Journal of Applied Mathematics and Physics, 2020, 8, 652-659
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Integral Basis of Affine Vertex Algebra $V_k(\mathfrak{sl}_2)$ and Virasoro Vertex Algebra $V_{Vir}(2k,0)$

Ashuai Wang

School of Mathematics, South China University of Technology, Guangzhou, China
Email: 738274395@qq.com

How to cite this paper: Wang, A. (2020) Integral Basis of Affine Vertex Algebra $V_k(\mathfrak{sl}_2)$ and Virasoro Vertex Algebra $V_{Vir}(2k,0)$. *Journal of Applied Mathematics and Physics*, **8**, 652-659.
https://doi.org/10.4236/jamp.2020.84050

Received: March 5, 2020
Accepted: March 31, 2020
Published: April 3, 2020

Abstract

In this paper, we consider an integral basis for affine vertex algebra $V_k(\mathfrak{sl}_2)$ when the level k is integral by a direct calculation, then use the similar way to analyze an integral basis for Virasoro vertex algebra $V_{Vir}(2k,0)$. Finally, we take the combination of affine algebras and Virasoro Lie algebras into consideration. By analogy with the construction of Lie algebras over $\mathbb{Z}$ using Chevalley bases, we utilize the $\mathbb{Z}$-basis of L_{av} whose structure constants are integral to find an integral basis for the universal enveloping algebra of it.

Keywords

Vertex algebra, Integral Basis, Virasoro Algebra, Affine Algebra

1. Introduction

While vertex algebras are usually assumed to be vector spaces over $\mathbb{C}$, the most important formula Jacobi identity makes sense over any commutative ring, so it is natural to consider vertex algebras over $\mathbb{Z}$. An integral basis of a vertex algebra could be considered an analogue of the Chevalley basis in a Lie algebra. Similar to the construction of Lie algebras over $\mathbb{Z}$ using Chevalley bases, we can create vertex algebras over $\mathbb{Z}$. Integral bases for vertex operator algebras associated with lattices have been studied in [1] [2]. In this paper, we are going to investigate integral basis for affine vertex algebras and Virasoro vertex algebra. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\hat{\mathfrak{g}}$ be the corresponding affine Kac-Moody algebra. The vacuum module $V_{\hat{\mathfrak{g}}}(k,0)$ at level k has a vertex algebra structure [3] [4] [5], we call it affine vertex algebra. We want to find an integral basis for it when $\mathfrak{g}=\mathfrak{sl}_2$ and k is an integer in Section 3.

Next, we consider the Virasoro vertex algebra. Among the most important vertex algebras are those associated with the Virasoro Lie algebra. It has been studied in [6] [7] [8]. They show that it is generated by the conformal vector ω and is minimal in the sense that it does not have any proper vertex operator subalgebra. Besides, any minimal vertex operator algebras of the same central charge are isomorphic. The study of Virasoro vertex algebras is the algebraic foundation of the study of the "minimal modules" in conformal field theory [9]. We use the similar way to analyze an integral basis for Virasoso vertex algebra $V_{Vir}(2k,0)$ when the level is 2*k*, where k is an integer.

We know that affine Lie algebra and Virasoro Lie algebra have close relationship in physics, so we consider them simultaneously, *i.e.*, as one algebraic structure. Then the definition of affine-Virasoro was introduced [10] [11], which is the semidirect product of the Virasoro algebra and an affine Kac-Moody Lie algebra with a common center. In the last section, we get an integral basis for the universal enveloping algebra of it.

In this paper, we observe that the $\mathbb{C}$-basis of affine vertex algebra $V_k(\mathfrak{sl}_2)$ and Virasoro vertex algebra $V_{Vir}(2k,0)$ may be integral basis for them in certain conditions. We create the conditions and confirm that they are exactly the integral bases. Then we utilize the analogue of Chevalley bases for finite dimensional Lie algebras to get an integral basis for the universal enveloping algebra of affine-Virasoro algebra.

2. Preliminaries

We assume that the readers are familiar with the theory of vertex operator algebras [3] [6] [12] [13].

Given an (untwisted) affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket relation,

$$\left[a \otimes t^m, b \otimes t^n\right] = [a,b] \otimes t^{m+n} + m\langle a,b\rangle \delta_{m+n,0}\mathbf{k}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, together with the condition that $\mathbf{k}$ is a nonzero central element of $\hat{\mathfrak{g}}$. Let $k \in \mathbb{C}$, $\hat{\mathfrak{g}}_{(-)}$ and $\mathfrak{g}$ act trivially on $\mathbb{C}$ and let $\mathbf{k}$ act as the scalar *k*, making $\mathbb{C}$ a $\hat{\mathfrak{g}}_{(\le 0)}$-module, which we denote by $V_{\hat{\mathfrak{g}}}(k,0) = U(\hat{\mathfrak{g}}) \otimes_{U\left(\hat{\mathfrak{g}}_{(\le 0)}\right)} \mathbb{C}_k$.

By the Poincaré-Birkhoff-Witt theorem, we have that,

$$V_{\hat{\mathfrak{g}}}(k,0) = U\left(\hat{\mathfrak{g}}_{(+)}\right) \simeq S\left(\hat{\mathfrak{g}}_{(+)}\right) \tag{2.1}$$

as a $\mathbb{Z}$-graded vector space. Set

$$\mathbf{1} = 1 \in \mathbb{C} \subset V_{\hat{\mathfrak{g}}}(k,0).$$

Then

$$V_{\hat{\mathfrak{g}}}(k,0) = \coprod_{n \ge 0} V_{\hat{\mathfrak{g}}}(k,0)_{(n)}.$$

$V_{\hat{\mathfrak{g}}}(k,0)_{(n)}$ is spanned by the vectors

$$a^{(1)}(-m_1)\cdots a^{(r)}(-m_r)\mathbf{1}$$

for $r \geq 0, a^{(i)} \in \mathfrak{g}, m_i \geq 1$, with $n = m_1 + \cdots + m_r$. It can be proved that this is a vertex algebra, more detail can be found in [5]. The vertex algebra structure is determined by

$$Y(a(-1)\mathbf{1}, x) = a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}.$$

Now we have that,

$$\begin{aligned} &Y\left(a^{(1)}(-m_1)\cdots a^{(r)}(-m_r)\mathbf{1}, z\right) \\ &= \frac{1}{(m_1-1)!}\cdots\frac{1}{(m_r-1)!}:\partial_z^{m_1-1}a^{(1)}(z)\cdots\partial_z^{m_r-1}a^{(r)}(z): \end{aligned}$$

where $r \geq 0, m_r \geq 1$ for $i = 1, \cdots, r$.

In the next section, we will consider the integral basis of it when $\mathfrak{g} = \mathfrak{sl}_2$ and k is an integer.

3. Integral Basis of $V_k(\mathfrak{sl}_2)$

In this section shall find an integral basis for $V_k(\mathfrak{sl}_2)$. We consider the case when k is an integral number. Firstly, we recall the definition of integral basis of a vertex algebra.

Definition 3.1. Suppose that V is a vertex algebra (over $\mathbb{C}$), an integral basis of it is its $\mathbb{C}$-basis whose $\mathbb{Z}$-span can form a vertex algebra over $\mathbb{Z}$.

In order to find an integral basis for $V_k(\mathfrak{sl}_2)$, we may firstly find an integral basis for $\mathfrak{g}$. That is, we need to find a basis of $\mathfrak{g}$ whose $\mathbb{Z}$-span $\mathfrak{g}_{\mathbb{Z}}$ is closed under the bracket. When $\mathfrak{g} = \mathfrak{sl}_2$, it is easy to see that the standard basis elements x, y, h satisfy the condition. Now let x, y, h be the standard basis of $\mathfrak{sl}_2$, we choose an ordered basis of $\widehat{\mathfrak{sl}}_{2(+)}$, that is

$$y_{n_1}y_{n_2}\cdots y_{n_k}h_{m_1}h_{m_2}\cdots h_{m_s}x_{l_1}x_{l_2}\cdots x_{l_t},$$

where $n_p, m_q, l_r (\in \mathbb{Z}) \leq -1, p = 1, 2, \cdots, k; q = 1, 2, \cdots, s; r = 1, 2, \cdots, t$. Then by (2.1), we get that

$$y_{n_1}y_{n_2}\cdots y_{n_k}h_{m_1}h_{m_2}\cdots h_{m_s}x_{l_1}x_{l_2}\cdots x_{l_t}\mathbf{1} \tag{3.1}$$

is a $\mathbb{C}$-basis of $V_k(\mathfrak{sl}_2)$.

For convenience, we denote $V_k(\mathfrak{sl}_2)$ by V, denote the $\mathbb{Z}$-span of a $\mathbb{C}$-basis of V by $V_{\mathbb{Z}}$.

Theorem 3.2. The formula (3.1) is an integral basis of $V_k(\mathfrak{sl}_2)$.

Proof. It is known that formula (3.1) is a $\mathbb{C}$-basis of $V_k(\mathfrak{sl}_2)$. To check that $V_{\mathbb{Z}}$ is a vertex algebra, we need to prove that the coefficients are still in $V_{\mathbb{Z}}$ of any

$$Y(u, x)v \tag{3.2}$$

for $u, v \in V_{\mathbb{Z}}$. Since $V_{\mathbb{Z}}$ is spanned by (3.1) over $\mathbb{Z}$, we just need to check formula (3.2) for (3.1). So

$$
\begin{aligned}
&Y\left(y_{n_1}\cdots y_{n_k}h_{m_1}\cdots h_{m_s}x_{l_1}\cdots x_{l_t}\mathbf{1},z\right)y_{n'_1}\cdots y_{n'_p}h_{m'_1}\cdots h_{m'_q}x_{l'_1}\cdots x_{l'_r}\mathbf{1}\\
&=\frac{1}{(-n_1-1)!\cdots(-n_k-1)!(-m_1-1)!\cdots(-m_s-1)!(-l_1-1)!\cdots(-l_t-1)!}\\
&\quad :\partial_z^{-n_1-1}Y(y,z)\cdots\partial_z^{-n_k-1}Y(y,z)\partial_z^{-m_1-1}Y(h,z)\cdots\partial_z^{-m_s-1}Y(h,z)\\
&\quad \partial_z^{-l_1-1}Y(x,z)\cdots\partial_z^{-l_t-1}Y(x,z):y_{n'_1}\cdots y_{n'_p}h_{m'_1}\cdots h_{m'_q}x_{l'_1}\cdots x_{l'_r}\mathbf{1}\\
&=\sum_{\substack{a_1,\cdots,a_k\\ b_1,\cdots,b_s\\ c_1,\cdots,c_t\in\mathbb{Z}}}C_{-a_1-1}^{-n_1-1}\cdots C_{-a_k-1}^{-n_k-1}C_{-b_1-1}^{-m_1-1}\cdots C_{-b_s-1}^{-m_s-1}C_{-c_1-1}^{-l_1-1}\cdots C_{-c_t-1}^{-l_t-1}\\
&\quad :y_{a_1}\cdots y_{a_k}h_{b_1}\cdots h_{b_s}x_{c_1}\cdots x_{c_t}:y_{n'_1}\cdots y_{n'_p}h_{m'_1}\cdots h_{m'_q}x_{l'_1}\cdots x_{l'_r}\mathbf{1}\\
&\quad z^{n_1+\cdots+n_k+m_1+\cdots+m_s+l_1+\cdots+l_t-a_1-\cdots a_k-b_1-\cdots-b_s-c_1-\cdots-c_t}
\end{aligned}
$$

Since $[x,y]=h$, $[h,x]=2x$, $[h,y]=-2y$, $k\in\mathbb{Z}$, the expression

$$:y_{a_1}\cdots y_{a_k}h_{b_1}\cdots h_{b_s}x_{c_1}\cdots x_{c_t}:\mathbf{1},$$

where $a_1,\cdots,a_k,b_1,\cdots,b_s,c_1,\cdots,c_t\in\mathbb{Z}$, is a $\mathbb{Z}$-linear combination of (3.1), we get that (3.1) is an integral basis of $V_k(\mathfrak{sl}_2)$.

4. Integral Basis of Virasoro Vertex Algebra $V_{Vir}(2k,0)$

In this section we shall find an integral basis for the Virasoro vertex algebra $V_{Vir}(2k,0)$ when the level is 2*k*, where k is an integer.

Firstly we recall the definition of Virasoro vertex algebra [6] [14] [15]. As we know, any vertex operator algebra V has the vertex subalgebra $\langle\omega\rangle$ generated by the conformal vector ω, and this is in fact the smallest vertex operator subalgebra of V; it is exactly the submodule of V for the Virasoro algebra generated by $\mathbf{1}$:

$$\operatorname{span}\left\{L(n_1)\cdots L(n_r)\mathbf{1}\mid r\geq 0,n_j\in\mathbb{Z}\right\}. \tag{4.1}$$

The Virasoro algebra $\mathcal{L}$ is the Lie algebra with basis $\{L_m\mid m\in\mathbb{Z}\}\cup\{\boldsymbol{c}\}$ equipped with the bracket relations,

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{1}{12}(m^3-m)\delta_{m+n,0}\boldsymbol{c}, \tag{4.2}$$

together with the condition that $\boldsymbol{c}$ is a central element of $\mathcal{L}$. The Virasoro algebra $\mathcal{L}$ equipped with the $\mathbb{Z}$-grading

$$\mathcal{L}=\coprod_{n\in\mathbb{Z}}\mathcal{L}_{(n)}, \tag{4.3}$$

where

$$\mathcal{L}_{(0)}=\mathbb{C}L_0\oplus\mathbb{C}\boldsymbol{c}\ \text{ and }\ \mathcal{L}_{(n)}=\mathbb{C}L_{-n}\ \text{ for } n\neq 0, \tag{4.4}$$

is a $\mathbb{Z}$-graded Lie algebra, and this grading is given by $ad\,L_0$-eigenvalus. For the Virasoro algebra $\mathcal{L}$, we have the graded subalgebras

$$\mathcal{L}_{(\pm)}=\coprod_{n\geq 1}\mathcal{L}_{(\pm n)}=\coprod_{n\geq 1}\mathbb{C}L_{\mp n}. \tag{4.5}$$

We also have the graded subalgebras

$$\mathcal{L}_{(\le 1)} = \coprod_{n\le 1}\mathcal{L}_{(n)} = \mathcal{L}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}\boldsymbol{c} \oplus \mathbb{C}L_{-1},$$
$$\mathcal{L}_{(\ge 2)} = \coprod_{n\ge 2}\mathcal{L}_{(n)} = \coprod_{n\ge 2}\mathbb{C}L_{-n} \subset \mathcal{L}_{(+)}. \tag{4.6}$$

Let ℓ be any complex number. Consider $\mathbb{C}$ as an $\mathcal{L}_{(\le 1)}$-module with $\boldsymbol{c}$ acting as the scalar ℓ and with $\mathcal{L}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_{-1}$ acting trivially. Denote this $\mathcal{L}_{(\le 1)}$-module by $\mathbb{C}_\ell$. Then we form the induced module

$$V_{Vir}(\ell,0) = U(\mathcal{L}) \underset{U(\mathcal{L}_{(\le 1)})}{\otimes} \mathbb{C}_\ell. \tag{4.7}$$

From the Poincaré-Birkhoff-Witt theorem, as a vector space,

$$V_{Vir}(\ell,0) = U\left(\mathcal{L}_{(\ge 2)}\right) \simeq S\left(\mathcal{L}_{(\ge 2)}\right). \tag{4.8}$$

Set

$$\mathbf{1} = 1 \in \mathbb{C} \subset V_{Vir}(\ell,0).$$

Then

$$V_{Vir}(\ell,0) = \coprod_{n\ge 0} V_{Vir}(\ell,0)_{(n)}, \tag{4.9}$$

where $V_{Vir}(\ell,0)_{(n)}$ has a basis consisting of the vectors

$$L(-m_1)\cdots L(-m_r)\mathbf{1} \tag{4.10}$$

for $r \ge 0$, $m_1 \ge m_2 \ge \cdots \ge m_r \ge 2$, with $m_1 + \cdots + m_r = n$. It can be proved that this is a vertex operator algebra, details can be found in [3]. The vertex algebra structure is determined by

$$Y(L(-2)\mathbf{1}, z) = L(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

and we have

$$Y(L(-j_1)\cdots L(-j_m)\mathbf{1}, z) = \frac{1}{(j_1-2)!}\cdots\frac{1}{(j_m-2)!} : \partial_z^{j_1-2}L(z)\cdots\partial_z^{j_m-2}L(z) :$$

where $m \ge 0, j_1 \ge j_2 \ge \cdots \ge j_r \ge 2$.

Next, we will consider the integral basis of it when $\boldsymbol{c}$ acts as $2k$ and k is integral. We know that

$$L(-m_1)\cdots L(-m_r)\mathbf{1}$$

is a basis of $V_{Vir}(\ell,0)_{(n)}$. Then

$$L(-m_1)\cdots L(-m_r)\mathbf{1}$$

with $r \ge 0$, $m_1 \ge m_2 \ge \cdots \ge m_r \ge 2$, with $m_1 + \cdots + m_r = 1, 2, \cdots$, is a $\mathbb{C}$-basis of $V_{Vir}(2k,0)$. We want to check that it is a $\mathbb{Z}$-basis of $V_{Vir}(2k,0)$. When $\boldsymbol{c}$ acts as $2k$, we get that,

$$[L(m), L(n)]\mathbf{1} = (m-n)L(m+n)\mathbf{1} + \frac{1}{6}(m^3 - m)\delta_{m+n,0}k$$

for any $m, n \in \mathbb{Z}$.

Just like the affine case, we have that,

$$Y(L(-j_1)\cdots L(-j_m)\mathbf{1}, z) L(-k_1)\cdots L(-k_t)\mathbf{1}$$

$$= \frac{1}{(j_1-2)!} \cdots \frac{1}{(j_m-2)!} : \partial_z^{j_1-2} L(z) \cdots \partial_z^{j_m-2} L(z) : L(-k_1) \cdots L(-k_t) \mathbf{1}$$
$$= \sum_{n_1,\cdots,n_m \in \mathbb{Z}} C_{-n_1-2}^{j_1-2} \cdots C_{-n_m-2}^{j_m-2} : L(n_1) \cdots L(n_m) : L(-k_1) \cdots L(-k_t) \mathbf{1} z^{-n_1-\cdots-n_m+j_1+\cdots+j_m},$$

where $m,t \ge 0, j_1 \ge j_2 \ge \cdots \ge j_m \ge 2, k_1 \ge k_2 \ge \cdots \ge k_t \ge 2$. Since

$$[L(m), L(n)]\mathbf{1} = (m-n)L(m+n)\mathbf{1} + \frac{1}{6}(m^3-m)\delta_{m+n,0}k,$$

and

$$\frac{1}{6}(m^3-m) \in \mathbb{Z},$$

the expression

$$: L(n_1) \cdots L(n_m) : L(-k_1) \cdots L(-k_t) \mathbf{1}$$

is a $\mathbb{Z}$-linear combination of $L(-m_1)\cdots L(-m_r)\mathbf{1}$, where $r \ge 0$, $m_1 \ge m_2 \ge \cdots \ge m_r \ge 2$, with $m_1 + \cdots + m_r = 1, 2, \cdots$, so $L(-m_1)\cdots L(-m_r)\mathbf{1}$ is an integral basis of $V_{Vir}(2k, 0)$.

5. Integral Basis for the Universal Enveloping Algebra of Affine-Virasoro Algebra

In this section we take the combination of affine algebras and Virasoro Lie algebras into consideration. By analogy with the construction of Lie algebras over $\mathbb{Z}$ using Chevalley bases, we utilize the $\mathbb{Z}$-basis of it whose structure constants are integral to find an integral basis for the universal enveloping algebra of affine-Virasoro algebra $U(L_{av})$ when $L = \mathfrak{sl}_2$.

Firstly we recall the definition of the affine-Virasoro algebra [15].

Definition 5.1. Let L be a finite-dimensional Lie algebra with a non-degenerated invariant normalized symmetric bilinear form $(,)$, then the affine-Virasoro Lie algebra is the vector space

$$L_{av} = L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \oplus \bigoplus_{i \in Z} \mathbb{C}d_i$$

with Lie bracket:

$$\begin{aligned} &[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x, y)\delta_{m+n,0}C, \\ &[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C, \\ &[d_i, x \otimes t^m] = mx \otimes t^{m+i}, \ [C, L_{av}] = 0, \end{aligned} \tag{5.1}$$

where $x, y \in L, m, n, i, j \in \mathbb{Z}$.

Now we consider the case of $L = \mathfrak{sl}_2$. Then by Definition (5.1), we get that the corresponding affine-Virasoro algebra $L_{av} = \mathbb{C}\{e_i, f_i, h_i, d_i, C \mid i \in \mathbb{Z}\}$, with Lie bracket:

$$\begin{aligned} &[e_i, f_j] = h_{i+j} + i\delta_{i+j,0}C, \\ &[h_i, e_j] = 2e_{i+j}, \ [h_i, f_j] = -2f_{i+j}, \\ &[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C, \end{aligned}$$

$$\left[d_i, h_j\right] = jh_{i+j},\ \left[h_i, h_j\right] = 2i\delta_{i+j,0}C,$$
$$\left[d_i, e_j\right] = je_{i+j},\ \left[d_i, f_j\right] = jf_{i+j},\ \left[C, \mathcal{L}\right] = 0,$$

where $i, j \in \mathbb{Z}$.

Lemma 5.2. If $\{a_i \mid i \in \mathbb{Z}\}$ is a $\mathbb{Z}$-basis of $\mathfrak{g}$, then

$$\left\{a_{k_1} \cdots a_{k_m} \mid k_1 \leq \cdots \leq k_m \in \mathbb{Z}\right\}$$

is an integral basis of $U(\mathfrak{g})$.

Proof. Since $\{a_i \mid i \in \mathbb{Z}\}$ is a $\mathbb{Z}$-basis of $\mathfrak{g}$, we can get that all of the structure constants are integral, then we get the conclusion.

Corollary 5.3. Let $L = \mathfrak{sl}_2$, the integral basis for the universal enveloping algebra of affine-Virasoro algebra is

$$\left\{\frac{1}{2}Ce_{i_1} \cdots e_{i_r} f_{j_1} \cdots f_{j_s} h_{k_1} \cdots h_{k_t} d_{l_1} \cdots d_{l_m}\right\}$$

where $i_1 \leq \cdots \leq i_r, j_1 \leq \cdots \leq j_s, k_1 \leq \cdots \leq k_t, l_1 \leq \cdots \leq l_m \in \mathbb{Z}$.

Proof. We only need to check that $\left\{e_i, f_i, h_i, d_i, \frac{1}{2}C \mid i \in \mathbb{Z}\right\}$ is a $\mathbb{Z}$-basis of L_{av}. By relations (5.1), all coefficients of these brackets are integral except $\left[d_i, d_j\right]$. Now we use $\frac{1}{2}C$ to replace C, we can get,

$$\left[d_i, d_j\right] = (j-i)d_{i+j} + \frac{1}{6}\left(j^3 - j\right)\delta_{i+j,0}\frac{1}{2}C.$$

Since

$$\frac{1}{6}\left(j^3 - j\right) \in \mathbb{Z},$$

we conclude that $\left\{e_i, f_i, h_i, d_i, \frac{1}{2}C \mid i \in \mathbb{Z}\right\}$ is a Chevalley basis of L_{av}, then we have proved this corollary.

6. Conclusion

In this paper, we get the integral basis for $V_k(\mathfrak{sl}_2)$, $V_{Vir}(2k, 0)$ and $U(L_{av})$. The constructions of affine vertex algebra and Virasoro vertex algebra are key to our proof. Lemma (5.2) is essential in finding integral basis for $U(L_{av})$. Approaches used here can be easily generalized to tensor product of vertex algebras and universal enveloping algebras. We can also generalize $V_k(\mathfrak{sl}_2)$ to general $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dong, C. and Griess, R. (2012) Integral Forms in Vertex Operator Algebras Which

Are Invariant under Finite Groups. *Journal of Algebra*, **365**, 184-198.
https://doi.org/10.1016/j.jalgebra.2012.05.006

[2] Prevost, S. (1992) Vertex Algebras and Integral Bases for the Enveloping Algebras of Affine Lie Algebras. Memoirs of the American Mathematical Society, Providence.
https://doi.org/10.1090/memo/0466

[3] Frenkel, E. and Ben-Zvi, D. (2001) Vertex Algebras and Algebraic Curves. Mathematical Surveys and Monographs, American Mathematical Society, Providence.

[4] Kac, V.G. (1998) Vertex Algebras for Beginners. 2nd Edition, Univ. Lecture Ser., American Mathematical Society, Providence. https://doi.org/10.1090/ulect/010

[5] Lepowsky, J. and Li, H. (2003) Introduction to Vertex Operator Algebras and Their Representations. Progress in Mathematics, Birkhäuser, Boston.
https://doi.org/10.1007/978-0-8176-8186-9

[6] Frenkel, I.B. and Zhu, Y. (1992) Vertex Operator Algebras Associated to Representations of Affine and Virasoro Algebras. *Duke Mathematical Journal*, **66**, 123-168.
https://doi.org/10.1215/S0012-7094-92-06604-X

[7] Li, H. (1996) Local Systems of Vertex Operators, Vertex Superalgebras and Modules. *Journal of Pure and Applied Algebra*, **109**, 143-195.
https://doi.org/10.1016/0022-4049(95)00079-8

[8] Wang, W. (1993) Rationality of Virasoro Vertex Operator Algebras. *Duke Mathematical Journal*, **7**, 197-211. https://doi.org/10.1155/S1073792893000212

[9] Belavin, A., Polyakov, A. and Zamolodchikov (1984) Infinite Conformal Symmetries in Two-Dimensional Quantum Field Theory. *Nuclear Physics*, **241**, 333-380.
https://doi.org/10.1016/0550-3213(84)90052-X

[10] Kac, V.G. and Raina, A.K. (1987) Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras. World Scientific, Singapore.
https://doi.org/10.1142/0476

[11] Kuroki, G. (1991) Fock Space Representations of Affine Lie Algebras and Integral Representations in the Wess-Zumino-Witten Models. *Communications in Mathematical Physics*, **142**, 511-542. https://doi.org/10.1007/BF02099099

[12] Dong, C. and Lepowsky, J. (1993) Generalized Vertex Algebras and Relative Vertex Operators. Birkhäuser, Boston. https://doi.org/10.1007/978-1-4612-0353-7

[13] Frenkel, I.B., Huang, Y.-Z. and Lepowsky, J. (1993) On Axiomatic Approaches to Vertex Operator Algebras and Modules. Memoirs of the American Mathematical Society, Providence. https://doi.org/10.1090/memo/0494

[14] Frenkel, I.B., Lepowsky, J. and Meurman, A. (1988) Vertex Operator Algebras and the Monster. Pure and Applied Mathematics Vol. 134. Academic Press, New York.

[15] Gao, Y., Hu, N. and Liu, D. (2016) Representations of the Affine-Virasoro Algebra of Type $A_1^{(1)}$. *Journal of Geometry and Physics*, **106**, 102-107.
https://doi.org/10.1016/j.geomphys.2016.03.009