

Exact Soliton-Like Spherical Symmetric Solutions of the Heisenberg-Ivanenko Type Nonlinear Spinor Field Equation in Gravitational Theory

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Abstract

In the present research work, we have obtained the exact spherical symmetric solutions of Heisenberg-Ivanenko nonlinear spinor field equations in the Gravitational Theory. The nonlinearity in the spinor Lagrangian is given by an arbitrary function which depends on the invariant generated from the bilinear spinor form $I_S = S^2$. We admit the static spherical symmetric metric. It is shown that a soliton-like configuration has a localized energy density and a finite total energy. In addition, The total charge and total spin are also finite. Role of the metric *i.e.* the proper gravitational field of elementary particles in the formation of the field configurations with limited total energy, spin and charge has been examined by solving the field equations in flat space-time. It has been established that the obtained solutions are soliton-like configuration with bounded energy density and finite total energy. In order to clarify the role of the nonlinearity in this model, we have obtained exact static symmetric solutions to the above spinor field equations in the linear case corresponding to Dirac's linear equation. It is proved that soliton-like solutions are absent.

Keywords

Lagrangian, Spherical Symmetric Metric, Soliton-Like Solution, Flat Space-Time

1. Introduction

The description of elementary particles by considering the nonlinear phenomena and taking into account the proper gravitational field has been one of the most century popular topics. Indeed, the theory which considers elementary particles as material points has shortcomings. With this theory, it is impossible to obtain a finite value of mass, charge and spin of elementary particles. In this approach, elementary particles are modeled by soliton-like solutions corresponding to nonlinear equations. In 1995, G. N. Shikin investigated the basics of soliton concept in General Relativity [1]. In 1998, A. Adomou and G. N. Shikin have obtained exact plane-symmetric solutions to the spinor field equations with nonlinear terms, which are arbitrary functions of the invariant $S = \bar{\psi}\psi$ [2]. Five years later, B. Saha and G. N. Shikin studied the system of nonlinear spinor and scalar fields with minimal coupling in general relativity [3]. In 2012, V. Adanhomè, A. Adomou, F. P. Codo and M.N. Hounkonnou extended the research work [Gravitation and Cosmology, Vol.4, 1998, pp.107-113] to exact spherical symmetric soliton-like solutions [4]. In a series of remarkable papers appearing in 2019, Alain Adomou, Jonas Edou and Siaka Massou obtained soliton-like solutions of the nonlinear spinor field equations in plane-symmetric metric and spherical symmetric metric [5] [6] [7]. In all these cases, the solutions are regular, the energy density is localized and the total energy is finite. In addition, the charge density and total spin are not bounded in the plane-symmetric metric but they are localized in the spherical symmetric metric. The geometrical symmetries of the space-time are very important in the gravitational theory. Let us emphasize that, the role of symmetries in general theory of relativity has been introduced by Katzin, Lavine and Davis in a series of papers [8] [9] [10] [11].

The aim of this paper is to describe the configuration of elementary particles by the soliton model in general relativity. In addition, the paper deals with the role of the proper gravitational field and the nonlinear terms in the formation of field configurations with limited total energy, spin and charge.

The paper is organized as follows. In Section 2, we established the basics equations and concepts using the variational principle and usual algebraic manipulations. Section 3 addresses the general analytical fundamental solutions. In Section 4, we analyzed and discussed in detail the principal results. Finally, concluding remarks are outlined in Section 5.

2. Lagrangian, Metric, Basics Equations and Concepts

The Lagrangian density of the nonlinear spinor and gravitational fields is given by:

$$L = \frac{R}{2\chi} + L_{S_p} \quad (1)$$

where the spinor Lagrangian density L_{S_p} is

$$L_{S_p} = \frac{i}{2} (\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi) - m\bar{\psi}\psi + L_N. \quad (2)$$

L_N is the nonlinear part of L_{S_p} . It describes the self-interaction of the spinor field. $L_N = F(I_S)$ is an arbitrary function depending on the invariant $I_S = S^2 = (\bar{\psi}\psi)^2$. $R = R_{\mu\nu}g_{\mu\nu}$ is the scalar curvature or the trace of Ricci's tensor. Then, $\chi = \frac{8\pi G}{c^4}$ is Einstein's gravitational constant, G is Newton's gravitational constant and c is the speed of light in vacuum. ψ is the 4-components Dirac's spinor with $\bar{\psi}$ its conjugate. In the sequel, we shall deal with the metric.

In this present analysis, we admit the static spherical symmetric metric in the form:

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} d\xi^2 - e^{2\beta} [d\theta^2 + \sin^2(\theta) d\phi^2]. \tag{3}$$

For simplicity reason, the velocity of light is chosen to equal to unity ($c = 1$) in natural unity. We define spatial variable as in $\xi = \frac{1}{r}$, where r stands for the radial component of the spherical symmetric metric. The metric functions, α , β and γ are stationary and are functions of ξ alone. They verify the harmonic coordinate condition given by the following expression:

$$\alpha = 2\beta + \gamma. \tag{4}$$

Applying the variational principle with respect to the function ψ and its conjugate $\bar{\psi}$, we get the nonlinear spinor field equations under the following form [12]:

$$i\gamma^\mu \nabla_\mu \psi - m\psi + 2\sqrt{I_S} \frac{dF}{dI_S} \psi = 0, \tag{5}$$

$$i\nabla_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} - 2\sqrt{I_S} \frac{dF}{dI_S} \bar{\psi} = 0, \tag{6}$$

Varying the lagrangian (1) with respect to the metric tensor $g_{\mu\nu}$ we obtain the general form of Einstein's equations:

$$G_\mu^\nu = R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R = -\chi T_\mu^\nu, \tag{7}$$

where G_μ^ν is the Einstein's tensor; R_μ^ν is the Ricci's tensor; δ_μ^ν is the Kronecker's symbol and T_μ^ν is the metric energy-momentum tensor of the spinor field. Then, taking into account (7), we find the components of the tensor G_μ^ν in the metric (3) under the coordinate condition (4) as follows:

$$G_0^0 = e^{-2\alpha} (2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -\chi T_0^0, \tag{8}$$

$$G_1^1 = e^{-2\alpha} (2\beta'\gamma' + \beta'^2) - e^{-2\beta} = -\chi T_1^1, \tag{9}$$

$$G_2^2 = e^{-2\alpha} (\beta'' + \gamma'' - 2\beta'\gamma' - \beta'^2) = -\chi T_2^2, \tag{10}$$

$$G_2^2 = G_3^3, \quad T_2^2 = T_3^3. \tag{11}$$

Prime (') in previous equations denotes differentiation with respect to ξ .

The corresponding metric energy-momentum tensor of the spinor field is

$$T_\mu^\nu = \frac{i}{4} g^{\nu\rho} (\bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi) - \delta_\mu^\nu L_{S_p}. \tag{12}$$

Introducing (5) and (6) into (2), L_{S_p} takes the following form:

$$L_{S_p} = \frac{1}{2} \bar{\psi} (i\gamma^\mu \nabla_\mu \psi - m\psi) - \frac{1}{2} (i\nabla_\mu \bar{\psi} \gamma^\mu + m\bar{\psi}) \psi + F(I_S), \tag{13}$$

$$= -2S^2 \frac{\partial F(I_S)}{\partial I_S} + F(I_S), \tag{14}$$

$$= -2I_S \frac{\partial F(I_S)}{\partial I_S} + F(I_S). \tag{15}$$

As the function $\psi = \psi(\xi)$, substituting (15) into (12), we define the nontrivial components of the metric T_μ^ν :

$$T_0^0 = T_2^2 = T_3^3 = -L_{S_p} = 2I_S \frac{\partial F(I_S)}{\partial I_S} - F(I_S), \tag{16}$$

$$T_1^1 = \frac{i}{2} (\bar{\psi} \gamma^1 \nabla_1 \psi - \nabla_1 \bar{\psi} \gamma^1 \psi) + 2I_S \frac{\partial F(I_S)}{\partial I_S} - F(I_S). \tag{17}$$

In the precedent equations γ^μ represent Dirac's matrices in curved space-time [12]. They are connected with Dirac's matrices in flat space-time $\bar{\gamma}_a$ by:

$$\begin{aligned} g_{\mu\nu}(\xi) &= e_\mu^a(\xi) e_\nu^b(\xi) \eta_{ab} \\ \gamma_\mu(\xi) &= e_\mu^a(\xi) \bar{\gamma}_a, \end{aligned} \tag{18}$$

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is Minkowski's metric and $e_\mu^a(\xi)$ are tetradic 4-vectors.

The relation (18) leads

$$\begin{aligned} \gamma^0(\xi) &= e^{-\gamma} \bar{\gamma}^0, \quad \gamma^1(\xi) = e^{-\alpha} \bar{\gamma}^1, \quad \gamma^2(\xi) = e^{-\beta} \bar{\gamma}^2, \\ \gamma^3(\xi) &= \frac{e^{-\beta} \bar{\gamma}^3}{\sin \theta}, \quad \gamma^5(\xi) = \bar{\gamma}^5 \end{aligned} \tag{19}$$

The matrices $\bar{\gamma}^\mu$ are chosen as in [4]

$$\begin{aligned} \bar{\gamma}^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \bar{\gamma}^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \quad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ \gamma^5 = \bar{\gamma}^5 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned}$$

In (2), (5)-(6) and (12), ∇_μ has the covariant derivative of the spinor meaning [13] [14]. It is linked to the spinor affine connection matrices $\Gamma_\mu(\xi)$ by:

$$\nabla_{\mu}\psi = \frac{\partial\psi}{\partial\xi^{\mu}} - \Gamma_{\mu}\psi \quad \text{or} \quad \nabla_{\mu}\bar{\psi} = \frac{\partial\bar{\psi}}{\partial\xi^{\mu}} + \Gamma_{\mu}\bar{\psi}. \tag{20}$$

As for Γ_{μ} , it takes the following general form:

$$\Gamma_{\mu}(\xi) = \frac{1}{4}g_{\rho\mu}(\partial_{\mu}e_{\sigma}^b e_a^{\rho} - \Gamma_{\mu\sigma}^{\rho})\gamma^{\delta}\gamma^{\sigma}, \tag{21}$$

where $\Gamma_{\mu\sigma}^{\rho}$ are Christoffel's symbols. From (21), we find

$$\begin{aligned} \Gamma_0 &= -\frac{1}{2}e^{-2\beta}\bar{\gamma}^0\bar{\gamma}^1\gamma', & \Gamma_1 &= 0, & \Gamma_2 &= \frac{1}{2}e^{-\beta-\gamma}\bar{\gamma}^2\bar{\gamma}^1\beta', \\ \Gamma_3 &= \frac{1}{2}(e^{-\beta-\gamma}\bar{\gamma}^3\bar{\gamma}^1\beta'\sin\theta + \bar{\gamma}^3\bar{\gamma}^2\cos\theta). \end{aligned} \tag{22}$$

Einstein's convention leads

$$\gamma^{\mu}\Gamma_{\mu} = -\frac{1}{2}(e^{-\alpha}\alpha'\bar{\gamma}^1 + \bar{\gamma}^2e^{-\beta}\cot\theta). \tag{23}$$

By substituting (20) and (23) into (5) and (6), we have

$$ie^{-\alpha}\bar{\gamma}^1\left(\partial_{\xi} + \frac{1}{2}\alpha'\right)\psi + \frac{i}{2}\bar{\gamma}^2e^{-\beta}\psi\cot\theta - \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)\psi = 0, \tag{24}$$

$$ie^{-\alpha}\bar{\gamma}^1\left(\partial_{\xi} + \frac{1}{2}\alpha'\right)\bar{\psi} + \frac{i}{2}\bar{\gamma}^2e^{-\beta}\bar{\psi}\cot\theta + \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)\bar{\psi} = 0. \tag{25}$$

Considering $\psi(\xi) = V_{\delta}(\xi)$ with $V_{\delta}(\xi) = \begin{pmatrix} V_1(\xi) \\ V_2(\xi) \\ V_3(\xi) \\ V_4(\xi) \end{pmatrix}$, for the components of

spinor field, we get the following system of equations from (24)

$$V_4' + \frac{1}{2}\alpha'V_4 - \frac{i}{2}e^{\alpha-\beta}V_4\cot\theta + ie^{\alpha}\left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)V_1 = 0, \tag{26}$$

$$V_3' + \frac{1}{2}\alpha'V_3 + \frac{i}{2}e^{\alpha-\beta}V_3\cot\theta + ie^{\alpha}\left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)V_2 = 0, \tag{27}$$

$$V_2' + \frac{1}{2}\alpha'V_2 - \frac{i}{2}e^{\alpha-\beta}V_2\cot\theta - ie^{\alpha}\left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)V_3 = 0, \tag{28}$$

$$V_1' + \frac{1}{2}\alpha'V_1 + \frac{i}{2}e^{\alpha-\beta}V_1\cot\theta - ie^{\alpha}\left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)V_4 = 0. \tag{29}$$

The functions V_1, V_2, V_3 and V_4 are connected by the relation

$$V_1^2 - V_2^2 - V_3^2 + V_4^2 = cste. \tag{30}$$

In Section 3, we shall resolve the fundamental fields equations.

3. General Analytical Fundamental Solutions

Summing the set of Equations (26)-(29), we obtain the first-order differential equation for the invariant function $I_S = S^2$ as follows:

$$\frac{dI_S}{d\xi} + 2\alpha'(\xi)I_S = 0. \quad (31)$$

The Equation (31) possesses the evident solution

$$I_S(\xi) = C_0 \exp[-2\alpha(\xi)], \quad C_0 = \text{const}. \quad (32)$$

The relation (32) reflects the natural link between the nonlinear spinor field of elementary particles and their proper gravitational field.

Using the spinor field Equations (24) and (25), the component T_1^1 of the metric energy-momentum tensor may be rewritten in the form:

$$T_1^1 = m\sqrt{I_S} - F(I_S). \quad (33)$$

In the paragraph to follow, we shall resolve Einstein's equations. To this end, since $T_0^0 = T_2^2$, we have $G_0^0 - G_2^2 = 0$. Therefore we obtain the following equation

$$\beta'' - \gamma'' = e^{2\beta+2\gamma}. \quad (34)$$

The transformation of the Equation (34) leads to a Liouville equation type having the solutions [1]:

$$\beta(\xi) = \frac{A}{4} \left(1 + \frac{2}{D}\right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right] = \left(1 + \frac{2}{D}\right) \gamma(\xi), \quad (35)$$

$$\gamma(\xi) = \frac{A}{4} \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right], \quad (36)$$

A and D are integration constants and T is a function.

The function T has the following form:

$$T(h, \xi + \xi_1) = \begin{cases} \frac{1}{h} \sinh[h(\xi + \xi_1)], & h > 0 \\ (\xi + \xi_1), & h = 0 \\ \frac{1}{h} \sin[h(\xi + \xi_1)], & h < 0 \end{cases} \quad (37)$$

where h and ξ_1 are integration constants.

By substituting the expressions (35) and (36) into (4), we get the metric function $\alpha(\xi)$ as follows:

$$\alpha(\xi) = \frac{A}{2} \left(\frac{3}{2} + \frac{2}{D}\right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right]. \quad (38)$$

Finally we define the relations between the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$:

$$\beta(\xi) = \frac{2+D}{4+3D} \alpha(\xi); \quad \gamma(\xi) = \frac{D}{4+3D} \alpha(\xi). \quad (39)$$

Equation (9) looks like the first integral of the Equations (8) and (10). It is also a first order differential equation. Then, introducing (33) and (39) into (9), we have

$$(\alpha')^2 = \frac{(4+3D)^2}{3D^2+8D+4} e^{2\alpha} \left[e^{\frac{-4-2D}{4+2D}\alpha} - \chi(m\sqrt{I_S} - F(I_S)) \right]. \tag{40}$$

Taking into account $\alpha' = -\frac{1}{2I_S} \frac{dI_S}{d\xi}$ and $I_S(\xi) = C_0 e^{-2\alpha(\xi)}$, from (40) we obtain

$$\frac{dI_S}{d\xi} = \pm \frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}} \sqrt{I_S} \sqrt{\left[\left(\frac{I_S}{C_0} \right)^{\frac{2+D}{4+3D}} - \chi(m\sqrt{I_S} - F(I_S)) \right]} \tag{41}$$

We obtain the general solutions of the Equation (41) as follows:

$$\int \frac{dI_S}{\sqrt{I_S} \sqrt{\left[\left(\frac{I_S}{C_0} \right)^{\frac{2+D}{4+3D}} - \chi(m\sqrt{I_S} - F(I_S)) \right]}} = \pm \frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}} (\xi + \xi_0) \tag{42}$$

The general solutions (42) depend on the arbitrary function $L_N = F(I_S)$. Thus, setting an analytical concrete form of the function $F(I_S)$, from (42) we can determine explicitly $I_S(\xi)$. Then, knowing $I_S(\xi)$, we can find out the metric function $\alpha(\xi)$ from (32). Finally, we can get the functions $\beta(\xi)$ and $\gamma(\xi)$ from (39).

Considering the invariant $I_S(\xi) = C_0 e^{-2\alpha(\xi)}$, we can establish the regularity properties of the obtained solutions. Studying the distribution of the energy per unit invariant volume $T_0^0 \sqrt{-3_g}$, we can establish the localization properties of the solutions.

The following paragraph addresses to the determination of the functions $V_\delta(\xi)$. In this optic, we must solve the set of Equations (26)-(29) in more compact form if we pass to the functions $W_\delta(\xi) = e^{\frac{\alpha}{2}} V_\delta(\xi)$, with $\delta = 1, 2, 3, 4$. We have:

$$W_4' - \frac{i}{2} e^{\alpha-\beta} W_4 \cot \theta + i e^\alpha \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) W_1 = 0, \tag{43}$$

$$W_3' + \frac{i}{2} e^{\alpha-\beta} W_3 \cot \theta + i e^\alpha \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) W_2 = 0, \tag{44}$$

$$W_2' - \frac{i}{2} e^{\alpha-\beta} W_2 \cot \theta - i e^\alpha \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) W_3 = 0, \tag{45}$$

$$W_1' + \frac{i}{2} e^{\alpha-\beta} W_1 \cot \theta - i e^\alpha \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) W_4 = 0, \tag{46}$$

where

$$W_\rho' = \left(V_\rho' + \frac{1}{2} \alpha' V_\rho \right) e^{\frac{1}{2}\alpha}. \tag{47}$$

With the set of Equations (43)-(46) where $W = W_\delta(\xi)$ let us pass to the system of equations depending on functions of the argument I_S , i.e. $W_\delta(I_S) = W_\delta(\xi)$,

$I_s(\xi) = C_0 e^{-2\alpha(\xi)}$. We obtain for $W_s(I_s)$ the set of equations as follows:

$$\frac{dW_4}{dI_s} - iE(I_s)W_4 + iK(I_s)W_1 = 0, \quad (48)$$

$$\frac{dW_3}{dI_s} + iE(I_s)W_3 + iK(I_s)W_2 = 0, \quad (49)$$

$$\frac{dW_2}{dI_s} - iE(I_s)W_2 - iK(I_s)W_3 = 0, \quad (50)$$

$$\frac{dW_1}{dI_s} + iE(I_s)W_1 - iK(I_s)W_4 = 0, \quad (51)$$

where

$$E(I_s) = \frac{1}{2} \frac{\left(\sqrt{\frac{C_0}{I_s}}\right)^{\frac{2+2D}{4+3D}} \cot \theta}{\frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}} \sqrt{I_s} \sqrt{\left[\left(\frac{I_s}{C_0}\right)^{\frac{2+D}{4+3D}} - \chi(m\sqrt{I_s} - F(I_s))\right]}}; \quad (52)$$

$$K(I_s) = \frac{\left(\sqrt{\frac{C_0}{I_s}}\right) \left(m - 2\sqrt{I_s} \frac{dF}{dI_s}\right)}{\frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}} \sqrt{I_s} \sqrt{\left[\left(\frac{I_s}{C_0}\right)^{\frac{2+D}{4+3D}} - \chi(m\sqrt{I_s} - F(I_s))\right]}} \quad (53)$$

In sequel, we shall transform the Equations (48)-(51) to the second order differential equations. In this perspective, differentiating the Equation (48) and substituting the expression of the function $W_1(I_s)$ and the expression of its derivative into the result, we obtain:

$$W_4'' - \frac{K'(I_s)}{K(I_s)} W_4' + \left[E^2(I_s) - K^2(I_s) + i \frac{K'(I_s)E(I_s) - K(I_s)E'(I_s)}{K(I_s)} \right] W_4 = 0. \quad (54)$$

Similarly differentiating the Equation (51) and introducing into the result the expression of $W_4(I_s)$ and the expression of its derivative, we obtain the second-order differential equation for the function $W_1(I_s)$:

$$W_1'' - \frac{K'(I_s)}{K(I_s)} W_1' + \left[E^2(I_s) - K^2(I_s) + i \frac{K(I_s)E'(I_s) - K'(I_s)E(I_s)}{K(I_s)} \right] W_1 = 0. \quad (55)$$

Doing the same operating on the Equations (49)-(50), we find the second-order differential equations obeyed by the functions $W_2(I_s)$ and $W_3(I_s)$ as follows:

$$W_3'' - \frac{K'(I_s)}{K(I_s)} W_3' + \left[E^2(I_s) - K^2(I_s) + i \frac{K(I_s)E'(I_s) - K'(I_s)E(I_s)}{K(I_s)} \right] W_3 = 0. \quad (56)$$

$$W_2'' - \frac{K'(I_S)}{K(I_S)}W_2' + \left[E^2(I_S) - K^2(I_S) + i \frac{K'(I_S)E(I_S) - K(I_S)E'(I_S)}{K(I_S)} \right] W_2 = 0. \tag{57}$$

By summing (54)-(55) and setting $U = W_1 + W_4$, we obtain the following second-order differential equations of the function $U(I_S)$:

$$U''(I_S) - \frac{K'(I_S)}{K(I_S)}U'(I_S) + 2[E^2(I_S) - K^2(I_S)]U(I_S) = 0. \tag{58}$$

The Equation (58) may be transformed to:

$$\frac{1}{K(I_S)\sqrt{2\varepsilon}} \frac{d}{dI_S} \left[\frac{U'(I_S)}{K(I_S)\sqrt{2\varepsilon}} \right] - U(I_S) = 0 \tag{59}$$

under the condition $E^2(I_S) = (1 - \varepsilon)K^2(I_S)$ with $0 < \varepsilon \leq 1$ [4].

The Equation (59) possesses the first integral

$$U'(I_S) = \pm \sqrt{U^2(I_S) + C_1 K(I_S) \sqrt{2\varepsilon}}, \quad C_1 = const. \tag{60}$$

If $C_1 = a_1^2 > 0$, then the Equation (60) has the solution

$$U(I_S) = a_1 \sinh N_1(I_S). \tag{61}$$

If $C_1 = -b_1^2 < 0$, the solution of the equation of (60) is given by:

$$U(I_S) = b_1 \cosh N_1(I_S). \tag{62}$$

with

$$N_1(I_S) = \sqrt{2\varepsilon} \int K(I_S) dI_S + R_1, \quad R_1 = const. \tag{63}$$

The difference of Equations (48) and (51), taking into account of (61) and (62), gives:

$$X(I_S) = W_1 - W_4 = -ia_1 \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(I_S), \tag{64}$$

or

$$X(I_S) = W_1 - W_4 = -ib_1 \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_1(I_S), \tag{65}$$

where a_1 and b_1 are integration constants.

Solving analogously the Equations (56) and (57), we obtain the following expressions for $Y(I_S) = W_2 + W_3$ as follows:

$$Y(I_S) = a_2 \sinh N_2(I_S), \quad \text{for } C_2 = a_2^2 > 0 \tag{66}$$

or

$$Y(I_S) = b_2 \cosh N_2(I_S), \quad \text{for } C_2 = -b_2^2 < 0. \tag{67}$$

In these conditions, it then follows from the expressions (66) and (67) that:

$$V(I_S) = W_2 - W_3 = ia_2 \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_2(I_S), \tag{68}$$

or

$$V(I_S) = W_2 - W_3 = ib_2 \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(I_S), \quad (69)$$

$$N_2(I_S) = \sqrt{2\varepsilon} \int K(I_S) dI_S + R_2, \quad (70)$$

where a_2 , b_2 and R_2 are integration constants.

Considering the cases where $C_1 = a_1^2 > 0$ and $C_2 = -b_2^2 < 0$, let us determine the expressions of the functions $W_\delta(I_S)$. We get for the functions $W_\delta(I_S)$ the following expressions:

$$W_1(I_S) = a_0 \left[\sinh N_1(I_S) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(I_S) \right], \quad (71)$$

$$W_2(I_S) = b_0 \left[\cosh N_2(I_S) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(I_S) \right], \quad (72)$$

$$W_3(I_S) = b_0 \left[\cosh N_2(I_S) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(I_S) \right], \quad (73)$$

$$W_4(I_S) = a_0 \left[\sinh N_1(I_S) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(I_S) \right], \quad (74)$$

with $a_0 = \frac{1}{2}a_1$ and $b_0 = \frac{1}{2}b_2$.

Let us note that we can also obtain the expressions of the functions $W_\delta(I_S)$ by choosing $C_1 = -b_1^2 < 0$ and $C_2 = a_2^2 > 0$. In addition, in the expressions (63) and (70), we can use the minus sign before the integral. By doing so, we don't lose generality [15]. We pass to the functions $V_\delta(\xi)$ by multiplying the functions $W_\delta(\xi)$ obtained in the expressions (71)-(74) by $e^{\frac{1}{2}\alpha(\xi)}$ as follows:

$$V_1(\xi) = a_0 \left[\sinh N_1(\xi) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(\xi) \right] \times \exp \left\{ -\frac{A}{4} \left(\frac{3}{2} + \frac{2}{D} \right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right] \right\} \quad (75)$$

$$V_2(\xi) = b_0 \left[\cosh N_2(\xi) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(\xi) \right] \times \exp \left\{ -\frac{A}{4} \left(\frac{3}{2} + \frac{2}{D} \right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right] \right\} \quad (76)$$

$$V_3(\xi) = b_0 \left[\cosh N_2(\xi) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(\xi) \right] \times \exp \left\{ -\frac{A}{4} \left(\frac{3}{2} + \frac{2}{D} \right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right] \right\} \quad (77)$$

$$V_4(\xi) = a_0 \left[\sinh N_1(\xi) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(\xi) \right] \times \exp \left\{ -\frac{A}{4} \left(\frac{3}{2} + \frac{2}{D} \right) \ln \left[\frac{A}{DT^2(h, \xi + \xi_1)} \right] \right\} \tag{78}$$

The following section deals with the analysis of the general results obtained previously by considering the concrete nonlinear terms of the arbitrary function $F(I_S)$ in the Lagrangian density.

4. Discussions

In this section, we choose the concrete type of nonlinear spinor field equations under the form:

$$F(I_S) = \lambda I_S = \lambda S^2 \tag{79}$$

where λ is nonlinearity parameter and $I_S = S^2$ is the invariant function. It is convenient to separately analyze the two cases $\lambda \neq 0$ and $\lambda = 0$.

- Firstly, we address the Ivanenko-Heisenberg type nonlinear spinor field equation corresponding to $F(I_S) = \lambda I_S$ and $\lambda \neq 0$. From (24) we have:

$$ie^{-\alpha} \bar{\gamma}^1 \left(\partial_{\xi} + \frac{1}{2} \alpha' \right) \psi + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot \theta - (m - 2\lambda \sqrt{I_S}) \psi = 0, \tag{80}$$

By substituting $F(I_S) = \lambda I_S = \lambda S^2$ into (42) and assuming that $\frac{2+D}{4+3D} = 1$, without loss of generality, we obtain:

$$S(\xi) = \left(\frac{m\chi C_0}{1 + \lambda\chi C_0} \right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1 + \lambda\chi C_0)(3D^2 + 8D + 4)}} (\xi + \xi_0) \right] \tag{81}$$

The invariant function $I_S(\xi)$ is given by the following expression:

$$I_S(\xi) = \left(\frac{m\chi C_0}{1 + \lambda\chi C_0} \right)^2 \cosh^4 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1 + \lambda\chi C_0)(3D^2 + 8D + 4)}} (\xi + \xi_0) \right] \tag{82}$$

From (32), we deduce the expression of the metric function $\alpha(\xi)$. Then, from (39) we deduce the expressions of the functions $\beta(\xi)$ and $\gamma(\xi)$:

$$\alpha(\xi) = \ln \left[\frac{\sqrt{C_0}}{\left\{ \left(\frac{m\chi C_0}{1 + \lambda\chi C_0} \right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1 + \lambda\chi C_0)(3D^2 + 8D + 4)}} (\xi + \xi_0) \right] \right\}} \right] \tag{83}$$

$$\beta(\xi) = \ln \left[\frac{\sqrt{C_0}}{\left\{ \left(\frac{m\chi C_0}{1 + \lambda\chi C_0} \right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1 + \lambda\chi C_0)(3D^2 + 8D + 4)}} (\xi + \xi_0) \right] \right\}} \right] \tag{84}$$

$$\gamma(\xi) = \left(\frac{D}{4+3D}\right) \ln \left[\frac{\sqrt{C_0}}{\left\{ \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] \right\}} \right] \tag{85}$$

Introducing (82) into (16), the energy density is defined as follows

$$T_0^0(\xi) = \lambda \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right)^2 \cosh^4 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] \tag{86}$$

Let us note that the energy density is bounded when $\xi \in [0, \xi_c]$. It takes the value

$\lambda \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right)^2$ when $\xi = \xi_0 = 0$. Then, it takes the value

$$\lambda \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right)^2 \cosh^4 \left[\frac{m\chi C_0(4+3D)\xi_c}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} \right] \text{ when } \xi = \xi_c \text{ and } \xi_0 = 0.$$

In virtue of (86), the energy density per unit invariant volume

$f(\xi) = T_0^0(\xi) e^{2\alpha-\gamma} \sin \theta$ is defined in the following way:

$$f(\xi) = \lambda \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right)^2 \cosh^4 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] \zeta(\xi) \sin \theta \tag{87}$$

where the function $\zeta(\xi)$ has the form:

$$\zeta(\xi) = \left[\frac{\sqrt{C_0}}{\left\{ \left(\frac{m\chi C_0}{1+\lambda\chi C_0}\right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] \right\}} \right]^{\frac{8+5D}{4+3D}} \tag{88}$$

From (88) the energy density per unit invariant volume of Heisenberg-Ivanenko type equation of a nonlinear spinor field is localized when $\xi \in [0, \xi_c]$. Therefore, the total energy $E = \int_0^{\xi_c} f(\xi) d\xi$ is finite.

Let us find an explicit form of the function $V_\delta(\xi)$, $\delta = 1, 2, 3, 4$. By doing so, we deduce from (63) and (70) the expressions of the functions $N_1(\xi)$ and $N_2(\xi)$ by substituting the obtained expression for $I_s(\xi)$ from (82) into (52) as follows:

$$N_{1,2}(\xi) = -2\lambda\sqrt{2\varepsilon}(\xi + \xi_0) + M \tanh \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] + R_{1,2} \tag{89}$$

where

$$M = \frac{2\sqrt{2\varepsilon}(1 + \lambda\chi C_0)^{\frac{3}{2}} \sqrt{3D^2 + 8D + 4}}{m\chi^2 C_0(4 + 3D)} = \text{const.}$$

Then, the expressions of $N_1(\xi)$ and $N_2(\xi)$ are substituted into (71)-(74) to produce the explicit form of the functions $W_\delta(\xi)$ that we multiply by $e^{-\frac{1}{2}\alpha(\xi)}$ to get explicitly $V_\delta(\xi)$ as follows:

$$V_1(\xi) = a_0 \left[\sinh N_1(\xi) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(\xi) \right] \left(\frac{m\chi\sqrt{C_0}}{1 + \lambda\chi C_0} \right)^{\frac{1}{2}} \times \cosh \left[\frac{\sqrt{2\varepsilon}(1 + \lambda\chi C_0)^2}{M\chi} (\xi + \xi_0) \right] \tag{90}$$

$$V_2(\xi) = b_0 \left[\cosh N_2(\xi) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(\xi) \right] \left(\frac{m\chi\sqrt{C_0}}{1 + \lambda\chi C_0} \right)^{\frac{1}{2}} \times \cosh \left[\frac{\sqrt{2\varepsilon}(1 + \lambda\chi C_0)^2}{M\chi} (\xi + \xi_0) \right] \tag{91}$$

$$V_3(\xi) = b_0 \left[\cosh N_2(\xi) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh N_2(\xi) \right] \left(\frac{m\chi\sqrt{C_0}}{1 + \lambda\chi C_0} \right)^{\frac{1}{2}} \times \cosh \left[\frac{\sqrt{2\varepsilon}(1 + \lambda\chi C_0)^2}{M\chi} (\xi + \xi_0) \right] \tag{92}$$

$$V_4(\xi) = a_0 \left[\sinh N_1(\xi) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh N_1(\xi) \right] \left(\frac{m\chi\sqrt{C_0}}{1 + \lambda\chi C_0} \right)^{\frac{1}{2}} \times \cosh \left[\frac{\sqrt{2\varepsilon}(1 + \lambda\chi C_0)^2}{M\chi} (\xi + \xi_0) \right] \tag{93}$$

Let us emphasize that the Equation (80) has soliton-like solutions. Here, the existence of the soliton-like configurations with localized energy density, finite total energy in Heisenberg-Ivanenko type nonlinear equation is an interesting result.

Using the solutions (90)-(93) we can determine the components of the spinor current vector $j^\mu = \bar{\psi}\gamma^\mu\psi$ under the general form

$$j^0 = (V_1^*V_1 + V_2^*V_2 + V_3^*V_3 + V_4^*V_4) e^{-(\alpha+\gamma)}, \tag{94}$$

$$j^1 = (V_1^*V_4 + V_2^*V_3 + V_3^*V_2 + V_4^*V_1) e^{-2\alpha}, \tag{95}$$

$$j^2 = -i(V_1^*V_4 - V_2^*V_3 + V_3^*V_2 - V_4^*V_1) e^{-(\alpha+\beta)}, \tag{96}$$

$$j^3 = (V_1^*V_3 - V_2^*V_4 + V_3^*V_1 - V_4^*V_2) e^{-(\alpha+\beta)}. \tag{97}$$

In the case of Heisenberg-Ivanenko type nonlinear equation, the components

of the spinor current vector may be rewritten in the following way:

$$j^0 = 2e^{-\alpha-\gamma} \left\{ a_0^2 \left[\sinh^2 N_1(\xi) + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right)^2 \cosh^2 N_1(\xi) \right] + b_0^2 \left[\cosh^2 N_2(\xi) + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right)^2 \sinh^2 N_2(\xi) \right] \right\} \tag{98}$$

$$j^1 = 2e^{-2\alpha} \left\{ a_0^2 \left[\sinh^2 N_1(\xi) - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right)^2 \cosh^2 N_1(\xi) \right] + b_0^2 \left[\cosh^2 N_2(\xi) - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right)^2 \sinh^2 N_2(\xi) \right] \right\} \tag{99}$$

$$j^2 = 4e^{-\alpha-\beta} \left[a_0^2 \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right) \cosh N_1(\xi) \sinh N_1(\xi) - b_0^2 \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}} \right) \cosh N_2(\xi) \sinh N_2(\xi) \right] \tag{100}$$

$$j^3 = 0 \tag{101}$$

As in this study the configuration is static, the components j^1 , j^2 and j^3 are evident. But only the component j^0 is nonzero. With this assumption, we get $a_0 = b_0 = a$, $R_1 = R_2 = R$, $N_1(\xi) = N_2(\xi) = N(\xi)$ and $\varepsilon = 1$. From the component j^0 , we define the charge density or the chronometric invariant of the spinor field as follows:

$$\rho(\xi) = (j_0 j^0)^{\frac{1}{2}} = 4a^2 \mathcal{G}(\xi) \cosh 2N(\xi) \tag{102}$$

where $N(\xi)$ is defined by the expression (89) and

$$\mathcal{G}(\xi) = e^{-\alpha(\xi)} = \left(\frac{m\chi\sqrt{C_0}}{1+\lambda\chi C_0} \right) \cosh^2 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right]. \tag{103}$$

The charge density is localized when $\xi \in [0, \xi_c]$. The total charge of the spinor field in the Heisenberg-Ivanenko type nonlinear equation is:

$$Q = \int_0^{\xi_c} \rho \sqrt{-3_g} d\xi = 4a^2 \int_0^{\xi_c} \cosh 2N(\xi) e^{\alpha-\gamma} \sin \theta d\xi < \infty, \tag{104}$$

ξ_c being the center of the field configuration and

$$e^{\alpha-\gamma} = \frac{C_0}{\left\{ \left(\frac{m\chi C_0}{1+\lambda\chi C_0} \right)^2 \cosh^4 \left[\frac{m\chi C_0(4+3D)}{2\sqrt{(1+\lambda\chi C_0)(3D^2+8D+4)}} (\xi + \xi_0) \right] \right\}} \tag{105}$$

From (104) the total charge is finite when $\xi \in [0, \xi_c]$.

Let us deal with the spin tensor of the nonlinear spinor field. Its general form is:

$$S^{\mu\nu,\lambda} = \frac{1}{4} \bar{\psi} \{ \gamma^\lambda \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\lambda \} \psi. \tag{106}$$

Using (106), the spatial density of the spin tensor $S^{ik,0}$, $i, k = 1; 2; 3$ is:

$$S^{ik,0} = \frac{1}{4} \bar{\psi} \{ \gamma^0 \sigma^{ik} + \sigma^{ik} \gamma^0 \} \psi = \frac{1}{2} \bar{\psi} \gamma^0 \sigma^{ik} \psi. \tag{107}$$

Thus, we have

$$S^{12,0} = S^{13,0} = 0. \tag{108}$$

$$S^{23,0} = 2a^2 \cosh 2N(\xi) e^{-\alpha}. \tag{109}$$

The relation (109) leads to the definition of the chronometric invariant of the spatial density as follows:

$$S_{chl}^{23,0} = \left(S_{23,0} S^{23,0} \right)^{\frac{1}{2}} = 2a^2 \cosh 2N(\xi) e^{-\alpha}. \tag{110}$$

Thus, the projection of the spin vector on the radial axis has the form:

$$S_1 = \int_0^{\xi_c} S_{chl}^{23,0} \sqrt{-3_g} d\xi = 2a^2 \int_0^{\xi_c} \cosh 2N(\xi) e^{\alpha-\gamma} \sin \theta d\xi. \tag{111}$$

Note that the spin tensor of the spinor field has a finite value and positive because the integrand is positive.

We can conclude that the Heisenberg-Ivanenko type nonlinear equation possesses soliton-like configuration with finite value of the total charge and the total spin. In addition, the metrics functions are stationary and regular. Therefore, these solutions must be used to describe the configuration of elementary particles with mass. In sequel, as mentioned in [2], we shall clarify the influence of nonlinear terms in the nonlinear field equations in the formation of regular localized soliton-like solutions. To this end, we must resolve Dirac's equation and compare its solutions with solutions to nonlinear spinor equations.

- In the linear case, the nonlinearity parameter is equal to zero *i.e.* $\lambda = 0 \Rightarrow L_N = 0$. Introducing $L_N = 0$ into the Equation (24) leads to the following expression:

$$ie^{-\alpha} \bar{\gamma}^1 \left(\partial_\xi + \frac{1}{2} \alpha' \right) \psi + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot \theta - m\psi = 0. \tag{112}$$

Then, according to (42), we get

$$S(\xi) = m\chi C_0 \cosh^2 \left[\frac{C_0(4+3D)}{2\sqrt{3D^2+8D+4}(\xi+\xi_0)} \right]. \tag{113}$$

The invariant function is given by the following expression:

$$I_S(\xi) = (m\chi C_0)^2 \cosh^4 \left[\frac{C_0(4+3D)}{2\sqrt{3D^2+8D+4}(\xi+\xi_0)} \right]. \tag{114}$$

As for the metric functions we have:

$$\alpha(\xi) = \ln \left[\frac{\sqrt{C_0}}{\left\{ m\chi C_0 \cosh^2 \left[\frac{C_0(4+3D)}{2\sqrt{3D^2+8D+4}(\xi+\xi_0)} \right] \right\}} \right] \tag{115}$$

$$\beta(\xi) = \ln \left[\frac{\sqrt{C_0}}{\left\{ m\chi C_0 \cosh^2 \left[\frac{C_0(4+3D)}{2\sqrt{3D^2+8D+4}}(\xi + \xi_0) \right] \right\}} \right] \tag{116}$$

$$\gamma(\xi) = \left(\frac{D}{4+3D} \right) \ln \left[\frac{\sqrt{C_0}}{\left\{ m\chi C_0 \cosh^2 \left[\frac{C_0(4+3D)}{2\sqrt{3D^2+8D+4}}(\xi + \xi_0) \right] \right\}} \right] \tag{117}$$

From (16), the energy density is

$$T_0^0 = 0 \tag{118}$$

Let us note that the invariant function $I_S(\xi) = S^2$ and the metric functions $g_{00} = e^{2\gamma}$, $g_{11} = -e^{2\alpha}$, $g_{22} = -e^{2\beta}$, $g_{33} = -e^{2\beta} \sin^2 \theta$ are regular. Then, the invariant function $I_S(\xi)$ is bounded when $\xi \in [0, \xi_c]$. But the energy density is unlimited. Here, it is clear that soliton-like solutions do not exist in the linear case. Moreover, from these results it is shown that to obtain the regular localized soliton-like solutions, the nonlinear terms are very important. In the following paragraph, we shall consider the case when the influence of the gravitational field is not taken into account for proving the importance of the proper gravitational field of elementary particles in the configuration of their geometrical structures.

In order to determine the role of the own gravitational field in the formation of regular localized solutions of soliton-like type to Heisenberg-Ivanenko type nonlinear spinor field equations, it is necessary to consider solutions to the Equation (5) in flat space-time when $\alpha = \beta = \gamma = 0$ in (2). It then follows that in the space-time without gravitation the set of equations for the functions $V_s(\xi)$ becomes

$$V_4' - \frac{i}{2} V_4 \cot \theta + i \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) V_1 = 0, \tag{119}$$

$$V_3' + \frac{i}{2} V_3 \cot \theta + i \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) V_2 = 0, \tag{120}$$

$$V_2' - \frac{i}{2} V_2 \cot \theta - i \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) V_3 = 0, \tag{121}$$

$$V_1' + \frac{i}{2} V_1 \cot \theta - i \left(m - 2\sqrt{I_S} \frac{dF}{dI_S} \right) V_4 = 0. \tag{122}$$

From $\alpha = \beta = \gamma = 0$, we obtain the expressions of the functions $I_S(\xi)$ and $F(I_S)$ as follows

$$I_S(\xi) = S^2 = C_0 = const, \quad F(I_S) = \lambda C_0 = const. \tag{123}$$

Taking into account the expression (123) and (16) we obtain the expression of

the energy density T_0^0 as follows:

$$T_0^0(\xi) = \lambda C_0 = \text{const.} \tag{124}$$

In this condition the distribution of the spinor field energy density per unit invariant volume takes the form

$$\varepsilon(\xi) = \lambda C_0 \sin \theta = \text{const.} \tag{125}$$

The total energy has finite value and given by the following expression:

$$E = \int_0^{\xi_c} \varepsilon(\xi) d\xi = \lambda C_0 \xi_c \sin \theta > 0. \tag{126}$$

In the flat space-time, we get the explicit expressions of the functions $N_1(\xi)$ and $N_2(\xi)$ under the form:

$$N_{1,2}(\xi) = m\sqrt{2\varepsilon}(\xi + \xi_0) + R. \tag{127}$$

Let us find the explicit form of the functions $V_\rho(\xi)$. In this perspective, by Substituting (127) into (90)-(93) the system of Equations (119)-(122) has the following solutions:

$$V_1(\xi) = a_0 \left[\sinh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) \right] \tag{128}$$

$$V_2(\xi) = a_0 \left[\cosh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) \right] \tag{129}$$

$$V_3(\xi) = a_0 \left[\cosh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) - i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \sinh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) \right] \tag{130}$$

$$V_4(\xi) = a_0 \left[\sinh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) + i \left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}} \right) \cosh(m\sqrt{2\varepsilon}(\xi + \xi_0) + R) \right] \tag{131}$$

Let us emphasize that in the space-time without gravitation, the obtained solutions are soliton-like configurations. Indeed, the solutions are regular with localized energy density and finite total energy. In addition to this, the metric functions are stationary. The consideration of the proper gravitational field and the geometrical proper of the metric are crucial and necessary to obtain the regular solutions having the soliton configuration type. We devote the last section to concluding remarks.

5. Concluding Remarks

In this manuscript, taking into account the proper gravitational field of elementary particles, we obtained the general solutions of Einstein and nonlinear spinor field equations. We analyzed in particular the Heisenberg-Ivanenko type nonlinear spinor field equations. We note that the solutions of Heisenberg-Ivanenko equation are regular and possess a bounded energy density and limited total energy. Similarly, the metric functions are stationary. The total charge and the total spin are finite quantities as well. We demonstrated that the soliton-like solutions exist in flat space-time and absent in linear case. The nonlinearity of the spinor field vanishes in the space-time without gravitation. Therefore, we note that, the gravitational field is nonlinear by nature and its nonlinearity induces the nonlinearity of the spinor field. In order to extend the present analysis, the forthcoming paper will address to Dirac equation of spinor field in curved space-time in gravitational theory: spherical symmetric soliton-like solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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