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Composite Minimization Problems in Hadamard Spaces

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Abstract

In this paper, we prove some Δ-convergence and strong convergence results for the sequence generated by a new algorithm to a minimizer of two convex functions and a common fixed point for quasi-pseudo-contractive mappings in Hadamard spaces. Our theorems improve and generalize some recent results in the literature.

Keywords

Hadamard Spaces, Composite Minimization, Common Fixed Points, Quasi-Pseudo-Contractive Mappings, Quasilinearization

1. Introduction

Let (X,d) be a metric space and $x,y\in X$ with $l=d(x,y)$. A *geodesic path* from x to y is an isometry $c:[0,l]\to X$ such that $c(0)=x, c(l)=y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a *geodesic space* if every two points of X are joined by a geodesic segment. A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1,x_2,x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\Delta(x_1,x_2,x_3)$ is the triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i,x_j)=d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j)$ for all $i,j=1,2,3$.

A geodesic space X is a *CAT*(0) *space* if for each geodesic triangle $\Delta:=\Delta(x_1,x_2,x_3)$ in X and its comparison triangle $\bar{\Delta}:=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$, the *CAT*(0) *inequality*

$$d(x,y)\le d_{\mathbb{R}^2}(\bar{x},\bar{y})$$

is satisfied by all $x,y\in\Delta$ and $\bar{x},\bar{y}\in\bar{\Delta}$. The meaning of the CAT(0) inequality is that a geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well-known that any complete and simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces, R-trees, Euclidean buildings. A complete CAT(0) space is called a *Hadamard space*.

Let C be a nonempty set and consider the following composite optimization problem: find $x^* \in C$ such that

$$f\left(x^*\right)+g\left(x^*\right)=\min_{x\in C}\left\{f\left(x\right)+g\left(x\right)\right\}, \tag{1}$$

where f,g are real-valued functions defined on C. This problem has a typical scenario in linear inverse problems, and it has applications in image reconstruction, machine learning, data recovering and compressed sensing (see [1]-[7] and the references therein).

In the case that X is a real Hilbert space or a real Banach space, problem (1) has been studied by many authors ([3] [5] [8]-[12]). For example, in 2019, Chang *et al.* [8] used a modified hybrid algorithm to find a minimizer for problem (1) in Banach spaces without the assumption that the potential function is Fréchet differentiable and its gradient is L-Lipschitz continuous.

Recently, many convergence results for solving optimization problems have been extended from the classical linear spaces to the setting of manifolds. For example, in 2015, Cholamjiak-Abdou-Cho [13] established strong convergence of the sequence to a minimizer of a convex function and to a fixed point of nonexpansive mappings in CAT(0) spaces. Also in 2019, Chang *et al.* [14] presented a new modified proximal point algorithm for solving the minimization of a convex function and the common fixed points problem for two k-strictly pseudononspreading mappings in Hadamard spaces.

Recall that a mapping $T:C\to C$ is said to be

(i) *nonexpansive*, if

$$d\left(Tx,Ty\right)\le d\left(x,y\right),\ \forall x,y\in C;$$

(ii) *quasi-nonexpansive*, if $Fix\left(T\right)\ne\varnothing$ and

$$d\left(Tx,x^*\right)\le d\left(x,x^*\right),\ \forall\ x\in C,\ \forall x^*\in Fix\left(T\right);$$

(iii) *k-strictly pseudononspreading*, if there exists a constant $k\in\left(0,1\right)$ such that for all $x,y\in C$

$$d^2\left(Tx,Ty\right)\le d^2\left(x,y\right)+kd^2\left(x,Tx\right)+kd^2\left(y,Ty\right)+2\left(1-k\right)\left\langle\overrightarrow{xT\left(x\right)},\overrightarrow{yT\left(y\right)}\right\rangle;$$

(iv) *demicontractive*, if $Fix\left(T\right)\ne\varnothing$ and there exists $k\in\left(0,1\right)$ such that

$$d^2\left(Tx,x^*\right)\le d^2\left(x,x^*\right)+kd^2\left(x,Tx\right),\ \forall x\in C,\ \forall x^*\in Fix\left(T\right).$$

Definition 1. An operator $T:C\to C$ is said to be *pseudo-contractive* if

$$\left\langle\overrightarrow{TxTy},\overrightarrow{xy}\right\rangle\le d^2\left(x,y\right),\ \forall x,y\in C.$$

Remark 1. *The interest of pseudo-contractive operators lies in their connection with monotone mappings, namely, T is a pseudo-contraction if and only if $I-T$ is a monotone mapping. It is well known that T is pseudo-contractive if and only if*

$$d^2(Tx,Ty) \le d^2(x,y) + d^2((I-T)x,(I-T)y), \forall x,y \in C.$$

Definition 2. An operator $T:C \to C$ is said to be *quasi-pseudo-contractive* if $Fix(T) \ne \varnothing$ and

$$d^2(Tx,x^*) \le d^2(x,x^*) + d^2(x,Tx), \forall x \in C, \forall x^* \in Fix(T). \quad (2)$$

From the above definitions, it is easy to see that the class of quasi-pseudo-contractive mappings is fundamental. It includes many kinds of nonlinear mappings such as the demicontractive mappings, the quasi-nonexpansive mappings and the k-strictly pseudononspreading with fixed points as special cases. Motivated by the researches above, we establish the convergent results to a minimizer of two convex functions and a common fixed point of quasi-pseudo-contractive mappings in Hadamard spaces. Thus our results generalize the corresponding results of Cholamjiak-Abdou-Cho [13], Chang *et al.* [14], Ariza-Ruiz *et al.* [15], Bačák [16], Dhompongsa *et al.* [17], Khan-Abbas [18] and many others.

2. Preliminaries and Lemmas

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results. In 1976, Lim [19] introduced the concept of Δ-convergence in a general metric space. Recall that a sequence $\{x_n\}$ in a CAT(0) space X is said to *Δ-converge to* $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. A geodesic space (X,d) is a CAT(0) space, if and only if

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \quad (3)$$

for all $x,y,z \in X$ and all $t \in [0,1]$. Berg and Nikolaev [20] introduced the concept of quasilinearization as follows. Denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle \cdot,\cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right]$$

for all $a,b,c,d \in X$. It is easy to see that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle, \langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$$

for all $a,b,c,d,x \in X$. It is proved in [20] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d), \forall a,b,c,d \in X.$$

Lemma 1. [14] *Let X be a Hadamard space. Then for all* $x,y,z,u,w \in X$ *and* $t,s \in [0,1]$, *we have*

(i) $d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z)$;

(ii) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) = |t-s|d(x,y)$;

(iii) $d(tx \oplus (1-t)y, tu \oplus (1-t)w) \le td(x,u) + (1-t)d(y,w)$.

Definition 3. [14] Let C be a nonempty subset of a Hadamard space X and let

$\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is Fejér monotone respect to C if

$$d(x_{n+1},x)\le d(x_n,x),\ \forall x\in C \text{ and } n\in\mathbb{N}.$$

Lemma 2. [21] *Let* $\{x_n\}$ *be a sequence in a Hadamard space X and let C be a nonempty subset of X. Suppose that* $\{x_n\}$ *is Fejér monotone with respect to C and that every* Δ*-sequential cluster point of* $\{x_n\}$ *belongs to C. Then* $\{x_n\}$ Δ*-converges to a point in C.*

Lemma 3. *Let C be a nonempty closed and convex subset of a Hadamard space X and* $T:C\to C$ *be an L-Lipschizian mapping with* $L\ge 1$. *Denote*

$$K:=(1-\xi)I\oplus\xi T\big((1-\eta)I\oplus\eta T\big). \tag{4}$$

If $0<\xi<\eta<\dfrac{1}{1+\sqrt{1+L^2}}$, then the following conclusions hold:

(i) $Fix(T)=Fix\big(T\big((1-\eta)I\oplus\eta T\big)\big)=Fix(K)$;

(ii) If $I-T$ is demiclosed at 0, then $I-K$ is also demiclosed at 0;

(iii) If T is quasi-pseudo-contractive, then the mapping K is quasi-nonexpansive, that is,

$$d(Kx,x^*)\le d(x,x^*),\ \forall x\in C \text{ and } x^*\in Fix(K).$$

Proof. (i) If $x^*\in Fix(T)$, it is obvious that $x^*\in Fix\big(T\big((1-\eta)I\oplus\eta T\big)\big)$. Conversely, if $x^*\in Fix\big(T\big((1-\eta)I\oplus\eta T\big)\big)$, *i.e.*, $x^*=T\big((1-\eta)x^*\oplus\eta Tx^*\big)$, letting $\mathcal{U}=(1-\eta)I\oplus\eta T$, then $T\mathcal{U}x^*=x^*$. Put $\mathcal{U}x^*=y^*$. Then $Ty^*=x^*$. Now we prove that $x^*=y^*$. In fact, we have

$$\begin{aligned} d(x^*,y^*)&=d(x^*,\mathcal{U}x^*)=d\big(x^*,\big((1-\eta)I\oplus\eta T\big)x^*\big)\\ &=\eta d(x^*,Tx^*)=\eta d(Ty^*,Tx^*)\le L\eta d(y^*,x^*).\end{aligned}$$

Since $0<L\eta<1$, we have $x^*=y^*$, *i.e.*, $x^*\in Fix(T)$. This shows that $Fix(T)=Fix\big(T\big((1-\eta)I\oplus\eta T\big)\big)$. It is obvious that $x\in Fix(K)$ if and only if $x\in Fix\big(T\big((1-\eta)I\oplus\eta T\big)\big)$. The conclusion (1) is proved.

(ii) For any sequence $\{x_n\}\subset C$ satisfying $x_n\rightharpoonup x^*$ and $d(x_n,Kx_n)\to 0$. Next we prove that $x^*\in Fix(K)$. From conclusion (1), we only need to prove that $x^*\in Fix(T)$. In fact, since T is L-Lipschizian, we get

$$\begin{aligned} d(x_n,Tx_n)&\le d\big(x_n,T\big((1-\eta)I\oplus\eta T\big)x_n\big)+d\big(T\big((1-\eta)I\oplus\eta T\big)x_n,Tx_n\big)\\ &\le\frac{1}{\xi}d\big(x_n,(1-\xi)x_n\oplus\xi T\big((1-\eta)I\oplus\eta T\big)x_n\big)\\ &\quad+Ld\big(x_n,(1-\eta)x_n\oplus\eta Tx_n\big)\\ &=\frac{1}{\xi}d(x_n,Kx_n)+L\eta d(x_n,Tx_n),\end{aligned}$$

which implies that

$$d(x_n,Tx_n)\le\frac{1}{\xi(1-L\eta)}d(x_n,Kx_n)\to 0.$$

Since T is demiclosed at 0, we have $x^*\in Fix(T)=Fix(K)$. The conclusion (2) is proved.

(iii) Since $x^* \in Fix(K) = Fix(T)$, we have from (2)

$$\begin{aligned} & d^2\left(T\left((1-\eta)I \oplus \eta T\right)x, x^*\right) \\ & \le d^2\left((1-\eta)x \oplus \eta Tx, x^*\right) + d^2\left((1-\eta)x \oplus \eta Tx, T\left((1-\eta)x \oplus \eta Tx\right)\right) \end{aligned} \tag{5}$$

for all $x \in X$. Since T is L-Lipschitzian, we get

$$d\left(Tx, T\left((1-\eta)x \oplus \eta Tx\right)\right) \le Ld\left(x, (1-\eta)x \oplus \eta Tx\right) = \eta Ld(x, Tx). \tag{6}$$

From (2) and (3), one has

$$\begin{aligned} & d^2\left((1-\eta)x \oplus \eta Tx, x^*\right) \\ & \le (1-\eta)d^2\left(x, x^*\right) + \eta d^2\left(Tx, x^*\right) - \eta(1-\eta)d^2(x, Tx) \\ & \le (1-\eta)d^2\left(x, x^*\right) + \eta d^2\left(x, x^*\right) + \eta d^2(x, Tx) - \eta(1-\eta)d^2(x, Tx) \\ & = d^2\left(x, x^*\right) + \eta^2 d^2(x, Tx). \end{aligned} \tag{7}$$

By (2) and (6), we obtain

$$\begin{aligned} & d^2\left((1-\eta)x \oplus \eta Tx, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \le (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) + \eta d^2\left(Tx, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \quad - \eta(1-\eta)d^2(x, Tx) \\ & \le (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) + \eta^3 L^2 d^2(x, Tx) - \left(\eta - \eta^2\right)d^2(x, Tx) \\ & = (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) - \left(\eta - \eta^2 - \eta^3 L^2\right)d^2(x, Tx). \end{aligned} \tag{8}$$

By (5), (7) and (8), we have

$$\begin{aligned} & d^2\left(T\left((1-\eta)I \oplus \eta T\right)x, x^*\right) \\ & \le d^2\left(x, x^*\right) + \eta^2 d^2(x, Tx) + (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \quad - \left(\eta - \eta^2 - \eta^3 L^2\right)d^2 * x, Tx) \\ & = d^2\left(x, x^*\right) + (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \quad - \eta\left(1 - 2\eta - \eta^2 L^2\right)d^2(x, Tx). \end{aligned} \tag{9}$$

Since $\eta < \dfrac{1}{1+\sqrt{1+L^2}}$, we deduce that $1 - 2\eta - \eta^2 L^2 > 0$. From (9), one gets

$$d^2\left(T\left((1-\eta)I \oplus \eta T\right)x, x^*\right) \le d^2\left(x, x^*\right) + (1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \tag{10}$$

for all $x \in X$ and $x^* \in Fix(T)$. Combing (2) and (10) one has

$$\begin{aligned} & d^2\left((1-\xi)x \oplus \xi T\left((1-\eta)x \oplus \eta Tx\right), x^*\right) \\ & \le (1-\xi)d^2(x, x^*) + \xi d^2\left(T\left((1-\eta)x \oplus \eta Tx\right), x^*\right) \\ & \quad - \xi(1-\xi)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \le (1-\xi)d^2\left(x, x^*\right) + \xi d^2\left(x, x^*\right) + \xi(1-\eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & \quad - \xi(1-\xi)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right) \\ & = d^2\left(x, x^*\right) + \xi(\xi - \eta)d^2\left(x, T\left((1-\eta)x \oplus \eta Tx\right)\right), \end{aligned}$$

which together with $\xi < \eta$ implies that

$$d\left(\left(1-\xi\right)x\oplus\xi T\left(\left(1-\eta\right)x\oplus\eta Tx\right),x^*\right)\le d\left(x,x^*\right),$$

that is,

$$d\left(Kx,x^*\right)\le d\left(x,x^*\right),\ \forall x\in C,\ \forall x^*\in Fix\left(K\right).$$

The proof is completed.

Now we consider the following problem: find a point $x^*\in C$ such that

$$\begin{cases} f\left(x^*\right)+g\left(x^*\right)=\min_{x\in C}\left\{f\left(x\right)+g\left(x\right)\right\}, \\ x^*=T_ix^*,\ i=1,2, \end{cases} \tag{11}$$

where C is a nonempty closed convex set of a Hadamard space X, $f,g:C\to(-\infty,+\infty]$ are proper convex functions and $T_i:C\to C,\ i=1,2$ is a quasi-pseudo-contractive mapping. Recall that a function $f:C\to(-\infty,+\infty]$ is said to be convex, if for any geodesic $[x,y]:=\left\{\gamma_{x,y}\left(\lambda\right):0\le\lambda\le1\right\}=\left\{\lambda x\oplus\left(1-\lambda\right)y:0\le\lambda\le1\right\}$ joining $x,y\in C$, the function $f\circ\gamma$ is convex. If we set

$$\Theta\left(x,y\right):=f\left(y\right)-f\left(x\right),$$

then the problem (11) is equivalent to the problem of finding $x^*\in\bigcap_{i=1}^2 Fix\left(T_i\right)$ such that

$$\Theta\left(x^*,y\right)+g\left(y\right)-g\left(x^*\right)\ge0,\ \forall y\in C.$$

Define

$$F\left(x,y\right):=\Theta\left(x,y\right)+g\left(y\right)-g\left(x\right),\ \forall x,y\in C.$$

It is easy to show that the bifunction $F\left(x,y\right):C\times C\to(-\infty,+\infty]$ has the following properties:

(A_1) $F\left(x,x\right)=0$;

(A_2) F is monotone, *i.e.*, $F\left(x,y\right)+F\left(y,x\right)\le0,\ \forall x,y\in C$;

(A_3) The function $y\mapsto F\left(x,y\right)$ is convex for all $x\in C$;

Define a mapping $T_r:X\to C$ by

$$T_r\left(x\right)=\left\{z\in C:F\left(z,y\right)+\frac{1}{r}\left\langle\overrightarrow{yz},\overrightarrow{zx}\right\rangle\ge0,\ \forall y\in C\right\},\ x\in X.$$

Lemma 4. *Let C be a nonempty closed convex subset of a Hadamard space X. Let F be a bifunction satisfying assumptions* (A_1)-(A_3) *and*

(A_4) *For each* $\bar{x}\in X$ *and* $r>0$, *there exists a compact subset* $D_{\bar{x}}\subset C$ *containing a point* $y_{\bar{x}}\in D_{\bar{x}}\subset C$ *such that* $F\left(x,y_{\bar{x}}\right)+\frac{1}{r}\left\langle\overrightarrow{xy_{\bar{x}}},\overrightarrow{\bar{x}x}\right\rangle<0$ *whenever* $x\in C/D_{\bar{x}}$.

Then, the following conclusions hold:

(a) T_r *is well defined in X and* T_r *is single-valued*;

(b) T_r *is firmly nonexpansive restricted to C, i.e.,* $\forall x,y\in C$,

$$d^2\left(T_rx,T_ry\right)\le\left\langle\overrightarrow{T_rxT_ry},\overrightarrow{xy}\right\rangle;$$

(c) $Fix\left(T_r\right)=\Omega$, *where* Ω *is the solution set of problem* (1) (*i.e., the set of* *minimizers of problem* (1));

(d) *For* $Fix(T_r) \neq \varnothing$, *one has*

$$d^2(T_r x, x) \leq d^2(x, p) - d^2(T_r x, p), \forall x \in C, \forall p \in Fix(T_r). \quad (12)$$

Proof. The result is a special case of Theorem 4 and Theorem 5 in [22], so we omit the proof here. □

3. Δ-Convergence Theorems

We are in a position to give our main theorems. Throughout this section we assume that

(1) (X, d) is a Hadamard space and C is a nonempty closed convex subset of X;

(2) $f, g: C \to (-\infty, +\infty]$ are proper convex functions and the bifunction $F(x, y)$ satisfies the assumption (A_4);

(3) $T_i: C \to C, i = 1,2$ is an L-Lipschitzian and quasi-pseudo-contractive mapping with $L \geq 1$, $I - T_i$ is demiclosed at 0;

(4) Denote

$$K_i := (1-\xi_i) I \oplus \xi_i T_i ((1-\eta_i) I \oplus \eta_i T_i)$$

with $0 < \xi_i < \eta_i < \dfrac{1}{1+\sqrt{1+L^2}}, i = 1,2$.

Theorem 1. *Let* $X, C, f, g, T_1, T_2, K_1, K_2$ *be the same above. For any given* $x_0 \in C$, *define the sequence* $\{x_n\} \subset C$ *as follows:*

$$\begin{cases} u_n = T_r x_n, \\ y_n = \beta_n K_1 u_n \oplus (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n K_2 y_n \oplus (1-\alpha_n) x_n, \end{cases} \quad (13)$$

where $r > 0$, $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0,1]$ *with* $0 < a \leq \alpha_n, \beta_n < b < 1, \forall n \geq 0$. *If the solution set* Γ *of problem* (11) *is nonempty, then the sequence* $\{x_n\}$ *Δ-converges to a point* $x^* \in \Gamma$, *which is a minimizer of* f, g *in C and also a common fixed point of* T_1, T_2 *in C.*

Proof. Step 1. It follows from Lemma 4 (c) that if $p \in \Gamma$, then $p \in \bigcap_{i=1}^{2} Fix(T_i) \cap Fix(T_r)$. Besides, by Lemma 3 (ii) we have $I - K_i$ $(i = 1,2)$ is demiclosed at 0.

Step 2. Next we prove that $\{x_n\}$ is Fejér monotone with respect to Γ. In fact, by Lemma 4 (b), T_r is firmly nonexpansive, then it is nonexpansive. Let $p \in \Gamma$, then one has

$$d(u_n, p) = d(T_r x_n, T_r p) \leq d(x_n, p). \quad (14)$$

It follows from (13) and (14) that

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n) x_n \oplus \beta_n K_1 u_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(K_1 u_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(u_n, p) \\ &\leq d(x_n, p). \end{aligned} \quad (15)$$

From (13), (14) and (15) we obtain

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)x_n \oplus \alpha_n K_2 y_n, p) \\ &\le (1-\alpha_n)d(x_n,p) + \alpha_n d(K_2 y_n, p) \\ &\le (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p) \\ &\le d(x_n,p), \end{aligned} \tag{16}$$

which implies that $\{d(x_n,p)\}$ is decreasing and bounded below. Thus the limit $\lim_{n\to+\infty} d(x_n,p)$ exists for each $p\in\Gamma$. It implies that $\{x_n\}$ is Fejér monotone with respect to Γ. Without loss of generality, we can assume that

$$\lim_{n\to+\infty} d(x_n,p) = c. \tag{17}$$

Therefore the sequence $\{x_n\}$ is bounded and so are the sequences $\{y_n\},\{u_n\},\{K_1u_n\},\{K_2y_n\}$.

Step 3. Now we prove that

$$\lim_{n\to+\infty} d(x_n,u_n) = 0. \tag{18}$$

In fact, it follows from (12) that

$$d^2(u_n,x_n) \le d^2(x_n,p) - d^2(u_n,p), \forall p\in\Gamma. \tag{19}$$

Hence in order to prove (18), it suffices to prove that $\lim_{n\to+\infty} d(u_n,p)=c$. Indeed, by (16) we get

$$d(x_{n+1},p) \le (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p),$$

which can be rewritten as

$$\begin{aligned} d(x_n,p) &\le \frac{1}{\alpha_n}\left(d(x_n,p) - d(x_{n+1},p)\right) + d(y_n,p) \\ &\le \frac{1}{a}\left(d(x_n,p) - d(x_{n+1},p)\right) + d(y_n,p), \end{aligned}$$

which together with (17) implies that

$$c = \liminf_{n\to+\infty} d(x_n,p) \le \liminf_{n\to+\infty} d(y_n,p). \tag{20}$$

Combing (15) and (17) we obtain

$$\limsup_{n\to+\infty} d(y_n,p) \le \limsup_{n\to+\infty} d(x_n,p) = c,$$

which together with (20) implies that

$$\lim_{n\to+\infty} d(y_n,p) = c. \tag{21}$$

Also, by (15) we have

$$d(y_n,p) \le (1-\beta_n)d(x_n,p) + \beta_n d(u_n,p).$$

Then one gets

$$\begin{aligned} d(x_n,p) &\le \frac{1}{\beta_n}\left(d(x_n,p) - d(y_n,p)\right) + d(u_n,p) \\ &\le \frac{1}{a}\left(d(x_n,p) - d(y_n,p)\right) + d(u_n,p), \end{aligned}$$

which together with (21) shows that

$$c = \liminf_{n\to+\infty} d(x_n, p) \le \liminf_{n\to+\infty} d(u_n, p).$$

On the other hand, it follows from (14) that

$$\limsup_{n\to+\infty} d(u_n, p) \le \limsup_{n\to+\infty} d(x_n, p) = c.$$

These imply that $\lim_{n\to+\infty} d(u_n, p) = c$. Thus by (19) one has that the equality (18) holds.

Step 4. In this step, we show that

$$\lim_{n\to+\infty} d(x_n, K_1u_n) = 0,\ \lim_{n\to+\infty} d(x_n, y_n) = 0 \text{ and } \lim_{n\to+\infty} d(x_n, K_2y_n) = 0.$$

In fact, it follows from (3), (13), (14) and Lemma 3 (iii) that

$$\begin{aligned} d^2(y_n, p) &= d^2((1-\beta_n)x_n \oplus \beta_n K_1u_n, p) \\ &\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(K_1u_n, p) - \beta_n(1-\beta_n)d^2(x_n, K_1u_n) \\ &\le (1-\beta_n)d^2(x_n, p) + \beta_n d^2(u_n, p) - \beta_n(1-\beta_n)d^2(x_n, K_1u_n) \\ &\le d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, K_1u_n), \end{aligned}$$

which together with (3), (13), (14) and Lemma 3 (iii) implies that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n K_2y_n, p) \\ &\le (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(K_2y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, K_2y_n) \\ &\le (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, K_2y_n) \\ &\le (1-\alpha_n)d^2(x_n, p) + \alpha_n\left[d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, K_1u_n)\right] \\ &\quad - \alpha_n(1-\alpha_n)d^2(x_n, K_2y_n) \\ &= d^2(x_n, p) - \alpha_n\beta_n(1-\beta_n)d^2(x_n, K_1u_n) - \alpha_n(1-\alpha_n)d^2(x_n, K_2y_n). \end{aligned}$$

After simplifying and by using the condition that $0 < a \le \alpha_n, \beta_n < b < 1$, one gets

$$\begin{aligned} &a^2(1-b)d^2(x_n, K_1u_n) + a(1-b)d^2(x_n, K_2y_n) \\ &\le d^2(x_n, p) - d^2(x_{n+1}, p) \to 0 \ (\text{as } n\to+\infty), \end{aligned}$$

which shows that

$$\lim_{n\to+\infty} d(x_n, K_1u_n) = 0 \text{ and } \lim_{n\to+\infty} d(x_n, K_2y_n) = 0. \tag{22}$$

Thus by (13) and (22), we get

$$d(y_n, x_n) = d((1-\beta_n)x_n \oplus \beta_n K_1u_n, x_n) = \beta_n d(x_n, K_1u_n) \to 0 \ (\text{as } n\to+\infty). \tag{23}$$

Furthermore, it follows form (18), (22) and (23) that

$$\begin{cases} d(u_n, K_1u_n) \le d(u_n, x_n) + d(x_n, K_1u_n) \to 0 \ (\text{as } n\to+\infty), \\ d(y_n, K_2y_n) \le d(y_n, x_n) + d(x_n, K_2y_n) \to 0 \ (\text{as } n\to+\infty), \\ d(T_r x_n, x_n) = d(u_n, x_n) \to 0 \ (\text{as } n\to+\infty). \end{cases} \tag{24}$$

Step 5. Finally, we prove that $\{x_n\}$ Δ-converges to some point $x^* \in \Gamma$. Since in the second step, we have shown that $\{x_n\}$ is bounded in C and it is Fejér monotone with respect to Γ. Then by Lemma 2, in order to prove $\{x_n\}$ Δ-converges to some point in Γ, it suffices to show that every Δ-sequential cluster point of $\{x_n\}$ belongs to Γ.

In fact, let x^* be a Δ-sequential cluster point of $\{x_n\}$, then there exits a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ Δ-converging to x^*. From (18) and (23), it follows that $\Delta - \lim_{j\to+\infty} u_{n_j} = x^*$ and $\Delta - \lim_{j\to+\infty} y_{n_j} = x^*$. Since T_r is nonexpansive, $I - T_r$ is demiclosed at 0. Note that $I - K_1$ and $I - K_2$ are also demiclosed at 0 by Lemma 3 (ii). Now by (24) and Lemma 3 (i), we obtain $x^* \in \bigcap_{i=1}^2 Fix(K_i) \cap Fix(T_r) = \bigcap_{i=1}^2 Fix(T_i) \cap Fix(T_r) = \Gamma$. Therefore, by Lemma 2, $\{x_n\}$ Δ-converges to some point in Γ. The proof is completed. □

4. Strong Convergence Theorems

Let (X,d) be a Hadamard space and C be a nonempty closed convex subset of X. Recall that a mapping $T: C \to C$ is said to be *demi-compact*, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to +\infty$), then there is a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly (*i.e.*, in metric topology) to some point in C.

Theorem 2. *Let all the conditions in Theorem* 1 *be satisfied and* T_r *be demi-compact restricted to C, then the sequence* $\{x_n\}$ *defined by* (13) *converges strongly to a point* $p^* \in \Gamma$.

Proof. Indeed, since T_r is demi-compact restricted to C, it follows from (24) that there is a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $p^* \in C$. Since $I - T_r$ is demiclosed at 0, we have $p^* \in Fix(T_r)$.

Moreover, it follows from (18) and (23) that $\{u_{n_i}\} \to p^*$ and $\{y_{n_i}\} \to p^*$ as $i \to +\infty$. Since $I - K_i$ $(i = 1,2)$ is demi-closed at 0, by (24) we have $p^* \in \bigcap_{i=1}^2 Fix(K_i) = \bigcap_{i=1}^2 Fix(T_i)$. Hence $p^* \in \Gamma$. Besides, it follows form (17) that $\lim_{n\to+\infty} d(x_n, p^*)$ exists. Thus we get $\lim_{n\to+\infty} d(x_n, p^*) = 0$. The proof is completed. □

Theorem 3. *Suppose that all the conditions in Theorem* 1 *are satisfied. Moreover, let* $\varphi: [0,+\infty) \to [0,+\infty)$ *be a nondecreasing function with* $\varphi(0) = 0, \varphi(r) > 0, \forall r > 0$ *and*

$$\varphi(d(x,\Gamma)) \le d(x, T_r x), \forall x \in C. \tag{25}$$

then the sequence $\{x_n\}$ *defined by* (13) *converges strongly to a point* $q^* \in \Gamma$.

Proof. It follows form (24) and (25) that

$$\lim_{n\to+\infty} \varphi(d(x_n,\Gamma)) = 0.$$

Since φ is nondecreasing with $\varphi(0) = 0$ and $\varphi(r) > 0, \forall r > 0$, we have

$$\lim_{n\to+\infty} d(x_n,\Gamma) = 0,$$

which implies that

$$\lim_{n,m\to+\infty} d(x_n, x_m) \le \lim_{n\to+\infty} d(x_n,\Gamma) + \lim_{m\to+\infty} d(x_m,\Gamma) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Noting that C is closed and convex in the Hadamard space X, C is also complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly to some point $q^* \in C$. Then

$d\left(q^*,\Gamma\right)=\lim_{n\to+\infty} d\left(x_n,\Gamma\right)=0$. Besides, since $K_i\ (i=1,2)$ is quasi-nonexpansive and T_r is nonexpansive, it is clear that $\Gamma=\bigcap_{i=1}^{2} Fix\left(T_i\right)\cap Fix\left(T_r\right)=\bigcap_{i=1}^{2} Fix\left(K_i\right)\cap Fix\left(T_r\right)$ is closed in *C*. Thus we get $q^*\in\Gamma$. The proof is completed. □

5. Conclusion and Remarks

Let us conclude this paper with some open questions whose answers might largely improve the applicability of the results in this present paper.

Question. Whether or not we can improve the (A_4) condition: For each $\overline{x}\in X$ and $r>0$, there exists a compact subset $D_{\overline{x}}\subset C$ containing a point $y_{\overline{x}}\in D_{\overline{x}}\subset C$ such that $F\left(x,y_{\overline{x}}\right)+\frac{1}{r}\left\langle \overrightarrow{xy_{\overline{x}}},\overrightarrow{\overline{x}x}\right\rangle<0$ whenever $x\in C/D_{\overline{x}}$, in order to obtain similar results regarding the resolvent operator T_r?

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Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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