Journal of Applied Mathematics and Physics, 2020, 8, 519-526
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Fekete-Szegö Estimate for a Class of Starlike Functions Involving Certain Analytic Multiplier Transform

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How to cite this paper: Makinde, D.O., Oyekunle, A.S. and Opoola, T.O. (2020) Fekete-Szegö Estimate for a Class of Starlike Functions Involving Certain Analytic Multiplier Transform. *Journal of Applied Mathematics and Physics*, **8**, 519-526.
https://doi.org/10.4236/jamp.2020.83041

Received: January 25, 2020
Accepted: March 14, 2020
Published: March 17, 2020

Abstract

In this paper, we investigated the coefficient estimates and the Fekete-Szegö problem for the subclass of analytic univalent functions involving the linear transformation $D^{s}_{\alpha,\beta,\gamma} f$ for the normalized analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.

Keywords

Analytic, Univalent, Starlike, Linear Transformation, Coefficient Estimates, Fekete-Szegö Inequality

1. Introduction

For the normalized analytic function *f* of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad a_n \in C \tag{1}$$

in the unit disk $U = \{z : |z| < 1\}$, Fekete and Szegö [1], proved that,

$$\left|a_3 - \lambda a_2^2\right| \le 1 + 2e^{-\frac{2\lambda}{1-\lambda}}, \quad 0 < \lambda \le 1. \tag{2}$$

And for the Schwarzian derivative S_f given by

$$S_f = \left(\frac{f''}{f}\right) - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Simple calculation shows that the coefficient functional $\phi_f(\lambda) = a_3 - \lambda a_2^2$ is related to the Schwarzian derivative by

$$\phi_f(\lambda) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}\left(f''(0)\right)^2 \right)$$

on normalized analytic functions f in the unit disk. Kanas and Darwish [2] remarked that, when $\lambda = 1, \phi_f(\lambda) = a_3 - a_2^2$, becomes $\frac{S_f(0)}{6}$, where S_f denotes the Schwarzian derivative given in Equation (3) and that if we consider the nth root transformation

$$\left(f\left(z^n\right)\right)^{\frac{1}{n}} = z + c_{n+1}z^{2n+1}c_{n+1}z^{2n+1} + \cdots$$

of the function in Equation (1), then $c_{n+1} = \frac{a_2}{2}$ and $c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}$, so that

$$a_3 - \lambda a_2^2 = n\left(c_{2n+1} - \mu c_{n+1}^2\right)$$

where $\mu = \lambda n + (n-1)/2$. Several authors have discussed the nature of $\phi_f(\lambda)$ for the normalized univalent functions in the unit disk. This is known as Fekkete-Szegö problem. Several authors have discussed the nature of $\phi(f)$ for classes of normalized univalent functions in the unit disk and this is known as Fekkete-Szegö problem. For example, Choi, Kim and Sugawa [3], gave a generalized prestarlike function, while in [4] Fekete-Szegö problem was solved using subordination principle. Moreover, in [5] [6] [7] [8] and [9] Fekete-Szegö problems were solved for class of close-to-convex functions. Authors in [10] [11] [12] and [13] also solved Fekete-Szegö for classes of normalized analytic functions.

Now, we denote by *S*, the set of all functions of the form (1) that are normalized analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. Let $S^*(\alpha), S^c(\alpha)$ be the classes of starlike and convex univalent function of order α, of the form:

Now, we denote by *S*, the set of all functions of the form (1) that are normalized analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. Let $S^*(\alpha), S^c(\alpha)$ be the classes of starlike and convex univalent function of order α, of the form:

$$S^* = \left\{ f \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \beta, 0 \le \beta < 1, z \in U \right\} \quad (4)$$

Several authors have generalized notions of α-starlikeness and α-convexity onto a complex order α see [14] [15] [16]. When $\alpha = 0$ in Equations (4) and (5), the starlike, respectively, convex functions with respect to the origin are obtained. With the aid of Ruscheweyh derivative, Kumar *et al.* [17] introduced the class $S_n(b)$ of functions $f \in S$ as follows:

Definition 1 *Let b be a nonzero complex number, and let f be a univalent function of the form* (1) *such that* $D^n f(z) \ne 0$ *for* $z \in U - \{0\}$. *We say that f belongs to* $S_n(b)$ *if*

$$\operatorname{Re}\left\{ 1 + \frac{1}{b}\left(\frac{z\left(D^n f\right)'(z)}{D^n f(z)} - 1 \right) \right\} > 0 \quad (5)$$

Moreover, the author in [18] defined a linear transformation $D^s_{\alpha,\beta,\gamma} f$ by

$$D^s_{\alpha,\beta,\gamma} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta + n^2\gamma}{\alpha + \beta + \gamma} \right)^s a^i_n z^n,\ \beta,\gamma \geq 0; \alpha \geq 1; s \in \mathbb{N} \cup 0, i(1 \leq i \leq k). \quad (6)$$

where $k \in \mathbb{N}$.

Motivated by the work of Kanas and Darwish, using the subclass of Kumar et al, involving the linear transformation in Equation (6), we study the coefficient estimates and solved the Fekete-Szegö problem for the subclass $S_n(b)$ involving the linear transformation $D^s_{\alpha,\beta,\gamma}$.

Definition 2 *Let b be a nonzero complex number, and f a univalent function of the form* (1) *such that* $H^n(z) \neq 0$ *for* $z \in U - \{0\}$. *We say that f belongs to* $S_n(b)$ *if*

$$\operatorname{Re}\left\{ 1 + \frac{1}{b} \left(\frac{z(H^n)'(z)}{H^n(z)} - 1 \right) \right\} > 0,\ z \in U, \quad (7)$$

where $H = D^s_{\alpha,\beta,\gamma} f$ *is as given in Equation* (6).

The following results shall be employed in the proof of the main results of this study.

Lemma 1 [19] *Let P be the class of analytic functions in U with* $p(0) = 1$, *Re* $p(z) > 0$ *and of the form*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (8)$$

then

$$|c_n| \leq 2,\ n \geq 1.$$

If $|c_1| = 2$, *then* $p(z) \equiv p_1 = \frac{1+\gamma_1 z}{1-\gamma_1 z}$ *with* $\gamma_1 = \frac{c_1}{2}$. *Conversely, if* $p(z) \equiv p_1$ *for some* $\gamma_1 = 1$, *then* $c_1 = 2\gamma_1$ *and* $|c_1| = 2$. *Furthermore, we have*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ *and* $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$, *then* $p(z) \equiv p_2$, *where*

$$p_2(z) = \frac{1 + z\dfrac{\gamma_2 z + \gamma_1}{1 + \gamma_1\gamma_2 z}}{1 - z\dfrac{\gamma_2 z + \gamma_1}{1 + \gamma_1\gamma_2 z}}$$

and $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. *Conversely, if* $p(z) = p_2$ *for some* $\gamma_1 < 1$ *and* $\gamma_2 = 1$, *then* $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ *and* $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$.

In what follows, we give the statement and proof of the results of this study.

2. Coefficient Estimates for...

Theorem 1 *Let* $n \geq 0$ *and b a non-zero complex number. If f of the form* (1) *is in* $S_n(b)$, *then*

$$\left|a_2^i\right| \leq 2|b|\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s$$

and

$$\left|a_3^i\right| \leq |b|\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \max\left[1,|1+2b|\right], \beta,\gamma \geq 0; \alpha \geq 1; s \in \mathbb{N} \cup 0, i(1 \leq i \leq k).$$

Proof 1 *Let* $f \in S_n(b)$, *then by definition* 2, *there exist a class of analytic function p given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

satisfying $P(0) = 1$ *and* $\operatorname{Re}(p(z)) > 0$ *such that*

$$1+\frac{1}{b}\left(\frac{z\left(H^n\right)'(z)}{H^n(z)}-1\right) = 1 + c_1 z + c_2 z^2 + \cdots \tag{9}$$

where $H = D^s_{\alpha\beta\gamma}$.

From Equation (9), *we have*:

$$\frac{z\left(H^n\right)'(z)}{H^n(z)} = 1 + b\left(p(z)-1\right) \tag{10}$$

Equating coefficients in Equation (10) *using Equation* (6) *with* $D^s_{\alpha,\beta,\gamma} f(z) = z + A_2 z^2 + A_3 z^3 + \cdots$, *we have*

$$A_2 = bc_1 \tag{11}$$

$$A_3 = \frac{b}{2}\left[c_2 + bc_1^2\right] \tag{12}$$

$$\equiv \frac{b}{2}\left(c_2 - \frac{c_1^2}{2}\right) + (1+2b)\frac{bc_1^2}{4} \tag{13}$$

From Equations (12) *and* (13) *using Equation* (6), *we have*,

$$a_2 = b\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s c_1 \tag{14}$$

respectively

$$a_3 = \frac{b}{2}\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \left[c_2 + bc_1^2\right] \tag{15}$$

On the account of Equations (14) *and* (15) *using Lemma* 1, *we have*

$$|a_2| = \left| b\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s c_1 \right| \le 2|b|\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s$$

and

$$\begin{aligned} |a_3| &= \left| \frac{b}{2}\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \left[c_2 - \frac{c_1}{2} + \frac{1+2b}{2}c_1^2\right]\right| \\ &\le \frac{|b|}{2}\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \left[2 - \frac{|c_1|}{2} + \frac{|1+2b|}{2}|c_1|^2\right] \\ &= \frac{|b|}{2}\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \left[2 + \frac{|c_1|}{2}\left(|1+2b|-1\right)\right] \\ &\le |b|\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \left[1, 1+|1+2b|-1\right] \\ &= |b|\left(\frac{\alpha+\beta+\gamma}{\alpha+2\beta+4\gamma}\right)^s \max\left[1, |1+2b|\right]. \end{aligned}$$

which proves theorem 1.

3. The Fekete-Szegö Problem for the Subclasses $S_n(b)$

***Theorem* 2** *Let b be a nonzero complex number and $f \in S_n(b)$. Then $\mu \in C$, the following holds.*

$$|a_3 - \mu a_2^2| \le b\left(\frac{\alpha+\beta+\gamma}{\alpha+3\beta+9\gamma}\right)^s \max\left\{1, \left|1+2b-2b\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right|\right\}$$

***Proof* 2** *From Equations* (14) *and (*15), *we have*

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b}{2}t_3^{-s}\left[c_2 + bc_1^2\right] - \mu\left(bt_2^{-s}c_1\right)^2 \\ &= \frac{b}{2}t_3^{-s}\left[c_2 + bc_1^2 - 2b\mu t_3^{-s}t_2^{2s}c_1^2\right] \\ &= \frac{b}{2}t_3^{-s}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1+2b-2b\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right)\right] \end{aligned}$$

where $t_2 = \left(\frac{\alpha+2\beta+4\gamma}{\alpha+\beta+\gamma}\right)$ *and* $t_3 = \left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma}\right)$.

Applying Lemma 1 *to the above last inequality, we obtain*

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{b}{2}t_3^{-s}\left[2 + \frac{c_1^2}{2}\left(\left|1+2b-2b\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right|-1\right)\right] \\ &\le bt_3^{-s}\max\left\{1, \left(\left|1+2b-2b\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right|\right)\right\} \end{aligned} \tag{16}$$

This proves the theorem.

Theorem 3 *Let b be a nonzero complex number and $f \in S_n(b)$. Then $\mu \in \mathbb{R}$, the following holds.*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} bt_3^{-s}\left[\left|1+2b\left(1-\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right)\right|\right] & \text{if } \mu \le t_3^{-s} \\ bt_3^{-s} & \text{if } t_3^{-s} \le \mu \le t_3^{-s}\frac{1+2b}{2b} \\ bt_3^{-s}\left[\left|2b\left(\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}-1\right)-1\right|\right] & \text{if } \mu \ge t_3^{-s}\frac{1+2b}{2b} \end{cases} \quad (17)$$

where $t_3=\left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma}\right)$.

Proof 3 *Let* $\mu \le t_3^{-s}$. *From Equation* (17), *we have*

$$\left|a_3-\mu a_2^2\right| \le bt_3^{-s}\left[\left|1+2b\left(1-\mu\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}\right)\right|\right]$$

Now, using the above calculations with $t_3^{-s} \le \mu \le t_3^{-s}\frac{1+2b}{2b}$, *we have*

$$\left|a_3-\mu a_2^2\right| \le bt_3^{-s}$$

and conclusively, let $\mu \ge t_3^{-s}\frac{1+2b}{2b}$, *then*

$$\left|a_3-\mu a_2^2\right| \le \frac{b}{2}t_3^{-s}\left[2+\frac{|c_1|^2}{2}\left(2\mu b\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}-2-2b\right)\right]$$

$$\le bt_3^{-s}\left[2\mu b\frac{(\alpha+3\beta+9\gamma)^s}{(\alpha+2\beta+4\gamma)^{2s}}-1-2b\right]$$

This concludes the proof of the theorem 3.

4. Conclusions

The result in this paper extends the work of Kanas and Darwish as it is evident that for $s=1$ and $\alpha+\beta+\gamma=\frac{\alpha+2\beta+4\gamma}{2}$, $s=0$ in the first part, respectively second part of the theorem 1 yields the first part respectively second part of the theorem 2.2 for $n=0$ in Kanas and Darwish. Moreover when $n>1$ and $\beta,\gamma \ge 0; \alpha \ge 1$ in Equation (6), the result in this study gives finer initial coefficient estimates and bound for Fekete-Szegö problem.

It will also be interesting to check the effect of the linear transformation given in (6) on other subclasses of normalized analytic functions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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