

Some Chaotic Properties of a Kind of Coupled Map Lattices

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Abstract

This paper is concerned with some chaotic properties of a kind of coupled map lattices, which is proposed by Kaneko. First, this research discussed the sensitivity, infinite sensitivity, transitivity, accessibility, densely Li-Yorke sensitivity and exact of coupled map lattices. Then, some sufficient conditions under which $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is Kato chaotic, positive entropy chaotic and Ruelle-Takens chaos are obtained.

Keywords

Coupled Map Lattices, Sensitivity, Chaos, Accessibility

1. Introduction

In 1983, Kaneko [1] proposed coupled map lattices (Short for CMLs). Then, in biophysics, materials, chaos, image processing, CMLs are intensively discussed (Refer to literature [2]-[8] and others). In 2005, the literature [9] showed that CMLs have some topology and ergodic properties. In 2010, Juan Lu [10] presented a definition of distributional chaos on a sequence (DCS) for CML systems and stated two different sufficient conditions for having DCS. In 2010, Juan Luis [11] proved that this CML system has positive topological entropy for zero coupling constant. In 2016, Risong Li [12] [13] had obtained some relevant conclusions for the zero coupling constant and proved that the system has three kinds of chaos. In this paper, the following CML from [14] is considered.

$$x_{m+1,n} = (1 - \varepsilon) f(x_{m,n}) + \varepsilon f(x_{m,n-1}) \quad (1)$$

where $x_{m,n} \in I$, $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $n \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, I is a non-degenerate compact interval, f is a map on I , and $\varepsilon \in [0, 1]$ is a constant.

For $t \in \mathbb{Z}$, let $\mathbb{N}_t = \{t, t+1, \dots\}$ and $\Omega = \{(0, n) : n \in \mathbb{Z}\} = \{\dots, (0, -1), (0, 0), (0, 1), \dots\}$. For any sequence $\phi = \{\phi_{0,n}\}_\infty^\infty$ on Ω , by induction, one can obtain a double-indexed sequence $x = \{x_{m,n} : m = 0, 1, 2, \dots; n = \dots, -1, 0, 1, \dots\}$, which is said to be a solution of the above system (1) with initial condition ϕ .

Let I be a subset of real number set, write

$$I_\infty = \left\{ \{a_n\}_{n=-\infty}^\infty = (\dots, a_{-1}, a_0, a_1, \dots) : a_n \in I, n \in \mathbb{Z} \right\}$$

and

$$\Delta_\infty = \left\{ (\dots, a_{-1}, a_0, a_1, \dots) : a_i = a_j \in I, i, j \in \mathbb{Z} \right\}$$

which is called the diagonal set of I_∞ .

For arbitrary, two sequences $x_1 = \{x_{1,n}\}_{n=-\infty}^\infty, x_2 = \{x_{2,n}\}_{n=-\infty}^\infty \in I_\infty$, it is easy to prove that

$$d(x_1, x_2) = \sup \left\{ |x_{1,n} - x_{2,n}| : n = \dots, -1, 0, 1, \dots \right\} \tag{2}$$

is a metric on I_∞ .

Let $f : I \mapsto I$ be a continuous map and $x = \{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ be a solution of the above system (1) with initial condition $\phi = \{\phi_{0,n}\}_\infty^\infty \in I_\infty$.

Let

$$x_m = \{x_{m,n}\}_{n=-\infty}^\infty = (\dots, x_{m,-1}, x_{m,0}, x_{m,1}, \dots), \forall m \in \mathbb{N}_0,$$

and

$$x_{m+1} = \{x_{m+1,n}\}_{n=-\infty}^\infty = (\dots, x_{m+1,-1}, x_{m+1,0}, x_{m+1,1}, \dots) = F(x_m), \forall m \in \mathbb{N}_0,$$

where $x_0 = \phi = \{x_{0,n} = \phi_{0,n}\}_{n=-\infty}^\infty$ and

$$x_{m+1,n} = (1 - \varepsilon) f(x_{m,n}) + \varepsilon f(x_{m,n-1}), \forall m \in \mathbb{N}_0, n \in \mathbb{Z}.$$

Then, one can see that the above system (1) is equivalent to the following system

$$x_{m+1} = F(x_m), x_m \in I_\infty, m = 0, 1, 2, \dots \tag{3}$$

For the above system (3), the map F is said to be induced by the system (1). Obviously, a double-indexed sequence $\{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ is a solution of the above system (1) if and only if the sequence $\{x_m = \{x_{m,n}\}_{n=-\infty}^\infty : m \in \mathbb{N}_0\}_{m=0}^\infty$ is a solution of the above system (3).

Next section, the definitions of sensitive, infinite sensitive, transitive, accessibility, densely Li-Yorke sensitive and exact will be reviewed. And then, in section 3, it is proved that the system $(\Delta_\infty, d, F|_{\Delta_\infty})$ satisfies three definitions of chaos (Kato chaotic, positive entropy chaotic and Ruelle-Takens chaos) under the conditions that f is chaos in these sense.

2. Preliminaries

After T. Y. Li and J. A. Yorke [15] first put forward the mathematical definition of ‘‘chaos’’, many other definitions of chaos appeared later. For example, sensi-

tivity, infinite sensitivity, transitivity, accessibility, densely Li-Yorke sensitivity, Kato chaotic, positive entropy chaotic, Ruelle-Takens chaos, and so on.

Definition 1. Let (X, ρ) be a metric space and $f : X \mapsto X$ be a continuous function. f is said to be

1) transitive if for any nonempty open subsets $U_1, U_2 \subset Y$, $f^n(U_1) \cap U_2 \neq \emptyset$ for some integer $n \in \mathbb{N}$ (see [16]).

2) sensitive if there exist $\eta > 0$ such that for any $x \in X$ and $\varepsilon > 0$, there exists $y \in B(x, \varepsilon)$ and $n \in \mathbb{N}$ such that $\rho(f^n(x), f^n(y)) \geq \eta$ (see [17]).

3) infinitely sensitive if there exist $\eta > 0$ such that for any $x \in X$ and $\varepsilon > 0$, there exists $y \in B(x, \varepsilon)$ and $n \in \mathbb{N}$ such that $\limsup_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) \geq \eta$ (see [17]).

4) $n \rightarrow \infty$ accessible if for any $\varepsilon > 0$ and any two nonempty open subsets $U_1, U_2 \subset X$, there are two points $x \in U_1$ and $y \in U_2$ such that $\rho(f^n(x), f^n(y)) < \varepsilon$ for some integer $n > 0$ (see [16]).

5) exact if for any open subset $U \subset X$, there is $m \in \mathbb{N}$ such that $f^m(U) = X$ (see [18]).

Remark 1. [19] There is another equivalent definition of transitivity. $f : X \mapsto X$ is said to be transitivity, if there is an $x_0 \in X$ such that $\overline{\text{Orb}_f(x_0)} = X$. Where, $\overline{\text{Orb}_f(x_0)} = \{f^n(x_0) | n = 0, 1, 2, \dots\}$ is called the orbit of the point x_0 .

Definition 2. 1) A dynamic system (X, f) (or the map $f : X \rightarrow X$) is Li-Yorke sensitive, if for any $x \in X$ has $x \in Q_\delta(f)$ for some $\delta > 0$.

2) A dynamic system (X, f) (or the map $f : X \rightarrow X$) is densely Li-Yorke sensitive if $Q_\delta(f)$ is dense in X for some $\delta > 0$. Among them,

$$Q_\delta(f) = \{x \in X : \forall \varepsilon > 0, \exists y \in B(x, \varepsilon) \text{ such that } (x, y) \in LY_\rho(f, \delta)\}$$

$$LY_\rho(f, \delta) = \left\{ (x, y) \in X \times X : \limsup_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) > \delta \right. \\ \left. \text{and } \liminf_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) = 0 \right\}$$

Definition 3. 1) A dynamic system (X, f) (or the map $f : X \rightarrow X$) is Kato chaotic if it is sensitive and accessible (see [20]).

2) A dynamic system (X, f) (or the map $f : X \rightarrow X$) is chaotic in the sense of Ruelle and Takens (short for R-T chaotic) if it is transitive and sensitive (see [21]).

Proposition 1. A dynamic system (X, f) (or the map $f : X \rightarrow X$) is Li-Yorke sensitive if and only if $\overline{P_\delta(f)} = X$ for some $\delta > 0$. Among them,

$$P_\delta(f) = \{x \in X : \forall \varepsilon > 0, \exists y \in B(x, \varepsilon), \exists n \in \mathbb{N} \text{ such that } \rho(f^n(x), f^n(y)) > \delta\}$$

Proposition 2. [17] A dynamical system (X, f) is infinitely sensitive if and only if it is sensitive.

Proposition 3. [22] A dynamical system (X, f) is dense Li-Yorke sensitivity, then it is Topological mixing (or its topological entropy is positive).

3. Main Results

In this section, let $X = I$. The metric ρ in I is defined by $\rho(a, b) = |a - b| (\forall a, b \in I)$. The metric d in I_∞^∞ is defined by (2).

Theorem 1. *If f is transitive, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is transitive.*

Proof. Since f is transitive, then there exist $a \in I$ satisfying $\overline{Orb_f(a)} = I$. Then for any $b \in I$ and any $\varepsilon > 0$, $B(b, \varepsilon) \cap Orb_f(a) \neq \emptyset$. That is, there exists a $k_0 > 0$ such that $\rho(f^{k_0}(a), b) = |f^{k_0}(a) - b| < \varepsilon$. Take

$x_0 = (\dots, a, a, a, \dots) \in \Delta_\infty^\infty$. It is easy to see, for any $k \in \mathbb{N}$,

$F^k(x_0) = \{f^k(a)\}_{n=-\infty}^\infty$. Then,

$Orb_F(x_0) = \{f^k(a)\}_{n=-\infty}^\infty \Big|_{k \in \mathbb{N}}$. For any $y = (\dots, b, b, b, \dots) \in \Delta_\infty^\infty$ and above $k_0 > 0$,

$$d(F^{k_0}(x_0), y) = \sup\{|f^{k_0}(a) - b| : n \in \mathbb{N}\} = |f^{k_0}(a) - b| < \varepsilon.$$

So, $B(y, \varepsilon) \cap Orb_F(x_0) \neq \emptyset$.

Thus, the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is transitive.

Theorem 2. *If f is sensitive, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is sensitive.*

Proof. Take $\Delta_\infty^\infty = \{(\dots, x_{-1}, x_0, x_0, \dots) : x_n = a \in I, n \in \mathbb{Z}\} \subset I_\infty^\infty$,

$\forall x = \{\dots, a, a, a, \dots\}$, $y = \{\dots, b, b, b, \dots\} \in \Delta_\infty^\infty$, $x \neq y$. It is easy to know that, for $\forall k \in \mathbb{N}$,

$$F^k(x) = \{f^k(a)\}_{n=-\infty}^\infty, F^k(y) = \{f^k(b)\}_{n=-\infty}^\infty.$$

So, for $\forall k \in \mathbb{N}$,

$$\begin{aligned} d(F^k(x), F^k(y)) &= d(\{f^k(a)\}_{n=-\infty}^\infty, \{f^k(b)\}_{n=-\infty}^\infty) \\ &= \sup\{|f^k(a) - f^k(b)|, k = \dots, -1, 0, 1, \dots\} \\ &= |f^k(a) - f^k(b)|. \end{aligned}$$

Since f is Sensitive, so there exists a $\varepsilon_0 > 0$ such that for any $p \in I$ and any $\delta > 0$, there exists a $q_{p,\delta} \in B(p, \delta)$ and $n_{p,\delta} \in \mathbb{N}$ such that $|f^{n_{p,\delta}}(p) - f^{n_{p,\delta}}(q_{p,\delta})| > \varepsilon_0$. So for any fixed $x = (\dots, p, p, p, \dots) \in \Delta_\infty^\infty$ and any $\delta > 0$, taking $y = (\dots, q_{p,\delta}, q_{p,\delta}, q_{p,\delta}, \dots) \in \Delta_\infty^\infty$, one has that,

$$d(x, y) = \sup\{\dots, |p - q_{p,\delta}|, |p - q_{p,\delta}|, |p - q_{p,\delta}|, \dots\} = |p - q_{p,\delta}| < \delta,$$

that is $y \in B(x, \delta)$. And because

$$d(F^{n_{p,\delta}}(x), F^{n_{p,\delta}}(y)) = |f^{n_{p,\delta}}(p) - f^{n_{p,\delta}}(q_{p,\delta})| > \varepsilon_0,$$

so $F|_{\Delta_\infty^\infty}$ is sensitive.

Corollary 1. *If f is chaotic in the sense of Ruelle and Takens, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is chaotic in the sense of Ruelle and Takens.*

Proof. According to Theorem 1, Theorem 2 and the definition of R-T chaos,

the conclusion is obvious.

According to Proposition 2 and Theorem 2, the following Corollary is hold.

Corollary 2. *If f is infinitely sensitive, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is sensitive.*

In fact, there is a stronger conclusion.

Theorem 3. *If f is infinitely sensitive, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is infinitely sensitive.*

Proof. Since f is infinitely sensitive, then there exists a $\delta > 0$ such that for any $a \in I$ and any ε , there exists $b_{a,\varepsilon} \in B(a, \varepsilon)$ and $n_{a,\varepsilon} \in \mathbb{N}$ such that $\limsup_{n_{a,\varepsilon} \rightarrow \infty} \rho(f^{n_{a,\varepsilon}}(a), f^{n_{a,\varepsilon}}(b_{a,\varepsilon})) \geq \delta$. So for any fixed $x = (\dots, a, a, a, \dots) \in \Delta_\infty^\infty$, and any $\varepsilon > 0$, taking $x = (\dots, b_{a,\varepsilon}, b_{a,\varepsilon}, b_{a,\varepsilon}, \dots) \in \Delta_\infty^\infty$, one has that

$$d(x, y) = \sup\{\dots, |a - b_{a,\varepsilon}|, |a - b_{a,\varepsilon}|, |a - b_{a,\varepsilon}|, \dots\} = |a - b_{a,\varepsilon}| < \varepsilon,$$

that is $y \in B(x, \varepsilon)$. And because

$$\limsup_{n_{a,\varepsilon} \rightarrow \infty} d(F^{n_{a,\varepsilon}}(x), F^{n_{a,\varepsilon}}(y)) = \limsup_{n_{a,\varepsilon} \rightarrow \infty} d(f^{n_{a,\varepsilon}}(a), f^{n_{a,\varepsilon}}(b_{a,\varepsilon})) \geq \delta.$$

So $F|_{\Delta_\infty^\infty}$ is infinitely sensitive.

Theorem 4. *If f is accessible, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is accessible.*

Proof. For any open subset

$$(\Delta_1)_\infty^\infty = \{(\dots, x_{-1}, x_0, x_1, \dots), x_n = a \in U_1 \subset I, n \in \mathbb{Z}\} \subset I_\infty^\infty$$

and

$$(\Delta_2)_\infty^\infty = \{(\dots, y_{-1}, y_0, y_1, \dots), y_n = b \in U_2 \subset I, n \in \mathbb{Z}\} \subset I_\infty^\infty,$$

since f is accessible, then, for the above $U_1, U_2 \subset I$, there exist $a \in U_1, b \in U_2$ such that

$$\rho(f^k(a), f^k(b)) = |f^k(a) - f^k(b)| < \varepsilon$$

for some $k > 0$. Take

$$x = (\dots, a, a, a, \dots) \in (\Delta_1)_\infty^\infty, y = (\dots, b, b, b, \dots) \in (\Delta_2)_\infty^\infty,$$

then

$$d(F^k(x), F^k(y)) = |f^k(a) - f^k(b)| < \varepsilon.$$

So, the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is accessible.

Corollary 3. *If f is Kato chaotic, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is Kato chaotic.*

Proof. According to Theorem 2 and Theorem 4, the conclusion is obvious.

Theorem 5. *If f is exact, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is exact.*

Proof. Since f is exact, for any open subset $D \subset I$, there exist $m \in \mathbb{N}$ such that $f^m(D) = I$. That is, for any $a \in D$, there exists an $m > 0$ such that

$B(f^m(a), \varepsilon) \cap I \neq \emptyset$ for any $\varepsilon > 0$. So there is a $b \in X$ such that

$$\rho(f^m(a), \varepsilon) = |f^m(a) - b| < \varepsilon.$$

Take $(\Delta^*)_\infty^\infty$ is arbitrary open subset of Δ_∞^∞ , and $x_0 = (\dots, a, a, a, \dots) \in (\Delta^*)_\infty^\infty$. Clearly, for any $k \in \mathbb{N}$, $F^k(x_0) = \{f^k(a)\}_{n=-\infty}^\infty$. For any $y_0 = (\dots, b, b, b, \dots) \in \Delta_\infty^\infty$, $d(F^m(x_0), y_0) = |f^m(a) - b| < \varepsilon$. That is to say, there exist an $m \in \mathbb{N}$, $F^m((\Delta^*)_\infty^\infty) = \Delta_\infty^\infty$. So, the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is exact.

In [23] we had proved that, f is Li-Yorke sensitive implies that the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is Li-Yorke sensitive. Inspired by this, the following conclusion can be drawing.

Theorem 6. *If f is densely Li-Yorke sensitive, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is densely Li-Yorke sensitive.*

Proof. Since f is densely Li-Yorke sensitive, then for any $a \in Q_\delta(f)$ and any $\varepsilon > 0$. Then there exists a $b \in B(a, \varepsilon)$ such that $(a, b) \in LY_\rho(f, \delta)$. Take $x^* = \{x_n = a\}_{n=-\infty}^\infty$, $y^* = \{y_n = b\}_{n=-\infty}^\infty$. One has that

$$\limsup_{n \rightarrow \infty} d(F^n(x^*), F^n(y^*)) = \limsup_{n \rightarrow \infty} d(f^n(a), f^n(b)) > \delta$$

and

$$\liminf_{n \rightarrow \infty} d(F^n(x^*), F^n(y^*)) = \liminf_{n \rightarrow \infty} d(f^n(a), f^n(b)) = 0$$

Thus there is an $x^* \in Q_\delta(F)$.

Any fixed $x \in \Delta_\infty^\infty$, write $x = (\dots, x_{m-1}, x_{m,0}, x_{m,1}, \dots)$, where $x_{m,p} = x_{m,p+1}$, $p \in \mathbb{Z}$. Because $f: I \mapsto I$ is densely Li-Yorke sensitive, then for any $\varepsilon > 0$ and the above $x_{m,0}$, $B(x_{m,0}, \varepsilon) \cap Q_\delta(f) \neq \emptyset$. Take $a \in B(x_{m,0}, \varepsilon) \cap Q_\delta(f)$, then

$$d(x, x^*) = \sup\{|x_{m,p} - a|\} = |x_{m,p} - a| < \varepsilon.$$

So $x^* \in B(x, \varepsilon)$. This suggests that $\overline{Q_\delta(F)} = \Delta_\infty^\infty$.

So, the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is densely Li-Yorke sensitive.

According to Proposition 3 and Theorem 6 the following is right.

Corollary 4. *If f is dense Li-Yorke sensitivity, then the system $(\Delta_\infty^\infty, d, F|_{\Delta_\infty^\infty})$ is Topological mixing (or its topological entropy is positive).*

4. Conclusion

Inspired by the literature [23], this paper further studies the chaoticity of coupled map lattices. Some sufficient conditions of sensitivity, accessibility and transitivity are obtained. However, the study of coupled map lattices is still a hot topic. Based on the conclusions of this paper and others, one can consider some questions, such as the form of CMLs, the measurement of CMLs, and discuss the chaos of CMLs in other systems, which are worthy of studying.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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