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Multiplicity of Solutions for Fractional Hamiltonian Systems under Local Conditions

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Abstract

Under some local superquadratic conditions on $W(t,u)$ with respect to *u*, the existence of infinitely many solutions is obtained for the nonperiodic fractional Hamiltonian systems ${}_tD_\infty^\alpha\left({}_{-\infty}D_t^\alpha u(t)\right)+L(t)u(t)=\nabla W(t,u(t)), \forall t\in\mathbb{R}$, where $L(t)$ is unnecessarily coercive.

Keywords

Fractional Hamiltonian Systems, Local Conditions, Variational Methods

1. Introduction

In this paper, we consider the fractional Hamiltonian system

$$
{}_tD_\infty^\alpha\left({}_{-\infty}D_t^\alpha u(t)\right)+L(t)u(t)=\nabla W(t,u(t)),\ u\in H^\alpha\left(\mathbb{R},\mathbb{R}^N\right), \tag{1}
$$

where $\alpha\in(1/2,1)$, $t\in\mathbb{R}$, $L\in C\left(\mathbb{R},\mathbb{R}^{N^2}\right)$ is a symmetric and positive definite matrix for all $t\in\mathbb{R}$, $W\in C^1\left(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}\right)$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at *u*. In the following, $(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}^N\mapsto\mathbb{R}$ denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm.

Fractional calculus has received increased popularity and importance in the past decade, which is mainly due to its extensive applications in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, etc. (see [1]-[6]). Models containing left and right fractional differential operators have been recognized as best tools to describe long-memory processes and hereditary properties. However, compared with classical theories for integer-order differential equations, researches on fractional differential equations are only on their initial stage of development.

Recently, the critical point theory and variational methods have become effective tools in studying the existence of solutions to fractional differential equations with variational structures. In [7], for the first time, Jiao and Zhou used the critical point theory to tackle the existence of solutions to the following fractional boundary value problem

$$ {}_tD_T^{\alpha}\left({}_0D_t^{\alpha}u(t)\right)=\nabla F\left(t,u(t)\right),\ \text{a.e. } t\in[0,T],\ u(0)=u(T)=0. $$

Jiao and Zhou studied the problem by establishing corresponding variational structure in some suitable fractional space and applying the least action principle and Mountain Pass theorem. Then in [8], Torres proved the existence of solutions for the fractional Hamiltonian system (1) by using the Mountain Pass theorem. The author showed that (1) possesses at least one nontrivial solution by assuming that W satisfies the (AR) condition and L satisfies the following coercive condition:

(L) $L(t)$ is a positive definite symmetric matrix for all $t\in\mathbb{R}$, and there exists an $l\in C\left(\mathbb{R},(0,\infty)\right)$ such that $l(t)\to\infty$ as $|t|\to\infty$ and

$$ \left(L(t)x,x\right)\geq l(t)|x|^2,\ \forall t\in\mathbb{R},\ \forall x\in\mathbb{R}^N. $$

Subsequently, the existence and multiplicity of solutions for the fractional Hamiltonian system (1) have been extensively investigated in many papers; see [9]-[15] and the references therein. However, it is worth noting that in most of these papers, L is required to satisfy the coercivity condition (L). Recently, the authors in [16] proved the existence of one nontrivial solution for (1), where L does not necessarily satisfy the condition (L) and W satisfies some kind of local superquadratic condition:

(W) There exist $b_1,b_2\in\mathbb{R}\ (b_1<b_2)$ such that $\lim_{|x|\to\infty}|W(t,x)|/|x|^2=\infty$ uniformly with respect to $t\in(b_1,b_2)$.

Here W is only required to be superquadratic at infinitely with respect to x when the first variable t belongs to some finite interval.

Motivated by the above papers, in this note, we will consider the multiplicity of solutions for the fractional Hamiltonian system (1), where L is not necessarily coercive and W satisfies some local growth condition. The exact assumptions on L and W are as follows:

Theorem 1. *Assume the following conditions hold*:

(L_1) There exists $l_1>0$ such that

$$ l(t)\geq l_1,\ \forall t\in\mathbb{R}, $$

and

$$ \int_{\mathbb{R}}\left(l(t)\right)^{-1}\mathrm{d}t<\infty, $$

where $l(t)=\inf_{x\in\mathbb{R}^N,|x|=1}\left(L(t)x,x\right)$ is the smallest eigenvalue of $L(t)$;

(W_1) $W\in C^1\left(\mathbb{R}\times B_\delta(0),\mathbb{R}\right)$ is even in x and $W(t,0)=0$, where $B_\delta(0)$ denotes the ball in $\mathbb{R}^N$ centered at 0 with radius $\delta>0$;

(W_2) There are constants $c_1>0$ and $0<\theta<1$ such that

$$ \left|\nabla W(t,x)\right|\leq c_1|x|^{\theta},\ \forall(t,x)\in\mathbb{R}\times B_\delta(0); $$

(W_3) There exists a constant $p > 2$ such that

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^p} = 0 \text{ uniformly for } t \in \mathbb{R};$$

(W_4) $2W(t,x) - (\nabla W(t,x), x) < 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$;

(W_5) There exists a constant $\mu > 2$ such that

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^\mu} = \infty \text{ uniformly for } t \in \mathbb{R}.$$

Then problem (1) has a sequence of solutions $\{u_k\}$ such that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$.

Remark 1. *There exist L and W that satisfy all assumptions in Theorem* 1. *For example*, *let*

$$L(t) = \begin{cases} \left[\left(n^2+1\right)^2 (t-n) + c\right] I_N, & n \le t < n + \dfrac{1}{n^2+1}, \\ \left[\left(n^2+1\right)^2 + c\right] I_N, & n + \dfrac{1}{n^2+1} \le t < n + \dfrac{n^2}{n^2+1}, \\ \left[\left(n^2+1\right)^2 (n+1-t) + c\right] I_N, & n + \dfrac{n^2}{n^2+1} \le t < n+1, \end{cases}$$

and

$$W(t,x) = |x|^4 \text{ for } |x| < 1$$

with $\theta = 1/2$, $p = 3$, $\mu = 5$. Note that W is superquadratic near the origin and there are no conditions assumed on W for $|x|$ large. As far as the authors know, there is little research concerning the multiplicity of solutions for problem (1) simultaneously under local conditions and non-coercivity conditions, so our result is different from the previous results in the literature.

The proof is motivated by the argument in [17]. We will modify and extend W to an appropriate $\tilde{W}$ and show for the associated modified functional I the existence of a sequence of solutions converging to zero in L^∞ norm, therefore to obtain infinitely many solutions for the original problem.

2. Preliminary Results

In this section, for the reader's convenience, we introduce some basic definitions of fractional calculus. The left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as

$$_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-\xi)^{\alpha-1} u(\xi)\,\mathrm{d}\xi,$$

$$_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi-x)^{\alpha-1} u(\xi)\,\mathrm{d}\xi.$$

The left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as

$$_{-\infty}D_x^\alpha u(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty} I_x^{1-\alpha} u(x), \tag{2}$$

$$ {}_xD_\infty^\alpha u(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\int_x I_\infty^{1-\alpha} u(x). \tag{3} $$

The definitions of (2) and (3) may be written in an alternative form as follows:

$$ {}_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}\mathrm{d}\xi, $$

$$ {}_xD_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}\mathrm{d}\xi. $$

Moreover, recall that the Fourier transform $\hat{u}(w)$ of $u(x)$ is defined by

$$ \hat{u}(w) = \int_{-\infty}^{\infty} \mathrm{e}^{-iwx} u(x)\mathrm{d}x. $$

To establish the variational structure which enables us to reduce the existence of solutions of (1), it is necessary to construct appropriate function spaces. In what follows, we introduce some fractional spaces, for more details see [8] and [18]. Denote by $L^p \equiv L^p(\mathbb{R},\mathbb{R}^N)$ $(1 \le p < \infty)$ the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$ \|u\|_{L^p} = \left(\int_{\mathbb{R}} |u(t)|^p \,\mathrm{d}t\right)^{1/p}, $$

and $L^\infty(\mathbb{R},\mathbb{R}^N)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$ \|u\|_\infty = \operatorname{esssup}\{|u(t)| : t \in \mathbb{R}\}. $$

For $\alpha > 0$, define the semi-norm

$$ |u|_{I_{-\infty}^\alpha} = \left\| {}_{-\infty}D_x^\alpha u \right\|_{L^2}, $$

and the norm

$$ \|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2\right)^{1/2}. $$

Let

$$ I_{-\infty}^\alpha = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^N)}^{\|\cdot\|_{I_{-\infty}^\alpha}}, $$

where $C_0^\infty(\mathbb{R},\mathbb{R}^N)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity.

Now we can define the fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^N)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$ |u|_\alpha = \left\| |w|^\alpha \hat{u} \right\|_{L^2}, $$

and the norm

$$ \|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{1/2}. $$

Set

$$ H^\alpha = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^N)}^{\|\cdot\|_\alpha}. $$

Moreover, we note that a function $u \in L^2(\mathbb{R},\mathbb{R}^N)$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$|w|^{\alpha}\hat{u} \in L^2(\mathbb{R},\mathbb{R}^N).$$

Especially, we have

$$|u|_{I_{-\infty}^{\alpha}} = \left\| |w|^{\alpha}\hat{u} \right\|_{L^2}.$$

Therefore, $I_{-\infty}^{\alpha}$ and H^{α} are equivalent with equivalent semi-norm and norm. Analogous to $I_{-\infty}^{\alpha}$, we introduce I_{∞}^{α}. Define the semi-norm

$$|u|_{I_{\infty}^{\alpha}} = \left\| {}_xD_{\infty}^{\alpha}u \right\|_{L^2},$$

and the norm

$$\|u\|_{I_{\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{\infty}^{\alpha}}^2 \right)^{1/2}.$$

Let

$$I_{\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R},\mathbb{R}^N)}^{\|\cdot\|_{I_{\infty}^{\alpha}}}.$$

Then $I_{-\infty}^{\alpha}$ and I_{∞}^{α} are equivalent with equivalent semi-norm and norm (see [18]).

Let $C(\mathbb{R},\mathbb{R}^N)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following lemma.

Lemma 1. ([8], Theorem 2.1) *If* $\alpha > 1/2$, *then* $H^{\alpha} \subset C(\mathbb{R},\mathbb{R}^N)$ *and there is a constant* $C = C_{\alpha}$ *such that*

$$\|u\|_{\infty} = \sup_{x\in\mathbb{R}} |u(x)| \le C\|u\|_{\alpha}.$$

Remark 2. *From Lemma* 1, *we know that if* $u \in H^{\alpha}$ *with* $1/2 < \alpha < 1$, *then* $u \in L^p$ *for all* $p \in [2,\infty)$, *since*

$$\int_{\mathbb{R}} |u(x)|^p \,\mathrm{d}x \le \|u\|_{\infty}^{p-2} \|u\|_{L^2}^2.$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (1). Let

$$X^{\alpha} = \left\{ u \in H^{\alpha} : \int_{\mathbb{R}} \left(\left| {}_{-\infty}D_t^{\alpha}u(t) \right|^2 + \left(L(t)u(t),u(t) \right) \right) \mathrm{d}t < \infty \right\},$$

then X^{α} is a Hilbert space with the inner product

$$\langle u,v \rangle_{X^{\alpha}} = \int_{\mathbb{R}} \left(\left({}_{-\infty}D_t^{\alpha}u(t), {}_{-\infty}D_t^{\alpha}v(t) \right) + \left(L(t)u(t),v(t) \right) \right) \mathrm{d}t,$$

and the corresponding norm is

$$\|u\|_{X^{\alpha}}^2 = \langle u,u \rangle_{X^{\alpha}}.$$

Lemma 2. *If* $L(t)$ *satisfies* (L_1), *then* X^{α} *is continuously embedded in* H^{α}.

Proof. By (L_1) we have

$$\left(L(t)u,u \right) \ge l(t)|u|^2 \ge l_1|u|^2, \ \forall t \in \mathbb{R}.$$

Then

$$
\begin{aligned}
l_1\|u\|_\alpha^2 &= l_1\left(\int_{\mathbb{R}}\left(\left|{}_{-\infty}D_t^\alpha u(t)\right|^2+|u(t)|^2\right)\mathrm{d}t\right)\\
&\le l_1\int_{\mathbb{R}}\left|{}_{-\infty}D_t^\alpha u(t)\right|^2\mathrm{d}t+\int_{\mathbb{R}}\left(L(t)u,u\right)\mathrm{d}t.
\end{aligned}
$$

It implies that

$$\|u\|_\alpha^2 \le K\|u\|_{X^\alpha}^2,$$

where $K=\max\{1,1/l_1\}$. □

Lemma 3. *If* $L(t)$ *satisfies* (L_1), *then* X^α *is compactly embedded in* L^q *for* $1\le q<\infty$.

Proof. First, by (L_1) and the Hölder inequality, one has

$$
\begin{aligned}
\int_{\mathbb{R}}|u|\,\mathrm{d}t &= \int_{\mathbb{R}}\left|\left(L(t)\right)^{-1/2}\left(L(t)\right)^{1/2}u\right|\mathrm{d}t\\
&\le \int_{\mathbb{R}}\left(l(t)\right)^{-1/2}\left|\left(L(t)\right)^{1/2}u\right|\mathrm{d}t\\
&\le \left(\int_{\mathbb{R}}\left(l(t)\right)^{-1}\right)^{1/2}\left(\int_{\mathbb{R}}\left(L(t)u,u\right)\mathrm{d}t\right)^{1/2}\\
&\le \left(\int_{\mathbb{R}}\left(l(t)\right)^{-1}\mathrm{d}t\right)^{1/2}\|u\|_{X^\alpha},\ \forall u\in E.
\end{aligned}
$$

This implies that X^α is continuously embedded into L^1.

Next, we prove that X^α is compactly embedded into L^1. Let $\{u_n\}$ be a bounded sequence such that $u_n \rightharpoonup u$ in X^α. We will show that $u_n\to u$ in L^1. Obviously, there exists a constant $d_1>0$ such that

$$\|u_n\|_{X^\alpha}\le d_1,\ \forall n\in N. \tag{4}$$

By (L_1), for any $\varepsilon>0$ there exists T_ε such that

$$\left(\int_{|t|>T_\varepsilon}\left(l(t)\right)^{-1}\mathrm{d}t\right)^{1/2}<\frac{\varepsilon}{2\left(d_1+\|u\|_{X^\alpha}\right)}. \tag{5}$$

Since by Lemma 2 X^α is continuously embedded into H^α, the Sobolev embedding theorem implies $u_n\to u$ in $L^2_{\mathrm{loc}}\left(\mathbb{R},\mathbb{R}^N\right)$. Then for the T_ε above, there exists $N_\varepsilon\in N$ such that

$$\left(\int_{-T_\varepsilon}^{T_\varepsilon}|u_n-u|^2\,\mathrm{d}t\right)^{1/2}<\frac{\varepsilon}{4T_\varepsilon},\ \forall n\ge N_\varepsilon. \tag{6}$$

Combining (4)-(6) and the Hölder inequality, for each $n\ge N_\varepsilon$, we have

$$
\begin{aligned}
\int_{\mathbb{R}}|u_n-u|\,\mathrm{d}t &= \int_{-T_\varepsilon}^{T_\varepsilon}|u_n-u|\,\mathrm{d}t+\int_{|t|>T_\varepsilon}|u_n-u|\,\mathrm{d}t\\
&\le 2T_\varepsilon\left(\int_{-T_\varepsilon}^{T_\varepsilon}|u_n-u|^2\,\mathrm{d}t\right)^{1/2}+\int_{|t|>T_\varepsilon}\left|\left(L(t)\right)^{-1/2}\left(L(t)\right)^{1/2}(u_n-u)\right|\mathrm{d}t\\
&\le \frac{\varepsilon}{2}+\int_{|t|>T_\varepsilon}\left(l(t)\right)^{-1/2}\left|\left(L(t)\right)^{1/2}(u_n-u)\right|\mathrm{d}t\\
&\le \frac{\varepsilon}{2}+\left(\int_{|t|>T_\varepsilon}\left(l(t)\right)^{-1}\mathrm{d}t\right)^{1/2}\left(\int_{|t|>T_\varepsilon}\left(L(t)(u_n-u),u_n-u\right)\mathrm{d}t\right)^{1/2}\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2\left(d_1+\|u\|_{X^\alpha}\right)}\|u_n-u\|_{X^\alpha}\le\varepsilon.
\end{aligned}
$$

This means that $u_n\to u$ in L^1 and hence X^α is compactly embedded into

L^1.

Last, since for $1<q<\infty$ one has

$$\int_{\mathbb{R}}|u|^q\,\mathrm{d}t\le\|u\|_\infty^{q-1}\|u\|_{L^1},$$

it is easy to verify that the embedding of X^α in L^q is also continuous and compact for $q\in(1,\infty)$. The proof is completed. □

Remark 3. *By Lemma* 1 - 3 *we see that there exists a constant* $\gamma_q>0$ *such that*

$$\|u\|_{L^q}\le\gamma_q\|u\|_{X^\alpha},\ \forall u\in X^\alpha,\ \forall q\in[1,\infty]. \tag{7}$$

Lemma 4. *Assume that* (W_1)-(W_4) *are satisfied. There is* $0<r<\frac{\delta}{2}$ *and* $\tilde{W}\in C^1(\mathbb{R},\mathbb{R}^N)$ *such that*

i)

$$\left|\nabla\tilde{W}(t,x)\right|\le c_2\left(|x|^\theta+|x|^{p-1}\right),\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{8}$$

where c_2 is a constant;

ii)

$$\hat{W}(t,x):=2\tilde{W}(t,x)-\left(\nabla\tilde{W}(t,x),x\right)\le 0,\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N \tag{9}$$

and

$$\hat{W}(t,x)=0\ \text{ iff }\ |x|=0. \tag{10}$$

Proof. By (W_1) and (W_2) one has

$$|W(t,x)|\le c_1|x|^{\theta+1},\ \forall(t,x)\in\mathbb{R}\times B_\delta(0). \tag{11}$$

Next we modify $W(t,x)$ for x outside a neighborhood of the origin 0. Choose

$$0<\beta<\frac{1}{4\gamma_p^p},$$

where γ_p is the constant given in (7). By (W_3), there is a constant $r\in\left(0,\frac{\delta}{2}\right)$ such that

$$W(t,x)\le\beta|x|^p,\ \forall t\in\mathbb{R}\text{ and }|x|\le 2r. \tag{12}$$

Define a cut-off function $\rho\in C^1(\mathbb{R},\mathbb{R})$ satisfying

$$\rho(t)=\begin{cases}1, & 0\le t\le r,\\ 0, & t\ge 2r,\end{cases}$$

and $-\frac{2}{r}\le\rho'(t)<0$ for $r<t<2r$. Using ρ, we define

$$\tilde{W}(t,x):=\rho(|x|)W(t,x)+\left(1-\rho(|x|)\right)W_\infty(x),\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{13}$$

where $W_\infty(x)=\beta|x|^p,\ \forall x\in\mathbb{R}^N$. Then by direct computation we get

$$\nabla\tilde{W}(t,x)=\rho(|x|)\nabla W(t,x)+\rho'(|x|)W(t,x)\\ +(1-\rho(|x|))W_\infty'(x)-\rho'(|x|)W_\infty(x), \tag{14}$$

$$\hat{W}(t,x)=\rho(|x|)\big(2W(t,x)-(\nabla W(t,x),x)\big)+(2-p)(1-\rho(|x|))W_\infty(x)\\ -\rho'(|x|)\big(W(t,x)-W_\infty(x)\big)|x| \tag{15}$$

for $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. It follows from ($W_1$) and ($W_2$) that

$$\nabla\tilde{W}(t,0)=\hat{W}(t,0)=0,\ \forall t\in\mathbb{R}. \tag{16}$$

Then by (11), (14), (W_2) and the choice of the cut-off function ρ, we have

$$\left|\nabla\tilde{W}(t,x)\right|\le\beta p|x|^{p-1},\ \forall t\in\mathbb{R},|x|\ge 2r,$$

and

$$\left|\nabla\tilde{W}(t,x)\right|\le\left|\nabla W(t,x)\right|+\frac{2}{r}\left|W(t,x)\right|+W_\infty'(x)+\frac{2}{r}W_\infty(x)\\ \le c_1|x|^\theta+4c_1|x|^\theta+\beta p|x|^{p-1}+4\beta|x|^{p-1}\\ =5c_1|x|^\theta+(4+p)\beta|x|^{p-1},\ \forall t\in\mathbb{R},|x|<2r.$$

Therefore, (8) is satisfied if $c_2=\max\{5c_1,(4+p)\beta\}$.

Finally, we prove (9) and (10). On one hand, using (16) we know that $\hat{W}(t,x)=0$ whenever $x=0$. On the other hand, assume that $r<|x|<2r$. By (12), (15), (W_4) and the choice of the cut-off function ρ, we obtain

$$\rho(|x|)\big(2W(t,x)-(\nabla W(t,x),x)\big)<0,$$

$$(2-p)(1-\rho(|x|))W_\infty(x)\le 0,$$

and

$$-\rho'(|x|)\big(W(t,x)-W_\infty(x)\big)|x|\le 0.$$

The above estimates imply that $\hat{W}(t,x)<0$ if $r<|x|<2r$. Besides, when $|x|\ge 2r$, by (15) we have

$$\hat{W}(t,x)=(2-p)W_\infty(x)<0.$$

when $0<|x|\le r$, by (W_4) we get

$$\hat{W}(t,x)=2W(t,x)-(\nabla W(t,x),x)<0.$$

Thus (9) and (10) are verified. The proof is completed. □

We now consider the modified problem

$${}_tD_\infty^\alpha\left({}_{-\infty}D_t^\alpha u(t)\right)+L(t)u(t)=\nabla\tilde{W}(t,u(t)), \tag{17}$$

whose solutions correspond to critical points of the functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}}\left(\left|{}_{-\infty}D_t^\alpha u(t)\right|^2+(L(t)u,u)\right)\mathrm{d}t-\int_{\mathbb{R}}\tilde{W}(t,u)\mathrm{d}t\\ =\frac{1}{2}\|u\|_{X^\alpha}^2-\int_{\mathbb{R}}\tilde{W}(t,u)\mathrm{d}t$$

for all $u\in X^\alpha$. By (11) and (13) we have

$$\left|\tilde{W}(t,u)\right| \le c_1|u|^{\theta+1}+\beta|u|^p,\ \forall(t,u)\in\mathbb{R}\times\mathbb{R}^N. \tag{18}$$

Thus, I is well defined.

Rewrite I as follows:

$$I=I_1-I_2,$$

where

$$I_1=\frac{1}{2}\int_{\mathbb{R}}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2+\left(L(t)u,u\right)\right)\mathrm{d}t \text{ and } I_2=\int_{\mathbb{R}}\tilde{W}(t,u)\,\mathrm{d}t.$$

In the following, c will be used to denote various positive constants where the exact values are different.

Lemma 5. *Let* (L_1), (W_1) *and* (W_2) *be satisfied. Then* $I\in C^1(X^{\alpha},\mathbb{R})$ *and* I_2' *is compact with*

$$\langle I_2'(u),v\rangle=\int_{\mathbb{R}}\left(\nabla\tilde{W}(t,u),v\right)\mathrm{d}t$$

$$\langle I'(u),v\rangle=\int_{\mathbb{R}}\left(\left({}_{-\infty}D_t^{\alpha}u(t),{}_{-\infty}D_t^{\alpha}v(t)\right)+\left(L(t)u,v\right)-\left(\nabla\tilde{W}(t,u),v\right)\right)\mathrm{d}t$$

for $u,v\in X^{\alpha}$. Moreover, nontrivial critical points of I in X^{α} are solutions of problem (17).

Proof. It is easy to check that $I_1\in C^1(X^{\alpha},\mathbb{R})$ and

$$\langle I_1'(u),v\rangle=\int_{\mathbb{R}}\left(\left({}_{-\infty}D_t^{\alpha}u(t),{}_{-\infty}D_t^{\alpha}v(t)\right)+\left(L(t)u,v\right)\right)\mathrm{d}t.$$

For any $\eta\in[0,1],u,h\in X^{\alpha}$, by (8) we have

$$\left|\left(\nabla\tilde{W}(t,u+\eta h),h\right)\right|\le c\left(|u|^{\theta}|h|+|h|^{\theta+1}+|u|^{p-1}|h|+|h|^p\right),$$

where c is independent of η. Hence, for any $u,h\in X^{\alpha}$, by the mean value theorem and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}\lim_{s\to 0}\frac{I_2(u+sh)-I_2(h)}{s}&=\lim_{s\to 0}\int_{\mathbb{R}}\left(\nabla\tilde{W}(t,u+\tau(t)sh),h\right)\mathrm{d}t\\&=\int_{\mathbb{R}}\left(\nabla\tilde{W}(t,u),h\right)\mathrm{d}t:=W_0(u,h),\end{aligned}$$

where $\tau(t)\in[0,1]$ depends on u,h,s. Moreover, it follows from (8) and (9) that

$$\begin{aligned}\left|W_0(u,h)\right|&\le\int_{\mathbb{R}}\left|\left(\nabla\tilde{W}(t,u),h\right)\right|\mathrm{d}t\\&\le c\left(\|u\|_{L^{\theta+1}}^{\theta}\|h\|_{L^{\theta+1}}+\|u\|_{L^p}^{p-1}\|h\|_{L^p}\right)\\&\le c\left(\|u\|_{X^{\alpha}}^{\theta}+\|u\|_{X^{\alpha}}^{p-1}\right)\|h\|_{X^{\alpha}}.\end{aligned}$$

Therefore, $W_0(u,\cdot)$ is linear and bounded in h, and $I_2'(u)=W_0(u,\cdot)$ is the Gateaux derivative of I_2 at u.

Next we prove that I_2' is weakly continuous. Set $Bu:=\nabla\tilde{W}(t,u)$. There exist B_1,B_2 such that $B=B_1+B_2$, where B_1 is bounded and continuous from $L^{\theta+1}$ to $L^{\frac{\theta+1}{\theta}}$ and B_2 is bounded and continuous from L^p to $L^{\frac{p}{p-1}}$. For any $v,h\in X^{\alpha}$,

$$\begin{aligned}\left|\left\langle I_2'(u)-I_2'(v),h\right\rangle\right| &= \left|\int_{\mathbb{R}}(Bu-Bv,h)\mathrm{d}t\right| \\ &= \left|\int_{\mathbb{R}}(B_1u+B_2u-B_1v-B_2v,h)\mathrm{d}t\right| \\ &\le \int_{\mathbb{R}}|B_1u-B_1v||h|\mathrm{d}t+\int_{\mathbb{R}}|B_2u-B_2v||h|\mathrm{d}t \\ &\le c\|B_1u-B_1v\|_{L^{\frac{\theta+1}{\theta}}}\|h\|_{X^\alpha}+c\|B_2u-B_2v\|_{L^{\frac{p}{p-1}}}\|h\|_{X^\alpha},\end{aligned}$$

which implies that

$$\sup_{\|h\|_{X^\alpha}=1}\left|I_2'(u)-I_2'(v)\right|\le c\|B_1u-B_1v\|_{L^{\frac{\theta+1}{\theta}}}+c\|B_2u-B_2v\|_{L^{\frac{p}{p-1}}}.$$

Now suppose $u\rightharpoonup v$ in X^α, then by Lemma 3, $u\to v$ in $L^{\theta+1}$ and L^p. Combining the above arguments, we have that I_2' is weakly continuous. Therefore, I_2' is compact and $I\in C^1(X^\alpha,\mathbb{R})$.

Finally, by a standard argument, it is easy to show that the critical points of I in X^α are solutions of problem (18) with $u(\pm\infty)=0$. The proof is completed. □

Lemma 6. *Assume that* (L_1), (W_1)-(W_4) *are satisfied. Then* 0 *is the only critical point of I such that* $I(u)=0$.

Proof. By (W_1), (W_2) and Lemma 5, we know that 0 is a critical point of *I* with $I(0)=0$. Now let $u\in X^\alpha$ be a critical point of *I* with $I(u)=0$. Then we have

$$0=2I(u)-\left\langle I'(u),u\right\rangle=-\int_{\mathbb{R}}\hat{W}(t,u)\mathrm{d}t,$$

where $\hat{W}$ is defined in (9). This together with (ii) of Lemma 4 implies that $|u(t)|=0$ for all $t\in\mathbb{R}$. The proof is completed. □

3. Proof of Theorem 1

The following lemma is due to Bartsch and Willem [19].

Lemma 7. *Let E be a Banach space with the norm* $\|\cdot\|$ *and* $E=\overline{\oplus_{j\in N}E(j)}$, *where* $E(j)$ *are all finite dimensional subspaces of E. Let* $I\in C^1(E,\mathbb{R})$ *be an even functional and satisfy*

(F_1) For every $k\ge k_0$, there exists $R_k>0$ such that $I(u)\ge 0$ for every $u\in E_k:\overline{\oplus_{j=k}^{\infty}E(j)}$ with $\|u\|=R_k$, and $b_k:=\inf_{u\in B_k}I(u)\to 0$ as $k\to\infty$. Here $B_k:=\{u\in E_k \mid \|u\|\le R_k\}$;

(F_2) For every $k\in N$, there exist $r_k\in(0,R_k)$ and $d_k<0$ such that $I(u)\le d_k$ for every $u\in E^k:=\oplus_{j=1}^{k}E(j)$ with $\|u\|=r_k$;

(F_3) *I* satisfies $(PS)^*$ condition with respect to $\{E^m \mid m\in N\}$, *i.e.* every sequence $u_m\in E^m$ with $I(u_m)<0$ bounded and $\left(I|_{E^m}\right)'(u_m)\to 0$ as $m\to\infty$ has a subsequence which converges to a critical point of *I*.

Then for each $k\ge k_0$, *I* has a critical value $\xi_k\in[b_k,d_k]$, hence $\xi_k<0$ and $\xi_k\to 0$ as $k\to\infty$.

Let $\{e_j\}_{j=1}^{\infty}$ be the standard orthogonal basis of X^α and define $E(j)=\mathbb{R}e_j$ for each $j\in N$. Now we show that the functional *I* has the geometric property of Lemma 7 under the conditions of Theorem 1.

Lemma 8. *Assume that* (L_1), (W_1) *and* (W_2) *hold. Then there exist a positive integer* k_0 *and a sequence* $R_k\to 0^+$ *as* $k\to\infty$ *such that*

$$\inf_{u\in E_k,\|u\|_{X^\alpha}=R_k} I(u)\ge 0,\ \forall k\ge k_0$$

and

$$b_k := \inf_{u\in B_k} I(u)\to 0 \text{ as } k\to\infty,$$

where $E_k := \overline{\oplus_{j=k}^{\infty} E(j)}$ and $B_k := \left\{u\in E_k \mid \|u\|_{X^\alpha}\le R_k\right\}$ for all $k\in N$.

Proof. By (18) we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_R \tilde{W}(t,u)\mathrm{d}t \\ &\ge \frac{1}{2}\|u\|_{X^\alpha}^2 - c_1\|u\|_{L^{\theta+1}}^{\theta+1} - \beta\|u\|_{L^p}^p,\ \forall u\in E_k. \end{aligned} \tag{19}$$

Set

$$l_k = \sup_{u\in E_k,\|u\|_{X^\alpha}=1} \|u\|_{L^{\theta+1}},\ \forall k\in N. \tag{20}$$

Since X^α is compactly embedded into $L^{\theta+1}$, there holds (see [20])

$$l_k \to 0^+ \text{ as } k\to\infty. \tag{21}$$

For each $k\in N$, it follows from (7), (19), (20) and the choice of β that

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|_{X^\alpha}^2 - c_1 l_k^{\theta+1}\|u\|_{X^\alpha}^{\theta+1} - \beta\gamma_p^p\|u\|_{X^\alpha}^p \\ &\ge \frac{1}{2}\|u\|_{X^\alpha}^2 - c_1 l_k^{\theta+1}\|u\|_{X^\alpha}^{\theta+1} - \frac{1}{4}\|u\|_{X^\alpha}^p,\ \forall u\in E_k. \end{aligned} \tag{22}$$

For each $k\in N$, choose

$$R_k = 4c_1 l_k^{\theta+1}, \tag{23}$$

then by (20) one has

$$R_k \to 0^+ \text{ as } k\to\infty, \tag{24}$$

and hence there exists a positive integer k_0 such that

$$R_k < 1,\ \forall k\ge k_0. \tag{25}$$

Now by (22), (23) and (25), we have

$$\inf_{u\in E_k,\|u\|_{X^\alpha}=R_k} I(u)\ge \frac{1}{2}R_k^2 - \frac{1}{4}R_k^{\theta+2} - \frac{1}{4}R_k^p \ge 0,\ \forall k\ge k_0.$$

Noting that $I(0)=0$ and

$$I(u)\ge -c_1 l_k^{\theta+1}\|u\|_{X^\alpha}^{\theta+1} - \frac{1}{4}\|u\|_{X^\alpha}^p,\ \forall k\in N, u\in E_k,$$

we have

$$0\ge \inf_{u\in B_k} I(u)\ge -c_1 l_k^{\theta+1}R_k^{\theta+1} - \frac{1}{4}R_k^p,\ \forall k\in N,$$

which combined with (21) and (24) implies that

$$b_k := \inf_{u\in B_k} I(u)\to 0 \text{ as } k\to\infty.$$

The proof is completed. □

Lemma 9. *Assume that* (L_1), (W_1) *and* (W_5) *hold. Then for every* $k \in N$, *there exist* $r_k \in (0, R_k)$ *and* $d_k < 0$ *such that* $I(u) \le d_k$ *for every* $u \in E^k := \oplus_{j=1}^k E(j)$ *with* $\|u\|_{X^\alpha} = r_k$.

Proof. For a fixed $k \in N$, since E^k is finitely-dimensional, there is a constant $C_k > 0$ such that

$$C_k \|u\|_{X^\alpha}^\mu \le \|u\|_{L^\mu}^\mu, \forall u \in E^k. \tag{26}$$

Set $p_k = \min\left\{R_k, \frac{w_k}{\gamma_\infty}\right\}$. Then by (W_5), there exists a constant $0 < w_k < r$ such that

$$\tilde{W}(t,u) = W(t,u) \ge m_k |u|^\mu, \forall t \in \mathbb{R} \text{ and } |u| \le w_k, \tag{27}$$

where $m_k = \frac{1}{p_k^{\mu-2} C_k}$. Now by (7), (26), (27) and Lemma 3, for $u \in E^k$ with $\|u\|_{X^\alpha} \le \frac{w_k}{\gamma_\infty}$, we get

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} \tilde{W}(t,u)\mathrm{d}t \\ &\le \frac{1}{2}\|u\|_{X^\alpha}^2 - m_k \|u\|_{L^\mu}^\mu \\ &\le \frac{1}{2}\|u\|_{X^\alpha}^2 - m_k C_k \|u\|_{X^\alpha}^\mu \\ &= \frac{1}{2}\|u\|_{X^\alpha}^2 \left(1 - \frac{2}{p_k^{\mu-2}}\|u\|_{X^\alpha}^{\mu-2}\right). \end{aligned}$$

Choose

$$0 < r_k = \left(\frac{2}{3}\right)^{\frac{1}{\mu-2}} p_k < p_k,$$

and let

$$d_k = -\frac{r_k^2}{6} < 0.$$

If $u \in E^k$ with $\|u\|_{X^\alpha} = r_k$, we have

$$I(u) \le d_k.$$

The proof is completed. □

Lemma 10. *Assume that* (L_1), (W_1), (W_2) *and* (W_4) *hold. Then* I *satisfies* $(PS)^*$ *condition with respect to* $\{E^m \mid m \in N\}$.

Proof. Let $u_m \in E^m$ be a $(PS)^*$ sequence, that is,

$$I(u_m) \text{ is bounded and } \left(I|_{E^m}\right)'(u_m) \to 0 \text{ as } m \to \infty. \tag{28}$$

Then we claim that $\{u_m\}$ is bounded. If not, passing to a subsequence if necessary, we may assume that

$$\|u_m\|_{X^\alpha} \to \infty \text{ as } m \to \infty. \tag{29}$$

From (13), (14), (15), we have

$$\begin{aligned}
&2I(u_m)-\left\langle \left(I|_{E^m}\right)'(u_m),u_m\right\rangle \\
&=\int_{\mathbb{R}}\left[\left(\nabla\tilde{W}(t,u_m),u_m\right)-2\tilde{W}(t,u_m)\right]\mathrm{d}t \\
&\geq (p-2)\beta\int_{\{t\in\mathbb{R}|u_m(t)|\geq 2r\}}|u_m|^p\,\mathrm{d}t
\end{aligned} \tag{30}$$

for all $m\in N$. From (28), (29) and (30), it follows that

$$\frac{\int_{\{t\in\mathbb{R}|u_m(t)|\geq 2r\}}|u_m|^p\,\mathrm{d}t}{\|u_m\|_{X^\alpha}}\to 0 \tag{31}$$

as $m\to\infty$. By (8) we get

$$\left|\nabla\tilde{W}(t,x)\right|\leq c\left(1+|x|^{p-1}\right),\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N,$$

which combined with (7) implies that

$$\begin{aligned}
&\left\langle \left(I|_{E^m}\right)'(u_m),u_m\right\rangle \\
&\geq \|u_m\|_{X^\alpha}^2-\int_{\mathbb{R}}\left|\nabla\tilde{W}(t,u_m)\right||u_m|\,\mathrm{d}t \\
&\geq \|u_m\|_{X^\alpha}^2-c\int_{\mathbb{R}}|u_m|^p\,\mathrm{d}t-c\int_{\mathbb{R}}|u_m|\,\mathrm{d}t \\
&\geq \|u_m\|_{X^\alpha}^2-c\|u_m\|_\infty\int_{\{t\in\mathbb{R}|u_m(t)|\geq 2r\}}|u_m|^{p-1}\,\mathrm{d}t \\
&\quad -c(2r)^{p-1}\int_{\{t\in\mathbb{R}|u_m(t)|<2r\}}|u_m|\,\mathrm{d}t-c\|u_m\|_{L^1} \\
&\geq \|u_m\|_{X^\alpha}^2-c\|u_m\|_\infty(2r)^{-1}\int_{\{t\in\mathbb{R}|u_m(t)|\geq 2r\}}|u_m|^p\,\mathrm{d}t \\
&\quad -c(2r)^{p-1}\|u_m\|_{L^1}-c\|u_m\|_{L^1} \\
&\geq \|u_m\|_{X^\alpha}^2-c\gamma_\infty\|u_m\|_{X^\alpha}(2r)^{-1}\int_{\{t\in\mathbb{R}|u_m(t)|\geq 2r\}}|u_m|^p\,\mathrm{d}t \\
&\quad -c(2r)^{p-1}\gamma_1\|u_m\|_{X^\alpha}-c\gamma_1\|u_m\|_{X^\alpha}.
\end{aligned}$$

From this and (31) it follows that

$$1=\frac{\|u_m\|_{X^\alpha}}{\|u_m\|_{X^\alpha}}\to 0 \text{ as } m\to\infty,$$

which is a contradiction. Hence $\{u_m\}$ is bounded. Noting that by Lemma 5 $\{u_m\}$ has a subsequence converging to a critical point of I (see [21]). Hence, I satisfies the $(PS)^*$ condition. The proof is completed. □

Proof of Theorem 1. It follows from Lemma 8 - 10 that the functional I satisfies the conditions (F_1)-(F_3) of Lemma 7. Therefore, by Lemma 7, there exists a sequence of critical values $\xi_k<0$ with $\xi_k\to 0$ as $k\to\infty$. Let $\{u_k\}$ be a sequence of critical points of I corresponding to these critical values, *i.e.* $I(u_k)=\xi_k$ and $I'(u_k)=0$ for all k. Then by Lemma 5, $\{u_k\}$ is a sequence of solutions of problem (17). By Lemma 10 and Remark 3.19 in [20], I satisfies $(PS)^*$ condition and hence we may assume without loss of generality that $u_k\to u$ in X^α as $k\to\infty$. Evidently, u is a critical point of I with $I(u)=0$.

Then by Lemma 6, u must be 0. Thus $u_k \to 0$ in X^{α} as $k \to \infty$. By (7), we further have $u_k \to 0$ in $L^{\infty}(\mathbb{R}, \mathbb{R}^N)$ as $k \to \infty$. Therefore, for k large enough, they are solutions of problem (1). The proof is completed. □

4. Conclusions and Remarks

Let us conclude this paper with some open questions whose answers might largely improve the applicability of the results in this present paper.

Question. Whether or not can we improve the non-coercivity condition (L_1): There is $l_1 > 0$ such that $l(t) \geq l_1, \forall t \in \mathbb{R}$ and $\int_{\mathbb{R}} (l(t))^{-1} \mathrm{d}t < \infty$, in order to obtain similar results?

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Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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