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Double Elzaki Transform Decomposition Method for Solving Non-Linear Partial Differential Equations

Moh A. Hassan[1*], Tarig M. Elzaki[2]

[1]Mathematics Department, Faculty of Mathematical & Computer Science, Gezira University, Wad Madani, Sudan
[2]Mathematics Department, Faculty of Sciences and Arts, University of Jeddah, Jeddah, KSA
Email: *moh.hassan512@yahoo.com, Tarig.alzaki@gmail.com

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Abstract

In this paper, we discuss a new method employed to tackle non-linear partial differential equations, namely Double Elzaki Transform Decomposition Method (DETDM). This method is a combination of the Double ELzaki Transform and Adomian Decomposition Method. This technique is hereafter provided and supported with necessary illustrations, together with some attached examples. The results reveal that the new method is very efficient, simple and can be applied to other non-linear problems.

Keywords

Double Elzaki Transform, Adomian Decomposition Method, Non-Linear Partial Differential Equations

1. Introduction

The non-linear partial differential equations, appear in many applications of mathematics, physics, chemistry and engineering, for this reason that the researcher presents a number of methods for solving it, such as Adomian Decomposition Method (ADM) [1], Variation Iteration Method (VIM) [1], Homotopy Perturbation Method (HPM) [1].

A new option appearing recently, includes the composition of previous methods with some integral transforms namely Laplace transform, Sumudu transform, or Elzaki transform; these compositions resulted in number of methods such as Laplace Decomposition Method (LDM) [2] [3] [4], Laplace Variation Iteration Method (LVIM) [5], Sumudu Decomposition Method (SDM) [6]-[15], Sumudu Homotopy Perturbation Method (SHPM) [16] [17], Elzaki Variation

Iteration Method (EVIM) [18], Elzaki project Differential Transform Method (EPDTM) [19], Elzaki Homotopy Perturbation Method (EHPM) [20] [21], and Elzaki Decomposition Method (EDM) [22] [23].

The essential motivation of the present study is to extend the application of the Double ELzaki Transform by introducing a new method called Double ELzaki Transform Decomposition Method (DETDM) for solving non-linear partial differential equations.

The significance of this method is its capability of combining easy integral transform Double ELzaki Transform (DET) [24] and an effective method for solving non-linear partial differential equations, namely Adomian Decomposition Method [1].

Several examples are given as follows to illustrate this method to explain its effectiveness.

2. Basic Definitions of Double Elzaki Transform

Definition: Let $f(x,t), t,x \in R^+$ be a function which can be expressed as a convergent infinite series, then its Double Elzaki Transform given by:

$$E_2\left[f(x,t),u,v\right] = T(u,v) = uv\int_0^\infty\int_0^\infty f(x,t)\mathrm{e}^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\mathrm{d}x\mathrm{d}t,\ x,t>0. \tag{1}$$

where u,v are complex values.

To obtain double Elzaki transform of partial derivatives we use integration by parts [24], and then we have:

$$\begin{gathered}
E_2\left[\frac{\partial f}{\partial x}\right] = \frac{1}{u}T(u,v) - uT(0,v) \\
E_2\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{1}{u^2}T(u,v) - T(0,v) - u\frac{\partial}{\partial x}T(0,v) \\
E_2\left[\frac{\partial f}{\partial t}\right] = \frac{1}{v}T(u,v) - vT(u,0) \\
E_2\left[\frac{\partial^2 f}{\partial t^2}\right] = \frac{1}{v^2}T(u,v) - T(u,0) - v\frac{\partial}{\partial t}T(u,0) \\
E_2\left[\frac{\partial^2 f}{\partial x\partial t}\right] = \frac{1}{uv}T(u,v) - \frac{v}{u}T(u,0) - \frac{u}{v}T(0,v) + uvT(0,0)
\end{gathered} \tag{2}$$

Proof:

$$E_2\left[\frac{\partial f}{\partial x}\right] = uv\int_0^\infty\int_0^\infty \frac{\partial}{\partial x}f(x,t)\mathrm{e}^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\mathrm{d}x\mathrm{d}t = v\int_0^\infty \mathrm{e}^{-\frac{t}{v}}\left[u\int_0^\infty \mathrm{e}^{-\frac{x}{u}}\frac{\partial}{\partial x}f(x,t)\mathrm{d}x\right]\mathrm{d}t$$

The inner integral gives $\frac{1}{u}T(u,t) - uf(0,t)$

$$\Rightarrow E_2\left[\frac{\partial f}{\partial x}\right] = \frac{v}{u}\int_0^\infty \mathrm{e}^{-\frac{t}{v}}T(u,t)\mathrm{d}t - uv\int_0^\infty \mathrm{e}^{-\frac{t}{v}}f(0,t)\mathrm{d}t$$

$$\Rightarrow E_2\left[\frac{\partial f}{\partial x}\right]=\frac{1}{u}T(u,v)-uT(0,v)$$

Also $E_2\left[\frac{\partial f}{\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0)$

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=uv\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}\mathrm{e}^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\mathrm{d}x\mathrm{d}t=v\int_0^\infty\mathrm{e}^{-\frac{t}{v}}\left[u\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}\mathrm{e}^{-\frac{x}{u}}\mathrm{d}x\right]\mathrm{d}t$$

The inner integral: $u\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}\mathrm{e}^{-\frac{x}{u}}\mathrm{d}x=\frac{T(u,t)}{u^2}-f(0,t)-u\frac{\partial f(0,t)}{\partial x}$.

By taking Elzaki transform with respect to t for above integral we get:

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial}{\partial x}T(0,v)$$

Similarly:

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial}{\partial t}T(u,0)$$

3. Double Elzaki Transform Decomposition Method (DETDM)

To clarify the basic idea of this method, we consider a general partial differential equation with the initial condition of the following form:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \tag{3}$$

$$u(x,0)=h(x),\quad u_t(x,0)=f(x). \tag{4}$$

where, L is the second order linear differential operator $L=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order then L, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the double Elzaki Transform on both sides of Equation (3) and single Elzaki Transform of Equation (4), we get:

$$E_2\left(Lu(x,t)\right)+E_2\left(Ru(x,t)\right)+E_2\left(Nu(x,t)\right)=E_2\left(g(x,t)\right), \tag{5}$$

$$E\left(u(x,0)\right)=E\left(h(x)\right)=T(u,0)\ \text{ and }\ E\left(u_t(x,0)\right)=E\left(f(x)\right)=\frac{\partial}{\partial t}T(u,0). \tag{6}$$

To substitute Equation (6) in (5), after using Equation (2), we get:

$$\begin{aligned}E_2\left(u(x,t)\right)=&\,v^2E_2\left(g(x,t)\right)+v^2E\left(h(x)\right)+v^3E\left(f(x)\right)\\&-v^2E_2\left(Ru(x,t)\right)-v^2E_2\left(Nu(x,t)\right).\end{aligned} \tag{7}$$

Now, with the application of the inverse Double Elzaki Transform on both side of Equation (7) we get:

$$u(x,t)=G(x,t)-E_2^{-1}\left[v^2E_2\left[Ru(x,t)+Nu(x,t)\right]\right]. \tag{8}$$

where $G(x,t)$ represents the terms arising from the source term and the prescribed initial conditions.

After that we represent solution as an infinite series given below,

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{9}$$

and the nonlinear term can be written as follow,

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n(u), \tag{10}$$

where, $A_n(u)$ are Adomian polynomial and it can be calculated by formula given below:

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0},\quad n=0,1,2,3,\cdots \tag{11}$$

To substitute (9) and (10) in (8), we get:

$$\sum_{n=0}^{\infty}u_n(x,t)=G(x,t)-E_2^{-1}\left[v^2E_2\left(R\sum_{n=0}^{\infty}u_n(x,t)+\sum_{n=0}^{\infty}A_n\right)\right]. \tag{12}$$

Then from Equation (12) we get:

$$\begin{aligned}
&u_0(x,t)=G(x,t),\\
&u_1(x,t)=-E_2^{-1}\left[v^2E_2\left[Ru_0(x,t)+A_0\right]\right],\\
&u_2(x,t)=-E_2^{-1}\left[v^2E_2\left[Ru_1(x,t)+A_1\right]\right].
\end{aligned} \tag{13}$$

In general, the recursive relation is given by:

$$u_n(x,t)=-E_2^{-1}\left[v^2E_2\left[Ru_{n-1}(x,t)+A_{n-1}\right]\right],\quad n\geq 1. \tag{14}$$

Finally, we approximate the solution $u(x,t)$ by the series:

$$u(x,t)=\lim_{N\to\infty}\sum_{n=0}^{\infty}u_n(x,t). \tag{15}$$

4. Application of the (DETDM)

Now we, applying the double Elzaki transform decomposition method (DETDM) to solve non-linear partial differential equations.

Example 1: Consider the following nonlinear partial differential equations

$$u_t+uu_x-u_{xx}=0, \tag{16}$$

with initial condition:

$$u(x,0)=x. \tag{17}$$

Take the double Elzaki transform to both sides of Equation (16), we get:

$$\frac{T(u,v)}{v}-vT(u,0)=E_2\left(u_{xx}-uu_x\right), \tag{18}$$

Take single Elzaki transform to initial condition we get:

$$E\left(u(x,0)\right)=T(u,0)=E(x)=u^3, \tag{19}$$

Substitute Equation (19) in Equation (18), we obtain:

$$T(u,v)=v^2u^3+vE_2\left(u_{xx}-uu_x\right). \tag{20}$$

Take the inverse double Elzaki transform to both sides of Equation (20), we obtain:

$$u(x,t) = x + E_2^{-1}\left[vE_2\left(u_{xx} - uu_x\right)\right]. \tag{21}$$

From the Adomian decomposition method, rewrite (21) as follows,

$$\sum_{n=0}^{\infty} u_n(x,t) = x + E_2^{-1}\left[vE_2\left(\sum_{n=0}^{\infty}(u_n)_{xx} - \sum_{n=0}^{\infty} A_n(u)\right)\right]. \tag{22}$$

where, $A_n(u)$ are Adomian polynomials that represent the nonlinear terms.

The first few components of $A_n(u)$ are given by:

$$\begin{aligned}
&A_0(u) = u_0(u_0)_x, \\
&A_1(u) = (u_0)_x u_1 + u_0(u_1)_x, \\
&A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0, \\
&A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0, \\
&\quad \vdots
\end{aligned} \tag{23}$$

By comparing both sides of Equation (22), we get:

$$u_0(x,t) = x, \tag{24}$$

$$u_{n+1}(x,t) = E_2^{-1}\left[vE_2\left[(u_n)_{xx} - A_n(u)\right]\right], \quad n \geq 0. \tag{25}$$

Then:

$$\begin{aligned}
u_1(x,t) &= E_2^{-1}\left[vE_2\left[(u_0)_{xx} - A_0(u)\right]\right] \\
&= E_2^{-1}\left[vE_2(-x)\right] \\
&= -E_2^{-1}\left[v^3u^3\right] = -xt,
\end{aligned} \tag{26}$$

$$\begin{aligned}
u_2(x,t) &= E_2^{-1}\left[vE_2\left[(u_1)_{xx} - A_1(u)\right]\right] \\
&= E_2^{-1}\left[vE_2(2xt)\right] \\
&= -E_2^{-1}\left[2v^4u^3\right] = xt^2,
\end{aligned} \tag{27}$$

By similar way we get:

$$u_3(x,t) = -xt^3. \tag{28}$$

And so on. Then the first four terms of the decomposition series for Equation (16), is given by:

$$u(x,t) = x - xt + xt^2 - xt^3 + \cdots, \tag{29}$$

The solution in a closed form is given by:

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1. \tag{30}$$

Example 2: Consider the following nonlinear partial differential equations:

$$u_{tt} - \frac{2x^2}{t}uu_x = 0, \tag{31}$$

with initial condition:

$$u(x,0)=0,\quad u_t(x,0)=x. \tag{32}$$

Take the double Elzaki transform to both sides of Equation (31), we get:

$$\frac{T(u,v)}{v^2}-T(u,0)-v\frac{\partial}{\partial t}T(u,0)=E_2\left(\frac{2x^2}{t}uu_x\right), \tag{33}$$

Take single Elzaki transform to initial conditions, we get:

$$E\left(u(x,0)\right)=0 \text{ and } E\left(u_t(x,0)\right)=\frac{\partial}{\partial t}T(u,0)=E(x)=u^3, \tag{34}$$

Substitute Equation (34) in Equation (33) we obtain:

$$T(u,v)=v^3u^3+v^2E_2\left[\frac{2x^2}{t}uu_x\right]. \tag{35}$$

Take the inverse double Elzaki transform to both sides of Equation (35), we obtain:

$$u(x,t)=xt+E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}uu_x\right]\right], \tag{36}$$

From the Adomian decomposition method, rewrite (36) as follows:

$$\sum_{n=0}^{\infty}u_n(x,t)=xt+E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}\sum_{n=0}^{\infty}A_n(u)\right]\right]. \tag{37}$$

where $A_n(u)$ are Adomian polynomials that represent the nonlinear terms.

The first few components of $A_n(u)$ are given by:

$$\begin{aligned}
&A_0(u)=u_0(u_0)_x,\\
&A_1(u)=(u_0)_x u_1+u_0(u_1)_x,\\
&A_2(u)=(u_0)_x u_2+(u_1)_x u_1+(u_2)_x u_0,\\
&A_3(u)=(u_0)_x u_3+(u_1)_x u_2+(u_2)_x u_1+(u_3)_x u_0.\\
&\quad\vdots
\end{aligned} \tag{38}$$

By comparing both sides of Equation (37), we get:

$$u_0(x,t)=xt, \tag{39}$$

$$u_{n+1}(x,t)=E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}A_n\right]\right],\quad n\ge 0. \tag{40}$$

Then:

$$\begin{aligned}
u_1(x,t)&=E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}A_0\right]\right]=E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}\cdot xt^2\right]\right]\\
&=E_2^{-1}\left[12v^5u^5\right]=\frac{1}{3}x^3t^3,
\end{aligned} \tag{41}$$

By similar way we get:

$$u_2(x,t)=\frac{2}{15}x^5t^5, \tag{42}$$

$$u_3(x,t) = \frac{17}{315} x^7 t^7, \tag{43}$$

And so on. Then the first four terms of the decomposition series for Equation (31), is given by:

$$u(x,t) = xt + \frac{1}{3}(xt)^3 + \frac{2}{15}(xt)^5 + \frac{17}{315}(xt)^7 + \cdots, \tag{44}$$

The solution in a closed form is given by:

$$u(x,t) = \tan(xt). \tag{45}$$

5. Conclusion

The combination of Adomian Decomposition Method (ADM) and Double Elzaki Transform Method (DETM) can produce a very effective method to solve nonlinear partial differential equations. Simply, it can be applied to other nonlinear partial differential equations of higher order.

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Authors' Contributions

The authors read and agreed the final manuscript

Conflicts of Interest

The authors declare that they have no competing interests.

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