Journal of Applied Mathematics and Physics, 2020, 8, 927-932
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Fuzzy BCK-Algebras

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How to cite this paper: Ahmed, M.A. and Amhed, E.A. (2020) Fuzzy BCK-Algebras. *Journal of Applied Mathematics and Physics*, **8**, 927-932.
https://doi.org/10.4236/jamp.2020.85071

Received: January 29, 2020
Accepted: May 17, 2020
Published: May 20, 2020

Abstract

In handing information regarding various aspects of uncertainty, non-classical-mathematics (fuzzy mathematics or great extension and development of classical mathematics) is considered to be a more powerful technique than classical mathematics. The non-classical mathematics, therefore, has now days become a useful tool in applications mathematics and computer science. The purpose of this paper is to apply the concept of the fuzzy sets to some algebraic structures such as an ideal, upper semilattice, lower semilattice, lattice and sub-algebra and gives some properties of these algebraic structures by using the concept of fuzzy sets. Finally, related properties are investigated in fuzzy BCK-algebras.

Keywords

BCK-Algebras, Fuzzy Set, Ideal, Lower Semilattice, Upper Semilattice, Semilattice, Level Subset, Lattice and Lattice Filter

1. Introduction

In 1991, O. G. Xi [1] applied the concept of fuzzy sets to BCK-algebras. After that, concept of fuzzy sets to BCK-algebras has been considered by a number of authors, amongst them, Y. B. Jun [2]. E. Y. Deeba (1980) [3] introduced the concept of lattice filters. Y. B. Jun, S. M. Hong and J. Meng (1998) [4] introduced the concept of fuzzy BCK-Filter. For the general development of BCK-algebras, the filter theory plays an important role as well as ideal theory. The purpose of this paper is to give the fuzzification of BCK-algebras and we study relationships between BCK-filters and lattice filters in BCK-algebras.

2. Preliminaries

(For more details of BCK/BCI-algebras we refer to [5] [6] and [7].)

An algebra $(X;*,0)$ of type (2, 0) is said to be a BCK-algebra if it satisfies the following

(I) $((x*y)*(x*z))*(z*y)=0$

(II) $(x*(x*y))*y=0$

(III) $x*x=0$

(IV) $0*x=0$

(V) $x*y=0$ and $y*x=0$ imply $x=y$

A BCK-algebras can be (partially) ordered by $x \le y$ if and only if $x*y=0$. The following statements are true in any BCK-algebras: for all x,y,z,

(p_1) $(x;\le)$ is a partially ordered set.

(p_2) $x*0=x$.

(p_3) $(x*y)*z=(x*z)*y$.

(p_4) $x*(x*(x*y))=x*y$.

(p_5) $x \le y$ implies $x*z \le y*z$ and $z*y \le z*x$.

BCK-algebra *X* satisfying the identity $x \wedge y = y \wedge x$ where $x \wedge y = y*(y*x)$ for all $x,y \in X$ is said to be commutative. If there is an element ℓ of BCK-algebra *X* satisfying $x \le \ell$ for all *x* in *X*, the element ℓ is called the unit of *X*. A BCK-algebra with the unit is called to be bounded. In a bounded BCK-algebra, we denote $\ell * x$ by N_x for every $x \in X$. In bounded commutative BCK-algebra denote $x \vee y = N_{(N_x \wedge N_y)}$.

(p_6) In bounded BCK-algebra we have $N_x * N_y \le y*x$) [8].

3. Fuzzy BCK-Algebra

Definition 3.1. Zadeh [9] A fuzzy subset of a BCK-algebra *X* is a function $\mu : X \to [0,1]$.

Definition 3.2. [10] Let *X* be a BCK-algebra. A fuzzy set μ in *X* is called a fuzzy subalgebra of *X* if, for all $x,y \in X$,

$$\mu(x*y) \ge \min\{\mu(x),\mu(y)\}$$

Definition 3.3. [11] Let *X* be a BCK-algebra and let *I* be a non empty subset of *X*. Then *I* it is called to be an ideal of *X* if, for all x,y in *X*.

(a) $0 \in I$

(b) $x*y \in I$ and $y \in I$ imply $x \in I$.

Definition 3.4. [12]. Let μ be a fuzzy set of *X*. For $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \ge t\}$ is called a level subset of μ.

Theorem 3.5. Let *X* be a BCK-algebra and let μ be an arbitrary fuzzy subalgebra of *X*, then for every $t \in [0,1]$, μ_t is an ideal of *X*, when $\mu_t \ne \phi$.

Proof. For any element $x \in X$ we have

$$\mu(0)=\mu(x*x) \ge \min\{\mu(x),\mu(x)\}=\mu(x)=t, \text{ where } t \in [0,1],$$

so that $0 \in \mu_t$. Let $x*y \in \mu_t$ and $y \in \mu_t$, and for any x,y in *X*, denote

$$t=\min\{\mu(x),\mu(y)\}.$$

Then by the hypothesis μ is a fuzzy subalgebra of *X*, we have

$$\mu(x*y) \geq \min\{\mu(x), \mu(y)\} = t.$$

If $\min\{\mu(x), \mu(y)\} = \mu(y) = t$, then $\mu(x) \geq t$ so that $x \in \mu_t$. Or $\min\{\mu(x), \mu(y)\} = \mu(x) = t$ which implies $x \in \mu_t$. Hence μ_t is an ideal of *X*.

Definition 3.6. [13] A partially ordered set $\langle X, \leq \rangle$ is said to be a lower semilattice if every pair of elements in *X* has a greatest lower bound (meet ($\wedge$)); it is called to be an upper semilattice if every pair of elements in *X* has least upper bound (Join ($\vee$)). If $\langle X, \leq \rangle$ is both an upper and a lower semilattice, then it is called a lattice.

Definition 3.7. Let *X* be a bounded BCK-algebra, μ be a fuzzy set in *X*. Then μ is called a fuzzy BCK-filter of *X* if, for all $x, y \in X$

(FF_1) $\mu(\ell) \geq \mu(x)$

(FF_2) $\mu(x) \geq \min\left\{\mu\left(N_{(N_x * N_y)}\right), \mu(y)\right\}$

Theorem 3.8. A BCK-algebra $(X;*,0)$ is commutative if and only if it is lower semilattice with respect to BCK-order $\leq$.

Theorem 3.9. Let μ be a fuzzy BCK-filter of bounded commutative BCK-algebra. Then for every $t \in [0,1]$, μ_t is an upper semilattice with respect to BCK-order $\leq$, when $\mu_t \neq \phi$.

Proof. For bounded commutative BCK-algebra $x = N_{N_x}$ for all $x \in X$ (by Y.B.Jun) and $N_x \wedge N_y \leq N_x, N_y$ (by Theorem 3.7), then we have
$x = N_{N_x} \leq N_{(N_x \wedge N_y)} = x \vee y$ and $y = N_{N_y} \leq N_{(N_x \wedge N_y)} = x \vee y$ for all $x, y \in \mu_t$.

This shows that $x \vee y$ is a common upper-bounded of *x* and *y*. Next if $x, y \leq z$, then $N_z \leq N_x, N_y$. (by Theorem 3.7) we obtain
$N_z \leq N_x \wedge N_y, N_{(N_x \wedge N_y)} \leq N_{N_z}$ thus $x \vee y \leq z$. Hence $x \vee y$ is a least upper bound of *x* and *y*, so it remains to prove $x \vee y \in \mu_t$, from (p_6) we have $N_{(x \vee y)} * N_y \leq y*(x \vee y) = 0$, and so $N_{(x \vee y)} * N_y = 0$.

Hence $\mu(x \vee y) \geq \min\left\{\mu\left(N_{\left(N_{(x \vee y)}\right)} * N_y\right), \mu(y)\right\} = \min\{\mu(N_0), \mu(y)\} = \mu(y)$,

so that $x \vee y \in \mu_t$. Hence $(\mu_t, \leq)$ is upper semilattice. This completes the proof. □

Theorem 3.10. Let μ be a fuzzy BCK-filter of a bounded commutative BCK-algebra *X*. If $y \leq x$ for all $x, y \in X$, then for every $t \in [0,1]$, μ_t is a lower semilattice with respect to BCK-order $\leq$, when $\mu_t \neq \phi$.

Proof. Let $x, y \in \mu_t$. By (II) we know that $(x \wedge y) \leq x, y$. Let *z* be any element of *X* such that $z \leq x, y$. Then $z*x = z*y = 0$, so

$$z = z*0 = z*(z*x) = x*(x*z).$$

By the same reason, we have $z = y*(y*z)$, hence

$$z = x*(x*z) = x*\big(x*(y*(y*z))\big) \leq x*(x*y) = x \wedge y.$$

This says that for all x, y in μ_t, $x \wedge y$ it is the greatest lower bound, so it is remain to prove $x \wedge y \in \mu_t$. Then from (p_6) we have
$N_{(x \wedge y)} * N_y \leq y*(x \wedge y) = y*(y*(y*x)) = y*(y*0) = y*y = 0$, and so

$N_{(x\wedge y)} * N_y = 0$.

$$\mu(x\wedge y) \geq \min\left\{\mu\left(N_{N_{(x\wedge y)}} * N_y\right), \mu(y)\right\}$$

Hence
$$= \min\{\mu(N_0), \mu(y)\}$$
$$= \min\{\mu(\ell), \mu(y)\} = \mu,$$

so that $x \wedge y \in \mu_t(y)$. Hence $(\mu, \leq)$ is lower semilattice. This completes the proof. □

Theorem 3.11. Let μ be a fuzzy BCK-filter of a bounded commutative BCK-algebra *X*. If $y \leq x$ for all $x, y \in X$, then for every $t \in [0,1]$, μ_t a lattice with respect to BCK-order $\leq$, when $\mu_t \neq \phi$.

Proof. Since μ fuzzy BCK-filter of a bounded commutative BCK-algebra *X*. and $y \leq x$ for all $x, y \in X$, then for every $t \in [0,1]$, μ_t is lower semilattice with respect to BCK-order $\leq$, (from Theorem 3.10). also μ_t is upper semilattice with respect to BCK-order $\leq$, (from Theorem 3.9) combining with Definition 3.6, thus μ_t is a lattice with respect to BCK-order $\leq$.

Definition 3.12. [14] A non-empty subset *F* of a BCK-algebra *X* is said to be a lattice filter if it satisfies that (D_1) $x \in F$ and $x \leq y$ implies $y \in F$; (D_2) $x, y \in F$ g.L.b. $\{x, y\} \in F$.

Theorem 3.13. Let *X* be a bounded commutative BCK-algebra, μ be a fuzzy BCK-filter of *X*, then for every $t \in [0,1]$, μ_t is a lattice filter of *X*, when $\mu_t \neq \phi$ and $y \leq x$ for all x, y in *X*.

Proof. Assume μ is a fuzzy BCK-filter of *X*, then for all $t \in [0,1]$, let $x, y \in X$ be such that $y \in \mu_t$ and $y \leq x$. Then $N_x * N_y \leq y * x = 0$ and so $N_x * N_y = 0$. Hence

$$\mu(x) \geq \min\left\{\mu\left(N_{(N_x * N_y)}\right), \mu(y)\right\}$$
$$= \min\{\mu(N_0), \mu(y)\}$$
$$= \min\{\mu(\ell), \mu(y)\}$$
$$= \mu(y)$$

So that $x \in \mu_t$. This shows that μ_t satisfies (D_1). The proof of (D'_2) is similar to the proof of Theorem 3.10 and omitted. Hence μ_t is a lattice filter of *X*. This completes the proof.

Theorem 3.14. Let *X* be a BCK-algebra and μ be an arbitrary fuzzy subalgebra of *X*. Then for every *a*,*x* and *y* in *X* and for every $t \in [0,1]$, the following hold when $\mu_t \neq \phi$:

(a) $a * \left(a * \left(\left(\cdots\left((a * x) * 0\right) * \cdots\right) * 0\right)\right) \in \mu_t$

(b) if *X* it is positive implicative, then $\left(x *^n y\right) \in \mu_t$ where $x *^n y$ is recursively defined as follows $x *^1 y = x * y, x *^{n+1} y = \left(x *^n y\right) * y$ for any $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of all the natural numbers).

(c) $(x * y) * x \in \mu_t$

(d) if *X* is bounded commutative, then $N_x * N_y \in \mu_t$

(e) if *X* is positive implicative, then $(x * y) * y \in \mu_t$

(f) if X is implicative, then $(x*y)*(y*(x*y)) \in \mu_t$

(g) if X is commutative, them $x*(y*(y*x)) \in \mu_t$

Proof. For any x, y in X denote $t = \min\{\mu(x), \mu(y)\}$

$$\mu\Big(a*\Big(a*\big(\big(\cdots\big((a*x)*0\big)*\cdots\big)*0\big)\Big)\Big)$$
$$= \mu(a*x) \geq \min\{\mu(a), \mu(x)\} = t$$

So $a*\Big(a*\big(\big(\cdots\big((a*x)*0\big)*\cdots\big)*0\big)\Big) \in \mu_t$. Hence (a) holds. In any positive implicative BCK-algebra, the following identity hold $x*^n y = x*y$ for any $n \in \mathbb{N}$ thus $\mu(x*^n y) = \mu(x*y) \geq \min\{\mu(x), \mu(y)\} = t$ so $x*^n y \in \mu_t$. Hence (b) holds.

$$\mu\big((x*y)*x\big) = \mu\big((x*x)*y\big) = \mu(0*x) = \mu(0)$$
$$= \mu(x*x) \geq \min\{\mu(x), \mu(x)\} = t,$$

so $(x*y)*x \in \mu_t$ and (c) holds.

$$\mu(N_x * N_y) = \mu\big((\ell*x)*(\ell*y)\big) = \mu\big((\ell*(\ell*y))*x\big)$$
$$= \mu\big(y*(y*\ell)*x\big) = \mu\big((y*0)*x\big)$$
$$= \mu(y*x) \geq \min\{\mu(x), \mu(y)\} = t,$$

thus $\mu(N_x * N_y) \geq t.$

Which implies $N_x * N_y \in \mu_t$, proving (d).

(e) Since X is a positive implicative BCK-algebra, it follows that

$$\mu\big((x*y)*y\big) = \mu\big((x*y)*(y*y)\big) = \mu\big((x*y)*0\big)$$
$$= \mu(x*y) \geq \min\{\mu(x), \mu(y)\} = t,$$

thus $(x*y)*y \in \mu_t$. Hence proving (e).

(f) Since X is an implicative BCK-algebra, it follows that

$$\mu\big((x*y)*(y*(x*y))\big) = \mu(x*y) \geq \min\{\mu(x), \mu(y)\} = t,$$

thus $\mu\big((x*y)*(y*(x*y))\big) \geq t$, so $(x*y)*(y*(x*y)) \in \mu_t$.

Hence (f) holds.

(g) Form commutative of X and it follows that

$$\mu\big(x*(y*(y*x))\big) = \mu\big(x*(x*(x*y))\big) = \mu(x*y)$$
$$\geq \min\{\mu(x), \mu(y)\} = t.$$

Thus $\mu\big(x*(y*(y*x))\big) \geq t$ so $x*(y*(y*x)) \in \mu_t$.

Hence (g) holds. This finishes the proof.

4. Conclusions

1) A level subset of a fuzzy sub-algebra of BCK-algebras X is an ideal X.

2) A level subset of the fuzzy BCK-filter of bounded commutative BCK-algebra X is an upper semilattice with respect to BCK-order $\leq$.

3) A level subset of a fuzzy BCK-filter of abounds commutative BCK algebras X is a lower semilattice with respect to BCK-order $\leq$ and If $y \leq x$ for all

$x, y \in X$.

4) A level subset of a fuzzy BCK-filter of abounds commutative BCK algebras X is a lattice with respect to BCK-order ≤.

5) A level subset of a fuzzy BCK-filter of a bounded commutative BCK-algebra X is a lattice filter with respect to BCK-order ≤ and if $y \le x$ for all $x, y \in X$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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