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Some Inequalities on *p*-Valent Functions Related to Geometric Structure Based on *q*-Derivative

Shahram Najafzadeh[1], Deborah Olufunmilayo Makinde[2*]

[1]Department of Mathematics, Payame Noor University, Tehran, Iran
[2]Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
Email: najafzadeh1234@yahoo.ie, *funmideb@yahoo.com, domakinde.comp@gmail.com

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Abstract

By applying the q-derivative, we introduce two new subclasses of p-valent functions with positive coefficients. By means of the well-known Jack's lemma, some inequalities related to starlike, convex and close-to-convex functions are also obtained.

Keywords

p-Valent Functions, Jack's Lemma, Starlike, Convex and Close-to-Convex Functions

1. Introduction

By $\mathcal{A}_p(n)$, we denote the class of functions of the type:

$$f(z) = z^p + \sum_{k=n+p}^{+\infty} a_k z^k, \quad (n, p \in \mathbb{N}), \tag{1}$$

which are p-valent and analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, see [1].

Now, we introduce some basic definitions and related details of the q-calculus, see [2] [3] [4].

The q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by:

$$(\alpha;q)_n = \begin{cases} 1, & n = 0, \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{2}$$

and according to the basic analogue of the gamma function, we get:

$$(q^\alpha;q)_n = \frac{(1-q)^n \Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}, \quad (n > 0), \tag{3}$$

where the q-gamma function is given by:

$$\Gamma_q(x)=\frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty},\quad (0<q<1). \tag{4}$$

If $|q|<1$ the relation (2) is meaningful for $n=\infty$ as a convergent product defined by:

$$(\alpha;q)_\infty=\prod_{j=0}^{\infty}(1-\alpha q^j). \tag{5}$$

Further, we conclude that

$$\Gamma_q(x+1)=\frac{(1-q^x)\Gamma_q(x)}{1-q}. \tag{6}$$

For $0<q<1$, the q-derivative of a function f is defined by:

$$\partial_q f(z)=\frac{f(qz)-f(z)}{z(q-1)},\quad (z\neq 0, q\neq 1). \tag{7}$$

A simple calculation yields that for $m\in\mathbb{N}$ and $\lambda>-1$,

$$\partial_q^m z^\lambda=\frac{\Gamma_q(\alpha)(1+\lambda)}{\Gamma_q(\alpha)(1+\lambda-m)}z^{\lambda-m}. \tag{8}$$

Also, in view of the following relation:

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n, \tag{9}$$

we note that the q-shifted factorial (2) reduces to the well-known Pochhammer symbol $(\alpha)_n$ [5], which is defined by:

$$(\alpha)_n=\begin{cases}1, & n=0,\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & n\in\mathbb{N}.\end{cases}$$

Differentiating (1) m times with respect to z (8), we conclude

$$\partial_q^m f(z)=\frac{\Gamma_q(1+p)}{\Gamma_q(1+p-m)}z^{p-m}+\sum_{k=n+p}^{\infty}\frac{\Gamma_q(1+k)}{\Gamma_q(1+k-m)}a_k z^{k-m}. \tag{10}$$

A function $f(z)\in\mathcal{A}_p(n)$ is said to be in the subclass $X_p(n,m)$ if it satisfies the inequality:

$$\left|\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\frac{\partial_q^m f(z)}{z^{p-m}}-1\right|<1, \tag{11}$$

where $z\in\mathbb{D}$, $p\in\mathbb{N}$, $0<q<1$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Indeed $f(z)\in\mathcal{A}_p(n)$ is said to be in the subclass $Y_p(n,m)$ if it satisfies the inequality:

$$\left|\frac{z\left(\partial_q^m f(z)\right)}{\partial_q^m f(z)}-(p-m)\right|<p-m. \tag{12}$$

For details see [6].

2. Main Results

To prove the main theorems related to $X_p(n,m)$ and $Y_p(n,m)$, we need the following lemma due to Jack [7] [8].

Lemma 1. *Let* $w(z)$ *e non-constant in* $\mathbb{D}$ *and* $w(0)=0$. *If* $|w|$ *attains its maximum value on the circle* $|z|=r<1$ *at* z_0, *then* $z_0 w'(z_0)=tw(z_0)$, *where* $t\ge 1$ *is a real number.*

A function $f(z)\in\mathcal{A}_p(n)$ is said to be in the subclass $\mathcal{A}_p\mathcal{K}(n)$ of *p*-valently close-to-convex functions with respect to the origin in $\mathbb{D}$ if

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>0,\quad (z\in\mathbb{D},p\in\mathbb{N}).$$

Also, $f(z)\in\mathcal{A}_p\mathcal{K}(n)$ is said to be in the subclass $\mathcal{A}_p\mathcal{S}(n)$ of *p*-valently starlike functions with respect to the origin in $\mathbb{D}$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad (z\in\mathbb{D},p\in\mathbb{N}).$$

Further $f(z)\in\mathcal{A}_p(n)$ is said to be in the subclass $\mathcal{A}_p\mathcal{C}(n)$ of *p*-valently convex functions with respect to the origin in $\mathbb{D}$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,$$

see [9] [10].

Theorem 2. *If* $f(z)\in\mathcal{A}_p(n)$ *satisfies the inequality:*

$$\left\{\frac{z\left(\partial_q^m f(z)\right)}{\partial_q^m f(z)}-(p-m)\right\}<\frac{1}{2},\tag{13}$$

then $f(z)\in X_p(n,m)$.

Proof. Let $f(z)\in\mathcal{A}_p(n)$, we define the function $w(z)$ by:

$$\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\frac{\partial_q^m f(z)}{z^{p-m}}=1+w(z),\quad (z\in\mathbb{D},p\in\mathbb{N},n\in\mathbb{N}_0).\tag{14}$$

with a simple calculation we have $w(0)=0$ (in $\mathbb{U}$).

For (14), we obtain:

$$\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\partial_q^m f(z)=z^{p-m}+z^{p-m}w(z),$$

or

$$\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\left(\partial_q^m f(z)\right)'=(p-m)z^{p-m-1}+(p-m)z^{p-m-1}w(z)+z^{p-m}w'(z),$$

or equivalently

$$\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\frac{\left(\partial_q^m f(z)\right)'}{z^{p-m-1}}=(p-m)(1+w(z))+zw'(z).\tag{15}$$

From (14) and (15), we get:

$$\frac{zw'(z)}{1+w(z)}=\frac{z\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)}-(p-m). \tag{16}$$

Now, let for $z_0\in\mathbb{D}$, $\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$, then by using Jack's lemma and putting $w(z_0)=\mathrm{e}^{i\theta}\neq -1$ in (16), we have:

$$\operatorname{Re}\left\{\frac{z\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)}-(p-m)\right\}=\left\{\frac{z_0 w'(z_0)}{1+w(z_0)}\right\}=\operatorname{Re}\left\{\frac{tw(z_0)}{1+w(z_0)}\right\}$$

$$=\operatorname{Re}\left\{\frac{t\mathrm{e}^{i\theta}}{1+\mathrm{e}^{i\theta}}\right\}=\operatorname{Re}\left\{\frac{t(\cos\theta+i\sin\theta)}{(1+\cos\theta)+i\sin\theta}\right\}$$

$$=\operatorname{Re}\left\{\frac{t(\cos\theta+i\sin\theta)\left((1+\cos\theta)-i\sin\theta\right)}{(1+\cos\theta)+i\sin\theta\left((1+\cos\theta)-i\sin\theta\right)}\right\}$$

$$=\operatorname{Re}\left\{\frac{t(1+\cos\theta+i\sin\theta)}{2+2\cos\theta}\right\}$$

$$=\operatorname{Re}\left\{\frac{t(1+\cos\theta)}{2+2\cos\theta}+\frac{it\sin\theta}{2+2\cos\theta}\right\}=\frac{t}{2}\geq\frac{1}{2},$$

which is a contradiction with (13). Thus we have $|w(z)|<1$ for all $z\in\mathbb{D}$, so from (14) we conclude:

$$\left|\frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)}\frac{\partial_q^m f(z)}{z^{p-m}}-1\right|=|w(z)|<1,$$

and this gives the result.

By letting $m=0$ and ($m=1, q\to 1$), we have the following corollaries which are due to Irmak and Cetin [11].

Corollary 3. *If* $f(z)\in\mathcal{A}_p(n)$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf'}{f}-p\right\}<\frac{1}{2},\quad (z\in\mathbb{D}, p\in\mathbb{N}),$$

then $\left|\frac{f(z)}{z^p}-1\right|<1$.

Corollary 4. *If* $f(z)\in\mathcal{A}_p(n)$ *satisfies the inequality*

$$\operatorname{Re}\left\{1+\frac{zf''}{f'}-p\right\}<\frac{1}{2},\quad (z\in\mathbb{D}, p\in\mathbb{N}),$$

then $f(z)\in\mathcal{A}_p\mathcal{K}(n)$ *and* $\left|\frac{f'}{z^{p-1}}-p\right|<p$.

Theorem 5. *If* $f(z)\in\mathcal{A}_p(n)$ *satisfies*

$$\left\{1+\left[\frac{\left(\partial_q^m f(z)\right)''}{\left(\partial_q^m f(z)\right)'}-\frac{\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)}\right]\right\}<\frac{1}{2},\quad (z\in\mathbb{D}, p\in\mathbb{N}, n\in\mathbb{N}_0), \tag{17}$$

then $f(z)\in Y_p(n,m)$.

Proof. Let the function $f(z) \in \mathcal{A}_p(n)$, we define the function $w(z)$ by

$$\frac{z\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)} = p\left(1 + w(z)\right). \tag{18}$$

It is easy to verify that $w(z)$ is analytic in $\mathbb{D}$ and $w(0) = 0$. By (18), we have:

$$z\left(\partial_q^m f(z)\right)' = p\partial_q^m f(z) + p\partial_q^m f(z) w(z),$$

or

$$\begin{aligned}&\left(\partial_q^m f(z)\right)' + z\left(\partial_q^m f(z)\right)'' \\ &= p\left(\partial_q^m f(z)\right)' + p\left(w'(z)\partial_q^m f(z) + w(z)\left(\partial_q^m f(z)\right)'\right),\end{aligned}$$

or

$$1 + \frac{z\left(\partial_q^m f(z)\right)''}{\left(\partial_q^m f(z)\right)'} = p\left(1 + w(z)\right) + pw'(z)\frac{\partial_q^m f(z)}{\left(\partial_q^m f(z)\right)'},$$

or by (18) we get

$$1 + \frac{z\left(\partial_q^m f(z)\right)''}{\left(\partial_q^m f(z)\right)'} = p\left(1 + w(z)\right) + \frac{zw'(z)}{1 + w(z)}.$$

Now, let for a point $z_0 \in \mathbb{D}$, $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$. By Jack's lemma and putting $w(z_0) = \mathrm{e}^{i\theta}$ we conclude:

$$\begin{aligned}&\mathrm{Re}\left\{1 + z\left[\frac{\left(\partial_q^m f(z)\right)''}{\left(\partial_q^m f(z)\right)'} - \frac{\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)}\right]\right\} \\ &= \mathrm{Re}\left\{\frac{z_0 w'(z_0)}{1 + w(z_0)}\right\} = \mathrm{Re}\left\{\frac{tw(z_0)}{1 + w(z_0)}\right\} = \mathrm{Re}\left\{\frac{t\mathrm{e}^{i\theta}}{1 + \mathrm{e}^{i\theta}}\right\} > \frac{t}{2} \ge \frac{1}{2},\end{aligned}$$

which is contradiction with (17). Thus for all $z \in \mathbb{D}$, $|w(z)| < 1$ and so from (18), we have:

$$\left|\frac{z\left(\partial_q^m f(z)\right)'}{\partial_q^m f(z)} - p\right| < p,$$

thus the proof is complete.

By letting $m = 0$ and ($m = 1, q \to 1$) we have the following corollaries that the first one is due to Irmak and Cetin [5].

Corollary 6. *If* $f(z) \in \mathcal{A}_p(n)$ *satisfies the inequality*

$$\mathrm{Re}\left\{1 + z\left(\frac{f''}{f'} - \frac{f'}{f}\right)\right\} < \frac{1}{2}, \quad \left(z \in \mathbb{D}, p \in \mathbb{N}\right),$$

then $f(z) \in \mathcal{A}_p\mathcal{S}(n)$ *and* $\left|\frac{zf'}{f} - p\right| < p$.

3. Conclusion

Studying the theory of analytic functions has been an area of concern for many authors. Literature review indicates lots of researches on the classes of p-valent analytic functions. The interplay of geometric structures is a very important aspect in complex analysis. In this study, two new subclasses of p-valent functions were defined by using q-analogue of the well-known operators and we gave some geometric structures like starlike, convex and close-to-convex properties of the subclasses. It is noted that the study is an extension of some previous studies as it is shown in corollaries 3, 4, 6.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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