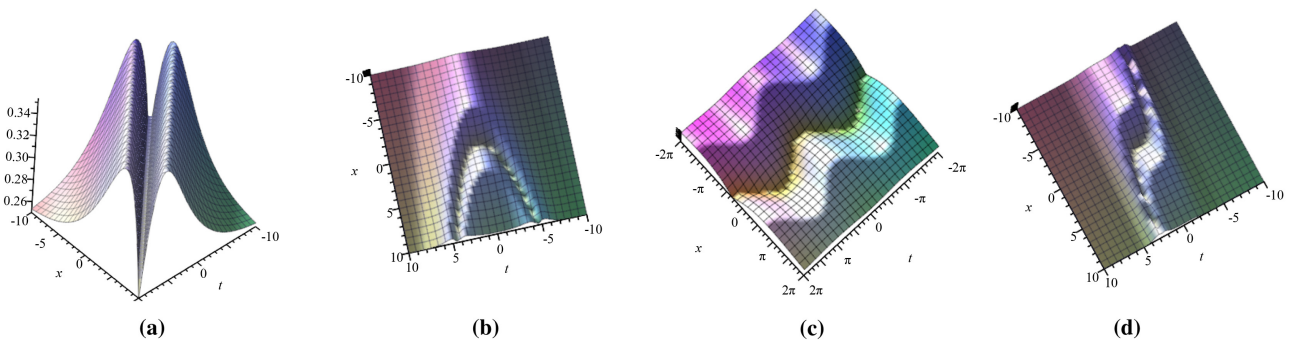
In **Figure 3**, we can obtain the similar results. **Figure 3** demonstrates that
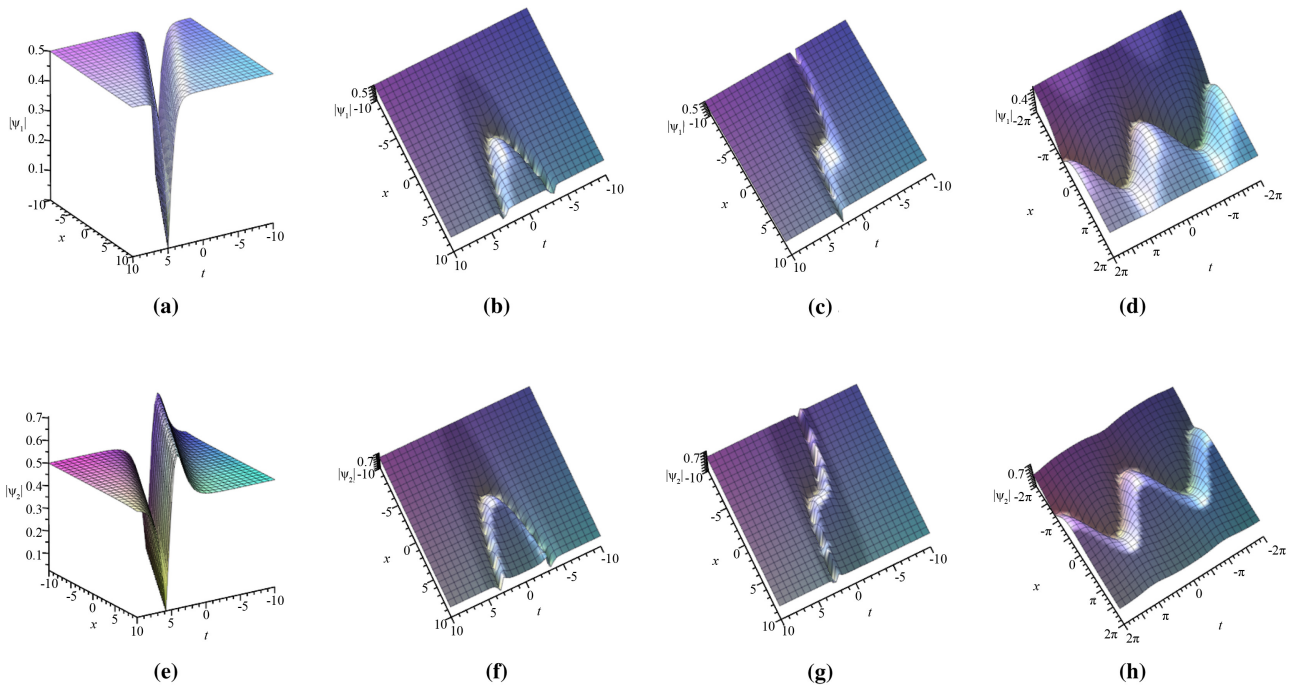


**Figure 1.** Evolution of dark-bright soliton solutions of Equation (26), plotted for different values of $a(t)$. (a) $a(t)=1$; (b) $a(t)=t$; (c) $a(t)=t^2$; (d) $a(t)=\sin(t)$; (e) $a(t)=1$; (f) $a(t)=t$; (g) $a(t)=t^2$; (h) $a(t)=\sin(t)$.
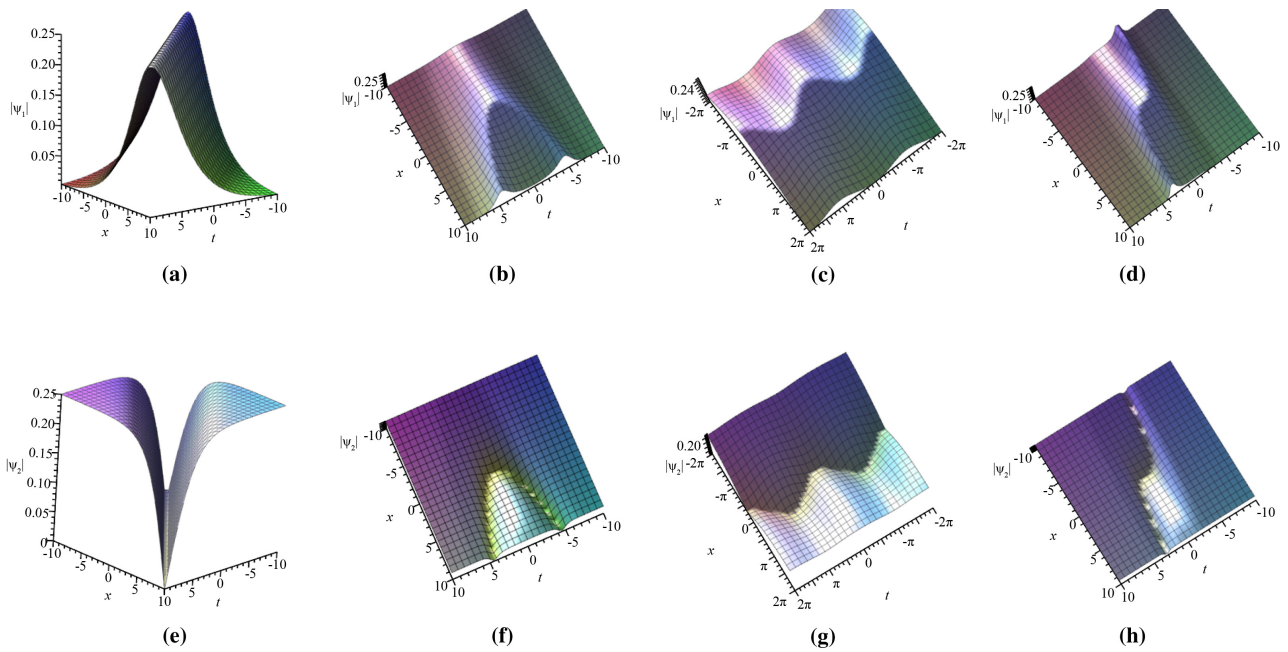
**Figure 2.** Evolution of bright-dark soliton solution of Equation (28), plotted for different values of $a(t)$. (a) $a(t)=1$; (b) $a(t)=t$; (c) $a(t)=t^2$; (d) $a(t)=\sin(t)$; (e) $a(t)=1$; (f) $a(t)=t$; (g) $a(t)=t^2$; (h) $a(t)=\sin(t)$.



**Figure 3.** Evolution and interaction of Equation (28) with the same parameters as **Figure 2**. (a) $a(t)=1$; (b) $a(t)=t$; (c) $a(t)=t^2$; (d) $a(t)=\sin(t)$.

each soliton shape keeps invariant after interaction, which denotes that the interaction is elasticwe. We can see that the solitons show a periodic property but the solitons are not symmetrical in the $t$ direction, and have the bell-shaped, parabolic, cubic or periodical-oscillating shapes.

## 5. Conclusion

In this paper, we have obtained some bright-dark soliton solutions and dark-dark soliton solutions of the coupled nonlinear Schrödinger equation with variable coefficients, using modified Sine-Gordon equation method. Then, we have discussed the effects of $a(t)$, which is the group velocity dispersion. For bright-dark soliton solutions 26 and 28, we have chosen $a(t)$ as the constant, linear, quadratic and trigonometric functions, respectively, and the bell-shaped, parabolic,

cubic and quasi-parabolic solitons have been obtained correspondingly, as shown in Figure 1 and Figure 2, and found that the propagation of the soliton solutions is determined by this value. We also investigated the evolution and interaction between the two solutions, and obtained that each solution shape keeps invariant after interaction and a periodic property in the $t$ direction, as presented in Figure 3. The results show that modified sine-Gordon method gives soliton solutions for variable coefficients systems directly, without difficult calculations and also could be applied to many coupled nonlinear models.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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