

Hamiltonian Polynomial Eigenvalue Problems

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Abstract

We present in this paper a new method for solving polynomial eigenvalue problem. We give methods that decompose a skew-Hamiltonian matrix using Cholesky like-decomposition. We transform first the polynomial eigenvalue problem to an equivalent skew-Hamiltonian/Hamiltonian pencil. This process is known as linearization. Decomposition of the skew-Hamiltonian matrix is the fundamental step to convert a structured polynomial eigenvalue problem into a standard Hamiltonian eigenproblem. Numerical examples are given.

Keywords

Hamiltonian Matrix, Polynomial Eigenvalue Problem, Skew-Hamiltonian/Hamiltonian Pencil, Cholesky Like-Decomposition

1. Introduction

In this work we propose a numerical approach for solving the k^{th} degree polynomial eigenvalue problem

$$P(\lambda)v = \sum_{i=0}^k \lambda^i M_i v = 0 \quad (\mathbf{P})$$

Problem **(P)** arises in many applications in science and engineering, ranging from the dynamical analysis of structural systems such as bridges and buildings to theories of elementary particles in atomic physics [1] [2]. It's also the most important task in the vibration analysis of buildings, machines, and vehicles [3]. We first transform our k^{th} degree polynomial eigenvalue problem **(P)** to an equivalent first-degree equation $(A - \lambda B)v = 0$ commonly called pencil problem. In the case when matrices M_i have symmetric/skew-symmetric structure, the problem **(P)** is transformed to a skew-Hamiltonian/Hamiltonian pencil [4]. The second step is to transform the skew-Hamiltonian/Hamiltonian pencil to a standard Hamiltonian eigenproblem $Hv = \lambda v$ [5]. This transformation is ob-

tained after decomposing B as $R^T J R$ where R is a permuted triangular matrix. The Hamiltonian matrix H is given by $J^T R^{-T} A R^{-1}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

It is known that any nonsingular skew-symmetric matrix has a decomposition of the form $B = R^T J R$ [6]. The real matrix $M = J^T B$ is skew-Hamiltonian and has the decomposition $J^T B = R^J R$ where R has the form of a permuted triangular matrix. We give here a new proof for this result and different algorithms that compute the decomposition $M = R^J R$.

2. Preliminaries

We give in this paragraph, new definitions and useful propositions.

Let $J = J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n denotes the $n \times n$ identity matrix. We will use J when the size is clear from the context. Recall that a matrix $M \in \mathbb{R}^{2n \times 2n}$ is skew-Hamiltonian if $M^J = M$, where the J -transpose of the matrix M is defined by $M^J = J^T M^T J$. Likewise, a Hamiltonian matrix H is written as $\begin{pmatrix} E & G \\ F & -E^T \end{pmatrix}$

where E, G and $F \in \mathbb{R}^{n \times n}$ with $G^T = G$ and $F^T = F$. We have $H^J = -H$. More general, the J -transpose of the rectangular $2p$ -by- $2q$ matrix N is defined by $2q$ -by- $2p$ matrix $N^J = J_{2q}^T N^T J_{2p}$.

The set $(E_i)_{1 \leq i \leq n}$ where $E_i = [e_i \ e_{n+i}]$ with e_i is denoting the i -th unit vector of length $2n$, satisfies $E_i J_2 = J_{2n} E_i$, $E_i^J = E_i^T$ and $E_i^T E_j = \delta_{ij} I_2$ where $E_i^J = J_2^T E_i^T J_{2n}$ and $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Let $U = [u_1, u_2] \in \mathbb{R}^{2n \times 2}$ where $u_1 = \sum_{i=1}^{2n} u_i^1 e_i$ and $u_2 = \sum_{j=1}^{2n} u_j^2 e_j$. Then U is written in a unique way as linear combination of $(E_i)_{1 \leq i \leq n}$ on $\mathbb{R}^{2 \times 2}$, $U = \sum_{i=1}^n E_i M_i$

where $M_i = \begin{pmatrix} u_i^1 & u_i^2 \\ u_{n+i}^1 & u_{n+i}^2 \end{pmatrix}$. Let $M \in \mathbb{R}^{2n \times 2n}$ be a $2n$ -by- $2n$ real matrix. Then M

is written as $M = \sum_{i=1}^n \sum_{j=1}^n E_i M_{ij} E_j^T$ where $M_{ij} = \begin{pmatrix} m_{ij} & m_{i,n+j} \\ m_{n+i,j} & m_{n+i,n+j} \end{pmatrix}$.

Definition 2.1. The $2n$ -by- $2n$ real matrix $L = \sum_{i=1}^n \sum_{j=1}^n E_i L_{ij} E_j^T$ is called lower J -triangular if $L_{ij} = 0_{2 \times 2}$ for $j > i$ and $L_{ii} = \begin{pmatrix} * & \mathbf{0} \\ * & * \end{pmatrix}$, (i.e., $L = \sum_{i=1}^n \sum_{j=1}^i E_i L_{ij} E_j^T$).

Definition 2.2. The $2n$ -by- $2n$ real matrix $U = \sum_{i=1}^n \sum_{j=1}^n E_i U_{ij} E_j^T$ is called upper J -triangular if $U_{ij} = 0_{2 \times 2}$ for $i > j$ and $U_{ii} = \begin{pmatrix} * & * \\ \mathbf{0} & * \end{pmatrix}$, (i.e., $U = \sum_{i=1}^n \sum_{j=i}^n E_i U_{ij} E_j^T$).

Proposition 2.1. Let M and N be two upper J -triangular (respectively, lower J -triangular) $2n$ -by- $2n$ real matrix. The product $P = MN$ remain as upper

J -triangular (respectively, as lower J -triangular).

Proof. Let $M = \sum_{i=1}^n \sum_{j=i}^n E_i M_{ij} E_j^T$ and $N = \sum_{i=1}^n \sum_{j=i}^n E_i N_{ij} E_j^T$ two upper J -triangular $2n$ -by- $2n$ real matrix. The matrix product of M and N is obtained by

$$P = \sum_{i=1}^n \sum_{j=i}^n E_i M_{ij} \sum_{k=j}^n N_{jk} E_k^T = \sum_{i=1}^n \sum_{k=i}^n E_i \underbrace{\left(\sum_{j=i}^k M_{ij} N_{jk} \right)}_{P_{ik}} E_k^T.$$

That proves $P = MN$ remain as upper J -triangular. (similarly, when M and N are lower J -triangular). \square

Definition 2.3. $U = [u_1, u_2] \in \mathbb{R}^{2n \times 2}$ is called J -isotropic if $U^J U = \begin{pmatrix} u_1^T J u_2 \\ 0 \end{pmatrix} I_2 = 0_2$.

Proposition 2.2. The inverse of a regular upper J -triangular $2n$ -by- $2n$ real matrix (respectively, lower J -triangular) is also upper J -triangular (respectively, also lower J -triangular).

Proof. Let $U = \sum_{i=1}^n \sum_{j=i}^n E_i U_{ij} E_j^T$ an upper J -triangular $2n$ -by- $2n$ real matrix. The following proposition $(P_k): U^{-1} E_k = \sum_{l=1}^k E_l C_l$, where $(C_l)_{1 \leq l \leq k}$ are 2-by-2 real matrix, holds for $k = 1$. Suppose $(P_i)_{1 \leq i \leq k}$ are true for $1 \leq i \leq k$, ($k < n$). For $i = k + 1$, we have $U^{-1} E_{k+1} = \sum_{i=1}^{k+1} E_i U_{ik+1}$, therefore

$$\begin{aligned} \underbrace{U^{-1} (U E_{k+1})}_{E_{k+1}} &= U^{-1} \left(\sum_{i=1}^k E_i U_{ik+1} + E_{k+1} U_{k+1k+1} \right) \\ &= \sum_{i=1}^k U^{-1} E_i U_{ik+1} + U^{-1} E_{k+1} U_{k+1k+1} \end{aligned}$$

Since $U_{k+1,k+1} = \begin{pmatrix} u_{k+1,k+1} & * \\ 0 & u_{n+k+1,n+k+1} \end{pmatrix}$ is invertible and by using recurrence hypothesis, then

$$\begin{aligned} U^{-1} E_{k+1} &= E_{k+1} U_{k+1k+1}^{-1} - \sum_{i=1}^k U^{-1} E_i (U_{ik+1} U_{k+1k+1}^{-1}) \\ &= E_{k+1} U_{k+1k+1}^{-1} - \sum_{i=1}^k \left(\sum_{j=1}^i E_j C_j \right) (U_{ik+1} U_{k+1k+1}^{-1}) \\ &= E_{k+1} U_{k+1k+1}^{-1} - \sum_{j=1}^k E_j C_j \left(\sum_{i=j}^k U_{ik+1} U_{k+1k+1}^{-1} \right) \\ &= \sum_{l=1}^{k+1} E_l G_l \end{aligned}$$

$$\text{with } G_{k+1,k+1} = U_{k+1,k+1}^{-1} = \begin{pmatrix} \frac{1}{u_{k+1,k+1}} & * \\ 0 & \frac{1}{u_{n+k+1,n+k+1}} \end{pmatrix}.$$

3. Cholesky Like-Decomposition for Skew-Hamiltonian Matrix

In this section, we study different ways to compute $R^J R$ decomposition of a real skew-Hamiltonian matrix $M \in \mathbb{R}^{2n \times 2n}$. We began first by giving some interesting theoretical results.

3.1. Definition and Theoretical Results

Definition 3.1. The $2n$ -by- $2n$ real skew-Hamiltonian matrix M is called J -definite if $U^J M U = \alpha I_2$ with $\alpha \neq 0$ for every non J -isotropic

$$U = [u_1, u_2] \in \mathbb{R}^{2n \times 2} \quad (\text{i.e., } u_1^T J u_2 \neq 0).$$

Remark 3.1 For $U = [u_1, u_2] \in \mathbb{R}^{2n \times 2}$ and a $2n$ -by- $2n$ real skew-Hamiltonian

$$\text{matrix } M, \quad U^J M U = \begin{pmatrix} -u_2^T J M u_1 & -u_2^T J M u_2 \\ \underbrace{u_1^T J M u_1}_0 & \underbrace{u_1^T J M u_2}_0 \end{pmatrix} = \alpha I_2 \quad \text{with } \alpha = u_1^T J M u_2.$$

Lemma 3.1. If M is a $2n$ -by- $2n$ real skew-Hamiltonian and J -definite matrix, then M is invertible.

Proof. If not, there exists $v \in \mathbb{R}^{2n}$ such that $Mv = 0_{\mathbb{R}^{2n}}$. Let $u \in \mathbb{R}^{2n}$ that verify $U = [u, v] \in \mathbb{R}^{2n \times 2}$ non J -isotropic (i.e., $u^T J v \neq 0$). Since $Mv = 0_{\mathbb{R}^{2n}}$, then $U^J M U = \underbrace{(u^T J M v)}_0 I_2 = 0_2$ which is contradictory with the hypothesis. \square

Theorem 3.2. If M is a $2n$ -by- $2n$ real skew-Hamiltonian, J -definite matrix, then M has an LUJ -factorization.

Proof. Let $W \in \mathbb{R}^{2k \times 2}$ ($1 \leq k \leq n$) be non J -isotropic. Suppose that

$$W = \sum_{i=1}^k E_i C_i \quad \text{where } C_i \in \mathbb{R}^{2 \times 2}. \text{ We construct an } U \in \mathbb{R}^{2n \times 2} \text{ such that}$$

$$U = \sum_{i=1}^n E_i C_i \quad \text{where } C_i = 0_{2 \times 2} \text{ for } i > k. \text{ We have, } U^J M U = W^J M^{[k,k]} W$$

where $M^{[k,k]} = \sum_{i=1}^k \sum_{j=1}^k E_i M_{ij} E_j^T$ as defined in theorem 2.2 given above. Then $2k$ -by- $2k$ matrix $M^{[k,k]}$ remains skew-Hamiltonian and J -definite and then invertible. \square

Corollary 3.3. If $M = LU$ is the LUJ -factorization of the real $2n$ -by- $2n$ skew-Hamiltonian, J -definite matrix M , then M has an $M = L\Delta L^J$ where

$$\Delta = \sum_{i=1}^n E_i \begin{pmatrix} u_{ii} & 0 \\ 0 & u_{ii} \end{pmatrix} E_i^T \quad (\text{here } u_{ii} \text{ is the } i\text{-th diagonal entry of } U).$$

Proof. Since the matrix M is skew-Hamiltonian, then by taking $U = \Delta N$ we obtain

$$M = LU = U^J L^J = N^J \Delta^J L^J = N^J (L\Delta)^J$$

$M = N^J (L\Delta)^J$ is nothing but the LUJ -factorization of M . Indeed, N^J is lower J -triangular with 1 in diagonal and $(L\Delta)^J$ is upper J -triangular. Thus, from the uniqueness of the LUJ -factorization, it follows that $N^J = L$. \square

Theorem 3.4. Let M be a $2n$ -by- $2n$ real skew-Hamiltonian J -definite matrix, then M has a Cholesky J -factorization $M = NN^J$ where N is lower J -triangular and in addition the $N_{ii} = \begin{pmatrix} n_{i,i} & 0 \\ 0 & n_{n+i,n+i} \end{pmatrix}$ are diagonal.

Proof. We proceed by induction on n . For $n=1$, the real 2-by-2 skew-Hamiltonian J -definite matrix $M = m_{11}I_2$ where $m_{11} \neq 0$. If we set

$$N = \begin{pmatrix} \sqrt{|m_{11}|} & 0 \\ 0 & \text{sign}(m_{11})\sqrt{|m_{11}|} \end{pmatrix}, \text{ the theorem holds trivially.}$$

Let's now $n \geq 2$. Since M is skew-Hamiltonian and J -definite, then $E_1^J M E_1 = m_{11}I_2 \neq 0_2$. We can write

$$\begin{aligned} M &= \sum_{i=1}^n \sum_{j=1}^n E_i M_{ij} E_j^T \\ &= E_1 M_{11} E_1^T + \sum_{i=2}^n E_i M_{i1} E_1^T + \sum_{i=2}^n E_i M_{i1}^J E_i^T \\ &\quad + \sum_{i=2}^n \sum_{j=2}^n E_i M_{ij} E_j^T. \end{aligned}$$

We set $W = \sum_{i=2}^n E_i M_{i1} E_1^T$ and $B = \sum_{i=2}^n \sum_{j=2}^n E_i M_{ij} E_j^T$. The J -transpose of W is given

$$\text{by } W^J = \sum_{i=2}^n E_i M_{i1}^J E_i^T. \text{ Let } K = \begin{pmatrix} \sqrt{|m_{11}|} & 0 \\ 0 & \text{sign}(m_{11})\sqrt{|m_{11}|} \end{pmatrix},$$

$$B_1 = E_1 K E_1^T + \sum_{i=2}^n E_i M_{i1} K^{-J} E_1^T + \sum_{i=2}^n E_i E_i^T \text{ and } A_1 = E_1 E_1^T + B - \frac{1}{m_{11}} W W^J. \text{ We}$$

$$\text{calculate } B_1 A_1 = E_1 K E_1^T + \sum_{i=2}^n E_i M_{i1} K^{-J} E_1^T + B - \frac{1}{m_{11}} W W^J. \text{ The } J\text{-transpose of } B_1$$

$$\text{is given by } B_1^J = E_1 K^J E_1^T + \sum_{i=2}^n E_i K^{-1} M_{i1}^J E_i^T + \sum_{i=2}^n E_i E_i^T. \text{ Finally, we obtain}$$

$$\begin{aligned} B_1 A_1 B_1^J &= m_{11} E_1 E_1^T + \sum_{i=2}^n E_1 \underbrace{(K K^{-1})}_{I_2} M_{i1}^J E_i^T + \sum_{i=2}^n E_i M_{i1} \underbrace{(K^{-J} K^J)}_{I_2} E_i^T \\ &\quad + \sum_{i=2}^n \sum_{j=2}^n E_i M_{i1} \underbrace{(K^{-J} K^{-1})}_{\frac{1}{m_{11}} I_2} M_{j1}^J E_j^T + B - \frac{1}{m_{11}} W W^J \end{aligned}$$

Since $W W^J = \sum_{i=2}^n \sum_{j=2}^n E_i M_{i1} M_{j1}^J E_j^T$, then $B_1 A_1 B_1^J = M$. By induction

$$B - \frac{1}{m_{11}} W W^J = L L^J, \text{ where } L = \sum_{i=2}^n \sum_{j=2}^i E_i L_{ij} E_j^T \text{ where } L \text{ is } 2(n-1)\text{-by-}2(n-1)$$

lower J -triangular matrix and in addition the $L_{ii} = \begin{pmatrix} l_{i,i} & 0 \\ 0 & l_{n+i,n+i} \end{pmatrix}$ are diagonal for

$i = 2, \dots, n$. Therefore, if we let $G = E_1 E_1^T + L L^J$, we obtain

$$G G^J = E_1 E_1^T + L L^J = A_1 \text{ and then finally}$$

$$M = B_1 A_1 B_1^J = B_1 (GG^J) B_1^J = (B_1 G)(B_1 G)^J = NN^J.$$

Since B_1 and G are lower J -triangular, then $N = B_1 G$ remain lower J -triangular and verify $N_{ii} = \begin{pmatrix} n_{i,i} & 0 \\ 0 & n_{n+i,n+i} \end{pmatrix}$ are diagonal. \square

3.2. Method 1

We construct an algorithm that gives decomposition $R^J R$ of skew-Hamiltonian matrices via a LU J -decomposition.

Proposition 3.5. Let M is a $2n$ -by- $2n$ real skew-Hamiltonian, J -definite matrix. If $M = LU$ its LU J -factorization, then $R = (LD)^J$ where D is a diagonal matrix defined by

$$D = \sum_{i=1}^n E_i \begin{pmatrix} \sqrt{\text{sign}(u_{ii})} u_{ii} & 0 \\ 0 & \text{sign}(u_{ii}) \sqrt{\text{sign}(u_{ii})} u_{ii} \end{pmatrix} E_i^T$$

(here u_{ii} is the i -th diagonal entry of U) is lower J -triangular and verify $M = R^J R$.

Proof. By corollary 3.3, $M = L\Delta L^J$. Since $\Delta = DD^J = \sum_{i=1}^n E_i \begin{pmatrix} u_{ii} & 0 \\ 0 & u_{ii} \end{pmatrix} E_i^T$

where D is as given above, then $R = (LD)^J$ is lower J -triangular and $R^J R = L \underbrace{DD^J}_{\Delta} L^J = M$. From the J -decomposition $M = LU$ given by algorithms in section, we set

$$D = \sum_{i=1}^n E_i \begin{pmatrix} \sqrt{\text{sign}(u_{ii})} u_{ii} & 0 \\ 0 & \text{sign}(u_{ii}) \sqrt{\text{sign}(u_{ii})} u_{ii} \end{pmatrix} E_i^T$$

where $u_{ii} = e_i^T U e_i$. We have $M = R^J R$ where $R = (LD)^J$. \square

3.3. Method 2

We study now a method that constructs decomposition $R^J R$ of skew-Hamiltonian J -definite matrices.

Let $M \in \mathbb{R}^{2n \times 2n}$ be a skew-Hamiltonian J -definite matrix.

$$M = \sum_{i=1}^n \sum_{j=1}^n E_i M_{ij} E_j^T$$

with $M_{ij} = \begin{pmatrix} m_{i,j} & m_{i,n+j} \\ m_{n+i,j} & m_{n+i,n+j} \end{pmatrix}$ and $E_i = [e_i \ e_{n+i}] \in \mathbb{R}^{2n \times 2}$. Let $M = LL^J$ where

L is lower J -triangular that verify $L_{ii} = \begin{pmatrix} l_{i,i} & 0 \\ 0 & l_{n+i,n+i} \end{pmatrix}$. The existence of L is guaranteed by theorem 4.4

$$L = \sum_{i=1}^n \sum_{j=i}^n E_i L_{ij} E_j^T.$$

Since

$$\begin{aligned}
 LL^J &= \left(\sum_{i=1}^n \sum_{j=1}^i E_i L_{ij} E_j^T \right) \left(\sum_{k=1}^n \sum_{s=1}^k E_s L'_{ks} E_k^T \right) = \sum_{i=1}^n \sum_{s=1}^i \sum_{k=s}^n E_i L_{is} L'_{ks} E_k^T \\
 &= \sum_{i=1}^n \sum_{j=1}^n E_i \left[\sum_{s=1}^i L_{is} L'_{js} \right] E_j^T = \sum_{i=1}^n \sum_{j=1}^n E_i M_{ij} E_j^T
 \end{aligned}$$

then

$$M_{ij} = \sum_{s=1}^{\min(i,j)} L_{is} L'_{js}$$

If $i=1, j=1$ then $s=1$. We obtain $L_{11} L'_{11} = M_{11}$. Since $L_{11} L'_{11} = \det(L_{11}) I_2$ and $M_{11} = m_{11} I_2$, then $l_{11} \times l_{n+1 n+1} = m_{11}$, then

$$\begin{cases} l_{11} = \sqrt{\text{sign}(m_{11}) m_{11}} \\ l_{n+1 n+1} = \text{sign}(m_{11}) \sqrt{\text{sign}(m_{11}) m_{11}} \end{cases}$$

And for $i=2,3,\dots,n$, $M_{i1} = L_{i1} L'_{11}$. Multiplying on the right by L_{11} , we obtain

$$M_{i1} L_{11} = L_{i1} L'_{11} L_{11} = \det(L_{11}) L_{i1}$$

Thus

$$L_{i1} = \frac{M_{i1} L_{11}}{\det(L_{11})} = M_{i1} \begin{pmatrix} \frac{\sqrt{\text{sign}(m_{11}) m_{11}}}{m_{11}} & 0 \\ 0 & \frac{\text{sign}(m_{11}) \sqrt{\text{sign}(m_{11}) m_{11}}}{m_{11}} \end{pmatrix}$$

Since

$$M_{22} = L_{21} L'_{21} + L_{22} L'_{22} = \det(L_{21}) I_2 + \det(L_{22}) I_2$$

then,

$$\det(L_{22}) I_2 = M_{22} - \det(L_{21}) I_2 = (m_{22} - \det(L_{21})) I_2$$

Since $\det(L_{22}) = l_{22} \times l_{n+2 n+2}$, then

If we set $\alpha_2 = m_{22} - \det(L_{21})$, then we obtain

$$L_{22} = \begin{pmatrix} \sqrt{\text{sign}(\alpha_2) \alpha_2} & 0 \\ 0 & \text{sign}(\alpha_2) \sqrt{\text{sign}(\alpha_2) \alpha_2} \end{pmatrix}.$$

However $M_{i2} = L_{i1} L'_{21} + L_{i2} L'_{22}$ for $i=2,3,\dots,n$, then $L_{i2} L'_{22} = M_{i2} - L_{i1} L'_{21}$.

Multiplying on the right by L_{22} we find $L_{i2} L'_{22} L_{22} = (M_{i2} - L_{i1} L'_{21}) L_{22}$ and finally

$$L_{i2} = \frac{(M_{i2} - L_{i1} L'_{21}) L_{22}}{\det(L_{22})}.$$

The method yield the following algorithm.

Algorithm:

for $j=1,2,\dots,n$

$$\alpha_j = m_{jj} - \sum_{s=1}^{j-1} \det(L_{js})$$

$$L_{jj} = \sqrt{\text{sign}(\alpha_j)\alpha_j} \begin{pmatrix} 1 & 0 \\ 0 & \text{sign}(\alpha_j) \end{pmatrix}$$

for $i = j + 1, j + 2, \dots, n$

$$L_{ij} = \frac{\left(M_{ij} - \sum_{s=1}^{j-1} L_{is}L'_{js} \right) L_{jj}}{\det(L_{jj})}$$

4. Polynomial Eigenvalue Problems

Many applications give rise to structured matrix polynomial eigenvalue problems

$$P(\lambda)v = \sum_{i=0}^k \lambda^i M_i v = 0$$

The numerical solution of this polynomial eigenvalue problem is one of the most important tasks in the vibration analysis of buildings, machines and vehicles [7]. In many applications, the coefficient matrices have particular structure and it is important that numerical methods respect this structure. A popular approach for solving the polynomial eigenvalue problem $P(\lambda)v = 0$ is to linearize to produce a generalized eigenproblem $(A - \lambda B)v = 0$ [8].

Theorem 4.1. [9] *Consider the polynomial eigenvalue problem $P(\lambda)v = 0$ with either $M_i^T = (-1)^i M_i$ or $M_i^T = (-1)^{i+1} M_i$ and with M_k nonsingular. Then solving problem $P(\lambda)v = 0$ is equivalent to solve $(A - \lambda B)v = 0$ where*

$$A = \begin{pmatrix} -M_0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & -M_2 & -M_3 & -M_4 & \dots & \dots & -M_k \\ 0 & M_3 & M_4 & & & & 0 \\ 0 & -M_4 & & & & & 0 \\ \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & & & & & \vdots \\ 0 & \pm M_k & 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} M_1 & M_2 & M_3 & \dots & \dots & M_{k-1} & M_k \\ -M_2 & -M_3 & -M_4 & \dots & \dots & -M_k & 0 \\ M_3 & M_4 & & & & & 0 \\ -M_4 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ \pm M_k & 0 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

We draw from this theorem that the polynomial eigenvalue problem (P) can be reduced to an eigenvalue pencil problem $Av = \lambda Bv$ where A is symmetric and B is skew-symmetric. The second step is to transform the skew-symmetric/symmetric pencil to a standard Hamiltonian eigenproblem $Hv = \lambda v$ by decomposing the skew-Hamiltonian matrix JB as $R^J R$. The Hamiltonian ma-

trix H is then obtained by $J^T R^T A R^{-1}$. Eigenvalue problems of this type arise property that all eigenvalues appear in quadruples $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$, the spectrum is symmetric with respect to the real and imaginary axes.

5. Numerical Examples

We present computed eigenvalues that solve the k^{th} degree polynomial eigenvalue problem $P(\lambda)v = \sum_{p=0}^k \lambda^p M_p v = 0$ of dimension $n \times n$ which is transforming to a standard eigenvalue problem of dimension $kn \times kn$. We also compute the error consisting in

$$err = \sum_{i=1}^{kn} \|P(\lambda_i)v^{(i)}\|_2$$

Example 1. [9]

Let us consider a quartic eigenvalue problem of the form $(\lambda^4 M_4 + \lambda^3 M_3 + \lambda^2 M_2 + \lambda M_1 + M_0)v = 0$.

We obtain a 144×144 quartic pencil, whose 576 eigenvalues are shown in **Figure 1** given above.

$$err = \sum_{i=1}^{576} \|P(\lambda_i)v^{(i)}\|_2 = 1.2471e - 014$$

Example 2. [10]

Now, let us consider the following quadratic eigenvalue problems given by $(\lambda^2 I_n + \lambda C + K)v = 0$. The 400 eigenvalues are shown in **Figure 2** below

$$err = \sum_{i=1}^{400} \|P(\lambda_i)v^{(i)}\|_2 = 3.1581e - 013$$

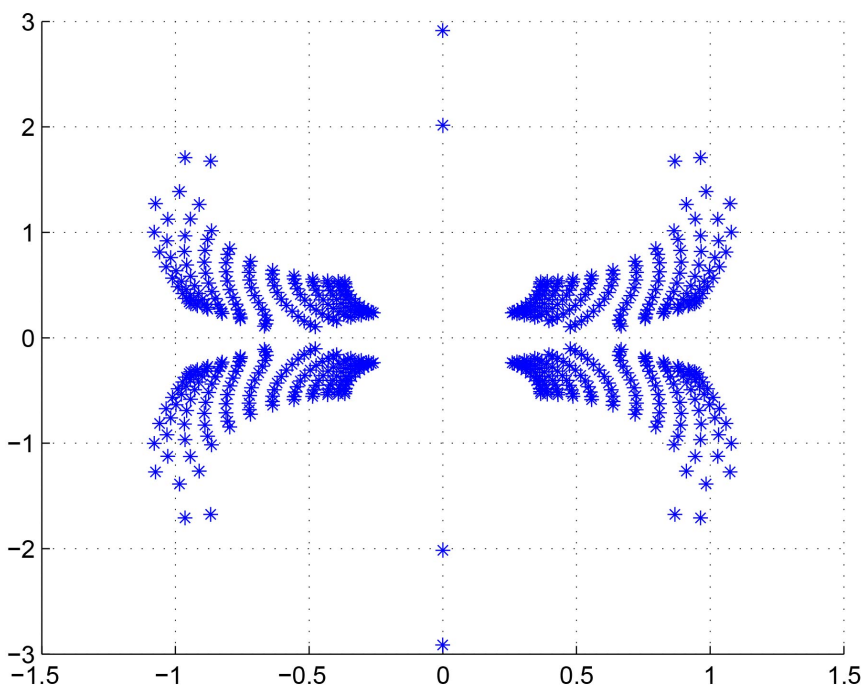


Figure 1. Eigenvalues of 144×144 polynomial problems.

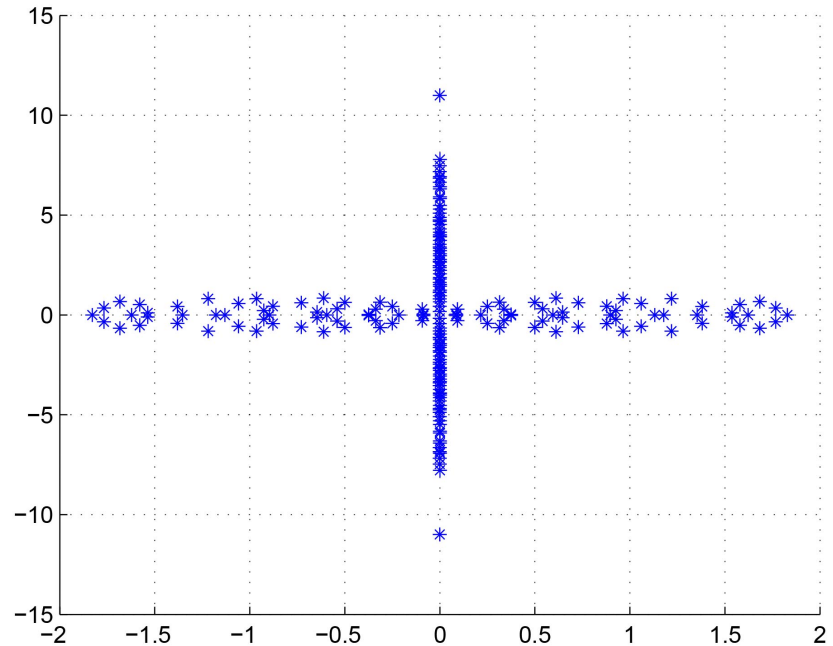


Figure 2. Eigenvalues of 400 quadratic polynomial problems.

6. Conclusion

We have proposed a numerical approach for solving polynomial eigenvalue problems structured. We first transform polynomial eigenvalue problem $P(\lambda)v = \sum_{i=0}^k \lambda^i M_i v = 0$ to a skew-Hamiltonian/Hamiltonian pencil $(A - \lambda B)v = 0$. The second step is to transform the pencil into a standard Hamiltonian eigenproblem $H z = \lambda z$. Numerical methods based on these structured linearizations are expected to be more effective in computing accurate eigenvalues in practical applications. My future work based on this current study is to solve the large matrix equations applied in signal processing, image restoration and model reduction in control theory.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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