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# Exact Solutions for (2 + 1)-Dimensional KdV-Calogero-Bogoyavlenkskii-Schiff Equation via Symbolic Computation

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## Abstract

This paper constructs exact solutions for the (2 + 1)-dimensional KdV-Calogero-Bogoyavlenkskii-Schiff equation with the help of symbolic computation. By means of the truncated Painlev expansion, the (2 + 1)-dimensional KdV-Calogero-Bogoyavlenkskii-Schiff equation can be written as a trilinear equation, through the trilinear-linear equation, we can obtain the explicit representation of exact solutions for the (2 + 1)-dimensional KdV-Calogero-Bogoyavlenkskii-Schiff equation. We have depicted the profiles of the exact solutions by presenting their three-dimensional plots and the corresponding density plots.

## Keywords

(2 + 1)-Dimensional KdV-Calogero-Bogoyavlenkskii-Schiff Equation, Trilinear Equation, Exact Solutions

## 1. Introduction

Over the past few decades, the research of the nonlinear evolution equations (NLEEs) is flourishing because of the rich findings of these equations. NLEEs display many interesting nonlinear dynamic behaviors, such as plasma physics, optical systems, ocean, superfluids, hydrodynamics and other nonlinear fields [1] [2] [3] [4]. The solutions of NLEEs can provide much physical information and more insight into the physical aspects and then lead to further applications, so seeking exact solutions is an important problem in nonlinear science. In fact, there are a variety of powerful methods have been used to solve the NLEEs. For instance, the Darboux transformation [5], the Lie group method [6], the Hirota bilinear method [7], the inverse scattering method [8] and other methods. Recent study for constructing lump solution by taking the function $f$ in its bilinear form as a positive quadratic function based on Hirota bilinear method was obtained in [9]. Then, by using this method, more and more the lump solutions and the interaction solutions have been perfectly obtained in NLEEs equations [10]-[26].

In this paper, we focus on the following (2 + 1)-dimensional KdV-Calogero-Bogoyavlenkskii-Schiff (KdV-CBS) equation [27] [28] [29].

$$4u_t - \alpha\left(4uu_y + 2u_x\partial_x^{-1}u_y + 2u_{xxy}\right) - \beta\left(6uu_x + u_{xxx}\right) = 0, \tag{1}$$

which with two arbitrary constant $\alpha$ and $\beta$, Equation (1) can be reduced to some other equations with physical meanings. If we setting $\alpha \neq 0, \beta = 0$, Equation (1) is reduced to Calogero-Bogoyavlenskii-Schiff equation [30].

$$4u_t - \alpha\left(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}\right) = 0. \tag{2}$$

If setting $\{\alpha = 0, \beta \neq 0\}$, Equation (1) is famous KdV equation

$$4u_t - \beta\left(u_{xxx} + 6uu_x\right) = 0. \tag{3}$$

Let $v_x = u_y$, Equation (1) is transformed into the following system

$$u_t - \alpha\left(4uu_y + 2u_x v + u_{xxy}\right) - \beta\left(6uu_x + u_{xxx}\right) = 0, \tag{4a}$$

$$v_x - u_y = 0. \tag{4b}$$

By applying Painlevé truncated extension, some types of explicit solutions (1) were studied in [31]. A direct bilinear Bäcklund transformation on the basis of quadrilinear representation was constructed in [32] [33]. Ref. [34] obtained soliton solutions, quasiperiodic wave solutions of the Equation (1) based on the Riemann-Bäcklunk method. Interaction solutions of Equation (1) were obtained under consistent Riccati expansion in [29].

The outline of this paper is as follows. In Section 2, we convert the original KdV-CBS equation to a trilinear equation by using the truncated Painlevé expansion. In Section 3, we derived exact solutions for the KdV-CBS equation by taking function $f$ as a positive quadratic function; another kind of exact solutions were got by taking function $f$ as a positive quadratic and an exponential function; by taking function $f$ as two exponential terms and the quadratic function, we obtained three kinds of exact solutions respectively via different parameters. A short conclusion is included in the last section.

## 2. Trilinear Equation

Based on the Painlevé analysis proposed in Ref. [4], Equation (4a) and Equation (4b) possess a truncated Painlevé expansion as follows:

$$\begin{aligned} u &= u_0 + \frac{u_1}{f} + \frac{u_2}{f^2}, \\ v &= v_0 + \frac{v_1}{f} + \frac{v_2}{f^2}, \end{aligned} \tag{5}$$

with $u_0, u_1, v_0, v_1, v_2, f$ being the function of $x, y$ and *t*. We have

$$\begin{aligned} &v_1 = f_{xy}, v_2 = -2 f_x f_y, u_2 = -2 f_x^2, u_0 = \lambda - \frac{f_{xxx}}{2 f_x} + \frac{f_{xx}^2}{4 f_x^2}, u_1 = 2 f_{xx}, \\ &v_0 = \frac{-3\beta\lambda}{\alpha} + \frac{2 f_t}{\alpha f_x} - 2\lambda \frac{f_y}{f_x} + \frac{3\beta}{4\alpha}\frac{f_{xx}^2}{f_x^2} + \frac{f_{xx} f_{xy}}{f_x^2} - \frac{f_{xxy}}{f_x} - \frac{\beta}{2\alpha}\frac{f_{xxx}}{f_x}, \end{aligned} \tag{6}$$

the Equation (6) is from Ref. [35]. The function *f* satisfies the following trilinear equation,

$$\begin{aligned} &\alpha f_{xxxy} f_x^2 + 4\alpha\lambda f_x^2 f_{xy} + \beta f_{xxxx} f_x^2 - \alpha f f_x f_{xxx} f_{xy} - 4\beta f_{xxx} f_x f_{xx} - 3\alpha f_x f_{xx} f_{xy} \\ &- 4\lambda\alpha f_{xx} f_x f_y + 3\alpha f_{xy} f_{xx}^2 + 3\beta f_{xx}^3 - 4 f_x^2 f_{xt} + 4\alpha 4 f_x^2 f_x f_x = 0. \end{aligned} \tag{7}$$

By solving the Equation (5), we can get

$$u = \lambda - \frac{1}{2}\frac{f_{xxx}}{f_x} + \frac{f_{xx}^2}{4 f_x^2}, \tag{8a}$$

$$v = \frac{-3\beta\lambda}{\alpha} + \frac{2 f_t}{\alpha f_x} - 2\lambda \frac{f_y}{f_x} + \frac{3\beta}{4\alpha}\frac{f_{xx}^2}{f_x^2} + \frac{f_{xx} f_{xy}}{f_x^2} - \frac{f_{xxy}}{f_x} - \frac{\beta}{2\alpha}\frac{f_{xxx}}{f_x}. \tag{8b}$$

Based on the idea in Ref. [9], the (2 + 1)-dimensional KdV-Calogero-Bogoy-avlenkskii-Schiff equation has a set of rational solutions determined by

$$\begin{aligned} f = &\left(a_1 x + a_2 y + a_3 t + a_4\right)^2 + \left(a_5 x + a_6 y + a_7 t + a_8\right)^2 + a_9 \\ &+ m \exp\left(k_1 x + k_2 y + k_3 t + k_4\right) + n \exp\left(-\left(k_1 x + k_2 y + k_3 t + k_4\right)\right), \end{aligned} \tag{9}$$

where $a_i, k, k_j, 1 \le i \le 9, 1 \le j \le 4$ are real parameters to be determined. Substituting function (9) into trilinear Equation (7), we give five kinds of exact solutions of Equation (4a) and Equation (4b) via function (9).

## 3. Lumps and Interaction Solutions

Case I

Substitution function (9) into trilinear Equation (7), a set of constraining equations for the parameters can be obtained as following

$$\begin{aligned} &a_1 = 0, a_2 = a_2, a_3 = a_2\lambda\alpha, a_4 = a_4, \\ &a_5 = a_5, a_6 = -\frac{a_5\beta}{\lambda}, a_7 = a_7, a_8 = a_8, a_9 = a_9, \\ &m = 0, n = 0, k_1 = k_1, k_2 = k_2, k_3 = k_3, k_4 = k_4, \beta = \beta, \lambda = \lambda \end{aligned} \tag{10}$$

where

$$\lambda \ne 0, a_9 > 0 \tag{11}$$

to guarantee that the corresponding *f* is positive. By substituting Equation (10) into function (9), then we get the function *f* reads

$$f = \left(a_2 y + a_2\lambda\alpha t + a_4\right)^2 + \left(a_5 x - \frac{a_5\beta}{\lambda} y + a_7 t + a_8\right)^2 + a_9. \tag{12}$$

By means of Equation (8a) and Equation (8b), the solutions of (4a) and (4b) can be obtained

$$u=\lambda+\frac{a_5^2}{2A_1^2}, \tag{13a}$$

$$v=-\frac{3\beta\lambda}{\alpha}+\frac{2B_1}{A_1}-\frac{2\lambda C_1}{A_1}-\frac{\beta a_5^2}{\alpha A_1^2}, \tag{13b}$$

where

$$\begin{aligned}
&f=g^2+h^2+a_9,\\
&g=a_2y+a_2\lambda\alpha t+a_4,\\
&h=a_5x-\frac{a_5\beta}{\lambda}y+a_7t+a_8,\\
&A_1=2a_5f, B_1=2a_2\alpha\lambda f+2a_7g, C_1=2a_2f-2\frac{a_5\beta}{\lambda}g.
\end{aligned} \tag{14}$$

The solutions of $u$ via Equation (13a) which can be seen in Figure 1(a); Figure 1(b) is the corresponding density plot. The solutions of $v$ via Equation (13b) can be seen in Figure 1(c); Figure 1(d) is the corresponding density plot.

Case II

$$\begin{aligned}
&a_1=a_1, a_2=-\frac{a_1\beta}{\alpha}, a_3=a_3, a_4=a_4,\\
&a_5=0, a_6=a_6, a_7=a_6\alpha\lambda, a_8=a_8, a_9=a_9,\\
&m=m, k_1=0, k_2=k_2, k_3=k_2\alpha\lambda, k_4=k_4,
\end{aligned} \tag{15}$$

which should satisfy the constraint conditions

$$\alpha\neq 0, a_9>0, m>0, \tag{16}$$

to ensure that the corresponding $f$ is positive and well defined. By substituting Equation (15) into function (9), then the function $f$ reads

$$\begin{aligned}
f=&\left(a_1x-\frac{a_1\beta}{\alpha}y+a_3t+a_4\right)^2+\left(a_6y+a_6\alpha\lambda t+a_8\right)^2\\
&+m\exp\left(k_2y+k_2\alpha\lambda t+k_4\right)+a_9.
\end{aligned} \tag{17}$$

Using Equation (8a) and Equation (8b), the solutions for (4a) and (4b) can be obtained

$$u=\lambda+\frac{a_1^2}{4g^2}, \tag{18a}$$

$$v=-\frac{3\beta\lambda}{\alpha}+\frac{A_2\alpha\lambda B_2}{\alpha a_1 g}+\frac{3\beta a_1^2}{16\lambda g^2}-\frac{a_1^3\beta}{\alpha g^2}, \tag{18b}$$

where

$$\begin{aligned}
&f=g^2+h^2+m\exp(\xi)+a_9,\\
&g=a_1x-\frac{a_1\beta}{\alpha}y+a_3t+a_4,\\
&h=a_6y+a_6\alpha\lambda t+a_8,\\
&A_2=2a_3g+2a_6\alpha\lambda h+k_2m\alpha\lambda\exp(\xi),\\
&B_2=-2\frac{a_1\beta g}{\alpha}+2a_6h+k_2m\exp(\xi),\\
&\xi=k_2y+k_2\alpha\lambda t+k_4,
\end{aligned} \tag{19}$$

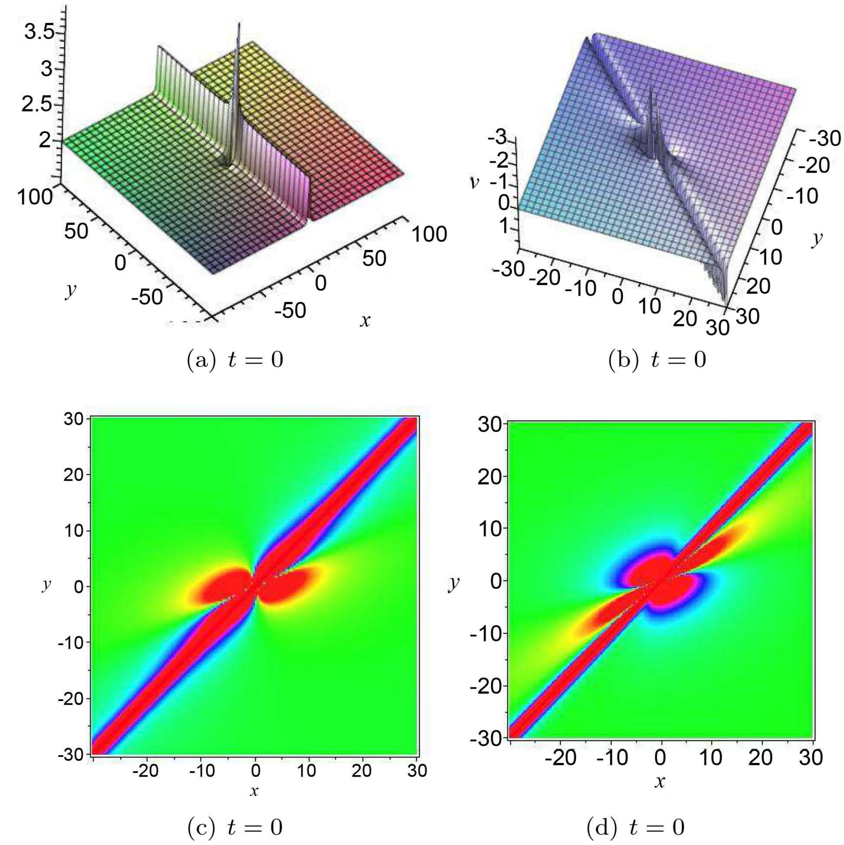



**Figure 1.** Profiles of the solution $u$ via Equation (13a) and profiles of the solution $v$ via Equation (13b) with $a_2 = 1$, $a_4 = 1$, $a_5 = 1$, $a_7 = 1$, $a_8 = 0$, $a_9 = 0.5$, $\beta = 1$, $\alpha = 1$, $\lambda = 2$, $t = 0$. (a) and (c) are the corresponding 3-dimensional plots; (b) and (d) are the corresponding density plots.

The solutions of $u$ via Equation (18a) can be seen in **Figure 2(a)**; **Figure 2(b)** is the corresponding density plot. The solutions of $v$ via Equation (18b) can be seen in **Figure 2(c)**; **Figure 2(d)** is the corresponding density plot.

Case III

$$\begin{aligned} &a_1 = a_1, a_2 = a_2, a_3 = a_3, a_4 = a_4, a_5 = 0, \\ &a_6 = a_6, a_7 = a_6\alpha\lambda, a_8 = a_8, a_9 = a_9, \alpha = \alpha, \beta = \frac{a_2\alpha}{a_1}, \\ &k_1 = 0, k_2 = k_2, k_3 = k_2\alpha\lambda, k_4 = k_4, \lambda = \lambda, m = m, n = n, \end{aligned} \tag{20}$$

where

$$a_1 \neq 0, a_9 > 0, m > 0, n > 0. \tag{21}$$

to ensure that the corresponding $f$ is positive and well defined. By substituting Equation (20) into function (9),

$$\begin{aligned} f = &\left(a_1x + a_2y + a_3t + a_4\right)^2 + \left(a_6x + a_6\alpha\lambda t + a_8\right)^2 + a_9 \\ &+ m\exp\left(k_2y + k_2\alpha\lambda t + k_4\right) + n\exp\left(-\left(k_2y + k_2\alpha\lambda t + k_4\right)\right), \end{aligned} \tag{22}$$

where $a_i, k_j, 1 \le i \le 9, 1 \le j \le 4$ are all real parameters to be determined. Then we get the solutions of the Equation (4a) and Equation (4b)

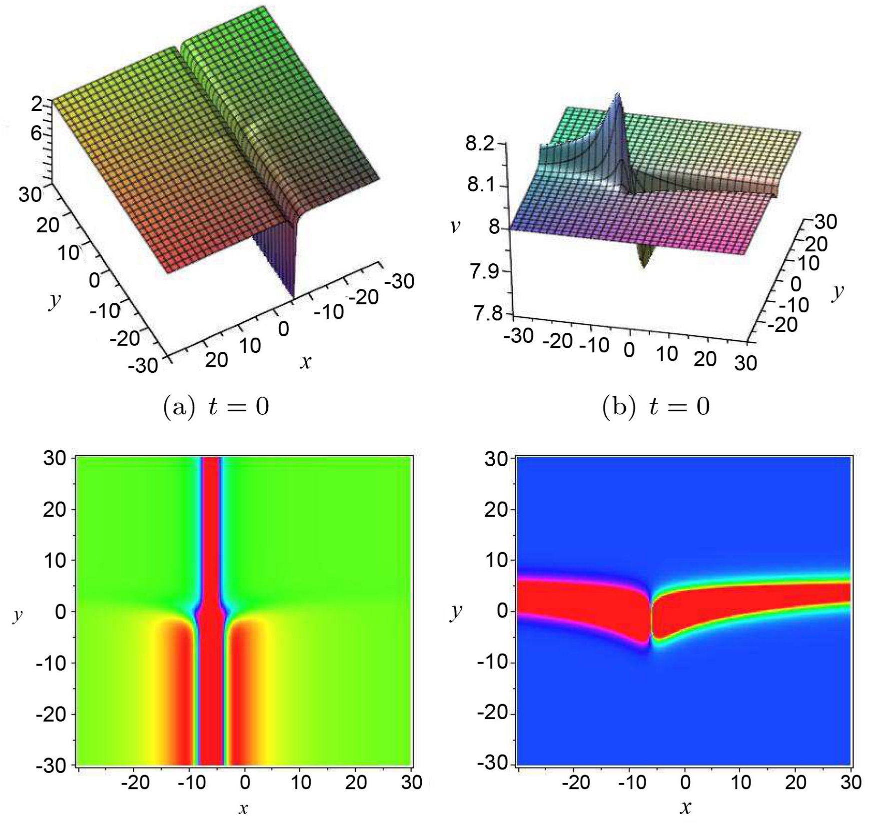



**Figure 2.** Profiles of the solution $u$ via Equation (18a) and the solution $v$ via Equation (18b) with $a_1 = 0.5$, $a_3 = 2$, $a_4 = 1$, $a_5 = 0$, $a_6 = 0$, $a_9 = 0.5$, $a_8 = 1$, $\beta = 0$, $\alpha = 1$, $\lambda = 1$, $m = 1$, $k_1 = 0$, $k_2 = 1$, $k_4 = 1$, $t = 1$. (a) and (c) are the corresponding 3-dimensional plots; (b) and (d) are the corresponding density plots.

$$u = \lambda + \frac{a_1^2}{4g^2}, \tag{23a}$$

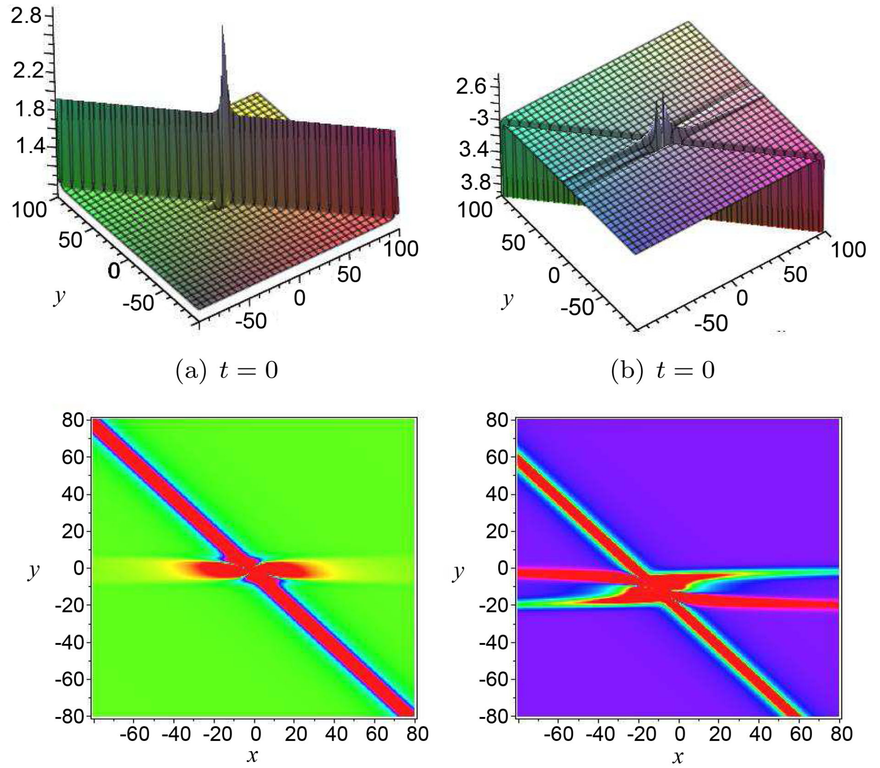
$$v = -\frac{3a_2\lambda}{a_1} + 2\frac{B_3 - \lambda\alpha C_3}{\alpha A_3} + \frac{a_1^3 a_2}{A_3^2}, \tag{23b}$$

with

$$\begin{aligned}
&f = h^2 + g^2 + a_9 + m\exp(\xi) + n\exp(-\xi),\\
&g = a_1x + a_2y + a_3t + a_4,\\
&h = a_6x + a_6\alpha\lambda t + a_8,\\
&\xi = k_2y + k_2\alpha\lambda t + k_4,\\
&A_3 = 2a_1g, C_3 = 2a_2g + 2a_6h + mk_2\exp(\xi) - nk_2\exp(-\xi)\\
&B_3 = 2a_3g + 2a_6\alpha\lambda h + mk_2\alpha\lambda\exp(\xi) - nk_2\alpha\lambda\exp(\xi)
\end{aligned} \tag{24}$$

The exact solutions for Equation (23a) and Equation (23b) can be seen in Figure 3(a) and Figure 3(c), Figure 3(b) and Figure 3(d) are the corresponding density plots.

Case IV




(a) $t = 0$ (b) $t = 0$

(c) $t = 0$ (d) $t = 0$

**Figure 3.** Profiles of the solution $u$ via Equation (23a) and $v$ via Equation (23b) with $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 2$, $a_5 = 0$, $a_6 = 1$, $a_8 = 1$, $a_9 = 1$, $k_1 = 0$, $k_2 = 1$, $k_4 = 1$, $m = 1$, $n = 1$, $\alpha = 1$, $\lambda = 1$, $t = 0$. (a) and (c) are the corresponding 3-dimensional plots; (b) and (d) are the corresponding density plots.

$$\begin{aligned} & a_1 = a_1, a_2 = -\frac{a_1\beta}{\alpha}, a_3 = -\frac{a_1 a_7}{\alpha a_6}, a_4 = -\frac{a_1 a_8}{a_6\alpha}, a_5 = -\frac{\alpha a_6}{\beta}, \\ & a_6 = a_6, a_7 = a7, a_8 = a_8, a_9 = a_9, m = m, n = n, \\ & \alpha = \alpha, \beta = \beta, k_1 = 0, k_2 = k_2, k_3 = k_2\alpha\lambda, k_4 = k_4, \lambda = \lambda \end{aligned} \tag{25}$$

where

$$\alpha\beta a_6 \neq 0, a_9 > 0, m > 0, n > 0. \tag{26}$$

to guarantee that the corresponding $f$ is positive and well defined. We substitute Equation (25) into Equation (9), hence, reinstall function $f$ as the following formula:

$$\begin{aligned} f = & \left(a_1 x - \frac{a_1\beta}{\alpha} y - \frac{a_1 a_7}{\alpha a_6} t - \frac{a_1 a_8}{a_6\alpha}\right)^2 + \left(a_5 x + a_6 y + a_7 t + a_8\right)^2 + a_9 \\ & + m\exp\left(k_2 y + k_2\alpha\lambda t + k_4\right) + n\exp\left(-\left(k_2 y + k_2\alpha\lambda t + k_4\right)\right), \end{aligned} \tag{27}$$

where $a_i, k_j, 1 \le i \le 9, 1 \le j \le 4$ are all real parameters to be determined. Then, we get

$$u = \lambda + \frac{D_4^2}{A_4^2}, \tag{28a}$$

$$v=-\frac{\beta\lambda}{\alpha}+\frac{2C_4}{\alpha A_4}+\frac{2\lambda B_4}{A_4}+\frac{D_4\left(3\beta D_4+4\alpha E_4\right)}{4\alpha A_4^2}, \tag{28b}$$

where

$$\begin{aligned}
&f=h^2+g^2+\alpha\exp(\xi)+\beta\exp(-\xi)+a_9,\\
&h=a_1x-\frac{a_1\beta}{\alpha}y-\frac{a_1a_7}{\alpha a_6}t-\frac{a_1a_8}{a_6\alpha},\\
&g=a_5x+a_6y+a_7t+a_8,\\
&\xi=k_2y+k_2\alpha\lambda t+k_4,\\
&A_4=2\left(a_1h+a_5g\right),\\
&B_4=-2\frac{a_1\beta}{\alpha}h+2a_6g+k_2\left(m\exp(\xi)-n\exp(-\xi)\right),\\
&C_4=-2\frac{a_1a_7}{\alpha a_6}h+2a_7g+k_2\alpha\lambda\left(m\exp(\xi)-n\exp(-\xi)\right),\\
&D_4=2\left(a_1^2+a_5^2\right),\\
&E_4=2\left(a_5a_6-\frac{a_1^2\beta}{\alpha}\right).
\end{aligned} \tag{29}$$

The exact solution for Equation (28b) can be seen in Figure 4(a), Figure 4(b) Figure 4(c) the corresponding density plots with the different time.

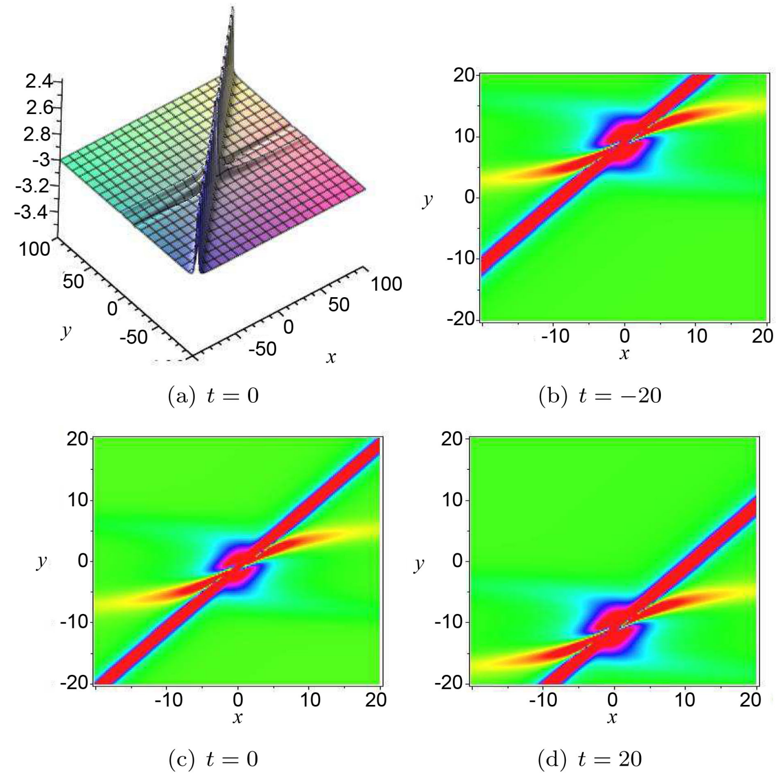



Figure 4. Profiles of the solution $v$ via Equation (28b) with $a_1=1$, $a_4=0$, $a_5=2$, $a_6=4$, $a_8=0$, $a_9=1$, $\alpha=1$, $\beta=1$, $k_1=1$, $k_2=1.5$, $k_3=2$, $k_4=0$. (a) 3-dimensional plot with the time $t = 0$; (b) (c) (b) the corresponding density plot with the different time.

Case V.

$$a_1 = 0, a_2 = a_2, a_3 = \alpha\lambda a_2, a_4 = a_4, a_5 = \frac{a_6 k_1}{k_2}$$
$$a_6 = a_6, a_7 = a_7, a_8 = a_8, a_9 = a_9, \alpha = -\frac{\beta k_1}{k_2}, \beta = \beta, \tag{30}$$
$$k_1 = k_1, k_2 = k_1, k_3 = \frac{a_7 k_2}{a_6}, k_4 = k_4, m = m, n = n.$$

with

$$k_2 a_6 \neq 0, a_9 > 0, m > 0, n > 0. \tag{31}$$

to guarantee that the corresponding $f$ is positive and well defined. We substitute Equation (30) into Equation (9), then function $f$ reads

$$\begin{aligned} f &= (a_2 y + \alpha\lambda a_2 t + a_4)^2 + \left(\frac{a_6 k_1}{k_2} x + a_6 y + a_7 t + a_8\right)^2 \\ &+ a_9 + m\exp\left(k_1 x + k_2 y + \frac{a_7 k_2}{a_6} t + k_4\right) \\ &+ n\exp\left(-\left(k_1 x + k_2 y + \frac{a_7 k_2}{a_6} t + k_4\right)\right), \end{aligned} \tag{32}$$

where $a_i, k_j, 1 \le i \le 9, 1 \le j \le 4$ are all real parameters to be determined. Then, the rational solution of system (4a) and (4a) can be got again.

$$\begin{aligned} u &= \lambda - \frac{k_1^3 \left(m\exp(\xi) - n\exp(-\xi)\right)}{2A_5} + \frac{D_5}{4A_5^2}, \\ v &= -\frac{3\beta\lambda}{\alpha} + 2\frac{B_5}{A_5} + 2\lambda\frac{C_5}{A_5}, + \frac{3\beta D_5^2}{4\alpha A_5^2} + \frac{D_5 E_5}{A_5^2} - \frac{F_5}{A_5} - \frac{\beta G_5}{2\alpha A_5}, \end{aligned} \tag{33}$$

and where

$$\begin{aligned} f &= g^2 + h^2 + m\exp(\xi) + n\exp(-\xi) + a_9, \\ g &= a_2 y + \alpha\lambda a_2 t + a_4, \\ h &= \frac{a_6 k_1}{k_2} x + a_6 y + a_7 t + a_8, \\ \xi &= k_1 x + k_2 y + \frac{a_7 k_2}{a_6} t + k_4, \\ A_5 &= 2\frac{a_6 k_1}{k_2} + k_1 h + m k_1 \exp(\xi) - n k_1 \exp(-\xi), \\ B_5 &= 2\alpha\lambda a_2 g + 2a_7 h + \frac{a_7 k_2}{a_6}\left(m\exp(\xi) + n\exp(-\xi)\right), \\ C_5 &= 2a_2 g + 2a_6 h + k_2\left(m\exp(\xi) + n\exp(-\xi)\right), \\ D_5 &= 2\frac{a_6^2 k_1^2}{k_2^2} + k_1^2\left(m\exp(\xi) + n\exp(-\xi)\right), \\ E_5 &= k_1 k_2\left(m\exp(\xi) + n\exp(-\xi)\right) + 2a_6\frac{a_6 k_1}{k_2}, \\ F_5 &= k_1^2 k_2\left(m\exp(\xi) + n\exp(-\xi)\right). \end{aligned} \tag{34}$$

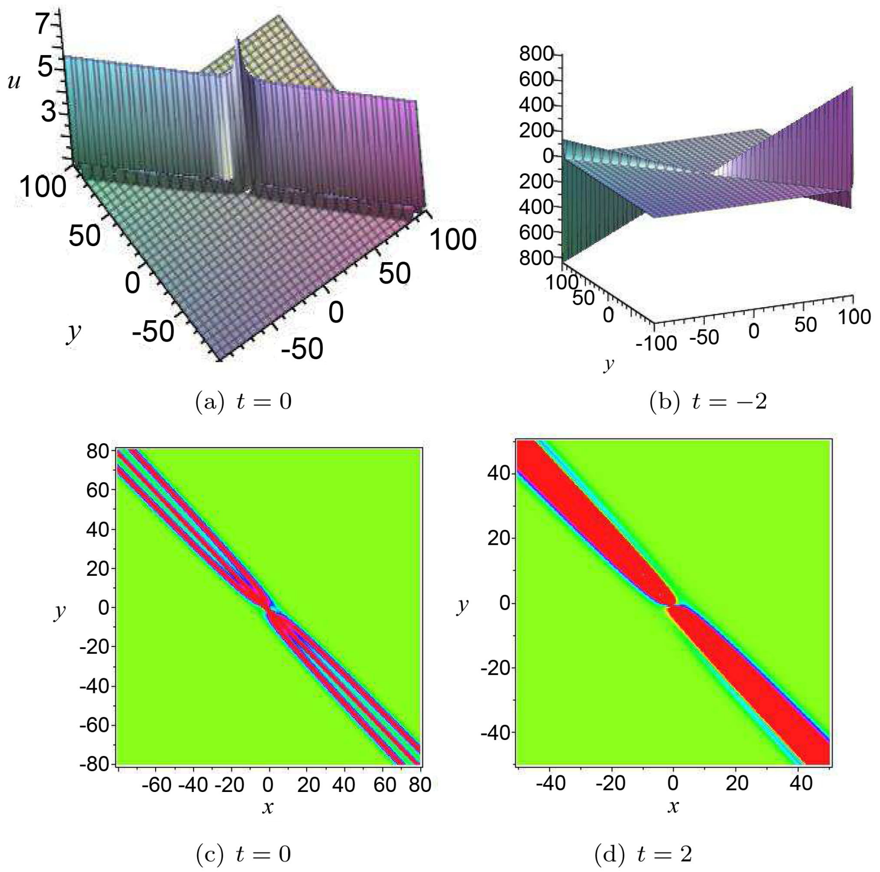



**Figure 5.** Profiles of the solution $u$ and $v$ via Equation (33) with $a_2 = 1$, $a_4 = 1$, $a_6 = 1$, $a_7 = 1$, $a_8 = 2$, $a_9 = 1$, $k_1 = 1$, $k_2 = 1$, $k_4 = 1$, $m = 1$, $n = 1$, $\lambda = 1$, $\beta = 1$, $t = 0$. (a) and (c) are the 3-dimensional plots; (b) and (d) are the corresponding density plots.

The exact solutions for Equation (33) can be seen in Figure 5(a) and Figure 5(c), Figure 5(b) and Figure 5(d) are the corresponding density plots.

## 4. Conclusion

In this paper, by using the truncated Painlev expansion, the (2 + 1)-dimensional KdV-CBS equation can be changed to a trilinear Equation (7), and by means of symbolic computations, we presented five types of exact solutions ((13a), (13b), (18a), (18b), (23a), (23b), (28a), (28b), (33)) to the (2 + 1)-dimensional KdV-CBS equation. Three-dimensional plots and the corresponding density plots of these exact solutions are given respectively in Figure 1 and Figure 5. This work shows the power of the methods, the exact solutions of some nonlinear equations can be obtained by using this means. It is expected that our results can enrich the exact solutions of the nonlinear equations.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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