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Pullback Random Attractors for Non-Autonomous Stochastic Fractional FitzHugh-Nagumo System

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Abstract

This paper is concerned with the asymptotic behavior of solutions for a class of non-autonomous fractional FitzHugh-Nagumo equations deriven by additive white noise. We first provide some sufficient conditions for the existence and uniqueness of solutions, and then prove the existence and uniqueness of tempered pullback random attractors for the random dynamical system generated by the solutions of considered equations in an appropriate Hilbert space. The proof is based on the uniform estimates and the decomposition of dynamical system.

Keywords

Non-Autonomous Stochastic Fractional FitzHugh-Nagumo System, Random Attractor, Additive White Noise

1. Introduction

In this paper, we investigate the random attractor of the non-autonomous stochastic fractional FitzHugh-Nagumo equations with additive white noise in bounded domains. Let $s \in (0,1)$, $\tau \in \mathbf{R}$ and $\mathbf{U}$ be a smooth bounded domain of $\mathbf{R}^n$. We consider the following stochastic system in U:

$$\mathrm{d}\tilde{u} + \left((-\Delta)^s \tilde{u} + \lambda_1 \tilde{u} + \alpha_1 \tilde{v}\right)\mathrm{d}t = f(x,\tilde{u})\mathrm{d}t + g(t,x)\mathrm{d}t + \phi_1(x)\mathrm{d}\omega_1,\ x \in \mathbf{U}, t > \tau, \quad (1)$$

$$\mathrm{d}\tilde{v} + (\lambda_2 \tilde{v} - \alpha_2 \tilde{u})\mathrm{d}t = h(t,x)\mathrm{d}t + \phi_2(x)\mathrm{d}\omega_2,\ x \in \mathbf{U}, t > \tau, \quad (2)$$

with initial conditions

$$\tilde{u}(\tau,x)=\tilde{u}_\tau(x),\ \tilde{v}(\tau,x)=\tilde{v}_\tau(x),\ x\in\mathbf{U}, \tag{3}$$

and boundary conditions

$$\tilde{u}(t,x)=\tilde{v}(t,x)=0,\ x\in\partial\mathbf{U},t>\tau, \tag{4}$$

where $\tilde{u}=\tilde{u}(t,x)$, $\tilde{v}=\tilde{v}(t,x)$ are real-valued functions on $[\tau,+\infty)\times\mathbf{U}$, $\lambda_1,\lambda_2,\alpha_1$ and α_2 are positive constants, $g(t,x)\in L^2_{loc}(\mathbf{R},L^2(\mathbf{U}))$, $h(t,x)\in L^2_{loc}(\mathbf{R},H^s(\mathbf{U}))$, $\phi_1\in D((-\Delta)^s)\cap W^{2s,p}(\mathbf{U})$ ($p\geq 1$ and the details of these spaces will be given later), $\phi_2\in L^2(\mathbf{U})$, $W_1(\cdot,\omega_1)$ and $W_2(\cdot,\omega_2)$ are independent two-sided real-valued Wiener processes with the standard process $\omega(\cdot):=(\omega_1(\cdot),\omega_2(\cdot))$ on a probability space which will be specified below. The nonlinear function $f(x,\tilde{u})$ is a differential function about two variables satisfying: for all $(x,s)\in\mathbf{U}\times\mathbf{R}$ and $\beta_i>0$, $(i=1,2)$

$$f(x,s)s\leq-\beta_1|s|^p+\psi_1(x), \tag{5}$$

$$|f(x,s)|\leq\beta_2|s|^{p-1}+\psi_2(x), \tag{6}$$

$$\frac{\partial f}{\partial s}(x,s)\leq\psi_3(x), \tag{7}$$

$$|f(x,s)-f(y,s)|\leq|\psi_4(x)-\psi_4(y)|, \tag{8}$$

where $p\geq 2$ are constant, $\psi_1\in L^1(\mathbf{U})$, $\psi_2\in L^q(\mathbf{U})$ with $\frac{1}{p}+\frac{1}{q}=1$, and $\psi_3\in L^\infty(\mathbf{U})$, $\psi_4\in H^s(\mathbf{U})$.

The FitzHugh-Nagumo system is a model for describing the signal transmission across axons in neurobiology, see [1] [2] [3]. The long term dynamics and inertial manifolds for the deterministic FitzHugh-Nagumo system have been extensively studied by many authors, see [4] [5] [6]. The existence of random attractor for the stochastic or lattice FitzHugh-Nagumo system has been investigated in [7] [8] [9] [10]. Recently, the fractional FitzHugh-Nagumo monodomain model is presented, the model consists of a coupled fractional nonlinear reaction-diffusion model and a system of ordinary differential equations. The stability and convergence are discussed using numerical method in [11]. To the best of our knowledge, there are some results on numerical calculation of deterministic fractional FitzHugh-Nagumo equation, but few results for theory study of stochastic fractional FitzHugh-Nagumo equation, especially for $s\in\left(0,\frac{1}{2}\right]$. As far as the author is aware, the attractors of the fractional stochastic equations are not well studied, it seems that the only publications [12] [13] in this respect, where the authors researched the existence of random attractors for the fractional stochastic equation with $s\in\left(\frac{1}{2},1\right)$. In [14] [15], the authors discussed the asymptotic behavior of fractional reaction-diffusion equation with $s\in(0,1)$.

In this paper, we explore the long time behavior of the solutions when the Equations (1)-(4) are perturbed by an additive white noise. There are several difficulties in this paper. Firstly, since the FitzHugh-Nagumo equation is a coupling equations, thus the uniform estimates of solutions are slightly different from the reaction-diffusion equation [15]. On the other hand, comparing with [15], we are concerned with the existence of random attractors of the fractional FitzHugh-Nagumo equation on U driven by additive noise rather than multiplicative noise, so new difficulties arise from the estimates for some terms, especially the nonlinearity *f*. Finally, the lack of higher regularity of the solution $\tilde{v}$ for the problem (2) which makes the difficulty to construct a compact attracting set of the random dynamical system. To achieve our goals, we must overcome these difficulties and establish the pullback asymptotic compactness of solutions in $L^2(\mathbf{U})\times L^2(\mathbf{U})$. We decompose the second component $\tilde{v}$ into a sum of two parts to overcome the lack of higher regularity like dealing with the wave equation in [16] [17].

This paper is organized as follows. In Section 2, we recall some basic concepts and define a continuous random dynamical system based on the solutions of the stochastic fractional FitzHugh-Nagumo Equations (1)-(4) in $L^2(\mathbf{U})\times L^2(\mathbf{U})$. We derive some uniform estimates for solutions and prove the existence of a pullback random attractor by pullback asymptotic compactness of solutions in Section 3.

2. Cocycles of the Stochastic Fractional FitzHugh-Nagumo System

In this section, we first collect some well-known results from the theory of random attractors and non-autonomous random dynamical systems. For further details, readers are also referred to [18] [19] [20] [21].

Let $\left(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbf{R}}\right)$ be a parametric dynamical system on the probability space $(\Omega,\mathcal{F},P)$, in which $\Omega=\left\{\omega=(\omega_1,\omega_2)\in C\left(\mathbf{R},\mathbf{R}^2\right):\omega(0)=0\right\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and P is the Wiener measure on $(\Omega,\mathcal{F})$, the group $\theta_t:\Omega\to\Omega$ defined by $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$ for $t\in\mathbf{R},\omega\in\Omega$. We usually write the norm of $L^2(\mathbf{R}^n)$ as $\|\cdot\|$ and the scalar product of $L^2(\mathbf{R}^n)$ as $(\cdot,\cdot)$. We also use $\|\cdot\|_X$ to denote the norm of a Banach space *X*.

In the following, "property holds for a.e. $\omega\in\Omega$ with respect to $(\theta_t)_{t\in\mathbf{R}}$" means that there is $\tilde{\Omega}\subset\Omega$ with $P(\tilde{\Omega})=1$ and $\theta_t\tilde{\Omega}=\tilde{\Omega}$ for all $t\in\mathbf{R}$ such that property holds for all $\omega\in\tilde{\Omega}$.

Definition 2.1. *Let $\mathcal{D}$ be a collection of some families of nonempty subset of X and Φ a continuous cocycle on X, $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-pullback random attractor for Φ if the following conditions are satisfied, for every $\tau\in\mathbf{R}$ and a.e. $\omega\in\Omega$,*

1) $\{\mathcal{A}(\tau,\omega)\}$ is compact in *X*, and $\omega\to d_X\left(x,\mathcal{A}(\tau,\omega)\right)$ is measurable;

2) $\{\mathcal{A}(\tau,\omega)\}$ is strictly invariant, *i.e.*, $\Phi\left(t,\tau,\omega,\mathcal{A}(\tau,\omega)\right)=\mathcal{A}(t+\tau,\theta_t\omega)$, for all $t\geq 0$;

3) $\{\mathcal{A}(\tau,\omega)\}$ attracts every member of $\mathcal{D}$ in *X*, *i.e.*, for all $B\in\mathcal{D}$, we have

$$\lim_{t\to+\infty} d_X\left(\Phi\left(t,\tau-t,\theta_{-t}\omega,B\left(\tau-t,\theta_{-t}\omega\right)\right),\mathcal{A}\left(\tau,\omega\right)\right)=0,$$

where d_X is the Hausdorff semi-distance in *X*.

Proposition 2.1. *Suppose X is a separable Banach spaces. Let* $\mathcal{D}$ *be an inclusion-closed collection of some families of nonempty subsets of X and* Φ *be a continuous cocycle on X over* $\left(\Omega,\mathcal{F},P,\left(\theta_t\right)_{t\in\mathbf{R}}\right)$. *For all* $t\in\mathbf{R}^+$, $\tau\in\mathbf{R}$, *and* $\omega\in\Omega$, Φ *has a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}\left(\tau,\omega\right)$ *in* $\mathcal{D}$ *given by,*

$$\mathcal{A}\left(\tau,\omega\right)=\bigcap_{s\ge0}\overline{\bigcup_{t\ge s}\Phi\left(t,\tau-t,\theta_{-t}\omega,K\left(\tau-t,\theta_{-t}\omega\right)\right)}^X,$$

if 1) Φ has a compact measurable $\mathcal{D}$-pullback absorbing set *K* in $\mathcal{D}$.

2) Φ is $\mathcal{D}$-pullback asymptotically compact in *X*.

To describe the main results of this raper, we review some concepts of the fractional Laplace operator on the bounded domain $\mathbf{U}$ (see [22] for details). Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbf{R}^n$, then for $0<s<1$, the fractional Laplace operator $(-\Delta)^s$ is given by, for $u\in\mathcal{S}$, $x\in\mathbf{R}^n$,

$$(-\Delta)^s u(x)=-\frac{1}{2}C(n,s)P.V.\int_{\mathbf{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\mathrm{d}y, \tag{9}$$

where $C(n,s)$ is a positive constant.

Let $H^s\left(\mathbf{R}^n\right)$ be the fractional Sobolev space defined by

$$H^s\left(\mathbf{R}^n\right)=\left\{u\in L^2\left(\mathbf{R}^n\right):\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y<\infty\right\},$$

which is equipped with the norm

$$\|u\|_{H^s(\mathbf{R}^n)}=\left(\int_{\mathbf{R}^n}|u(x)|^2\mathrm{d}x+\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

Note that $H^s\left(\mathbf{R}^n\right)$ is a Hilbert space with inner product given by

$$\begin{aligned}(u,v)_{H^s(\mathbf{R}^n)}&=\int_{\mathbf{R}^n}u(x)v(x)\mathrm{d}x\mathrm{d}y\\&+\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y,\ u,v\in H^s\left(\mathbf{R}^n\right).\end{aligned}$$

For convenience, we will also use the notation:

$$\|u\|^2_{\dot{H}^s(\mathbf{R}^n)}=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y,\ u\in H^s\left(\mathbf{R}^n\right).$$

By proposition 3.6 in [22], we find that:

$$\|u\|^2_{H^s(\mathbf{R}^n)}=\|u\|^2_{L^2(\mathbf{R}^n)}+\frac{2}{C(n,s)}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbf{R}^n)},\ \text{for all } u\in H^s\left(\mathbf{R}^n\right), \tag{10}$$

and hence $\left(\|u\|_{L^2(\mathbf{R}^n)}^2+\left\|(-\Delta)^{\frac{s}{2}}u\right\|_{L^2(\mathbf{R}^n)}^2\right)^{\frac{1}{2}}$ is an equivalent norm of $H^s\left(\mathbf{R}^n\right)$. Similar to $H^s\left(\mathbf{R}^n\right)$, we can define $H^{2s}\left(\mathbf{R}^n\right)$ and $W^{2s,p}\left(\mathbf{R}^n\right)$ for $s\in(0,1)$ (see [22]).

Since the fractional operator $(-\Delta)^s$ is non-local, we here interpret the boundary (4) as $\tilde{u}(t,x)=\tilde{v}(t,x)=0, x\in\mathbf{R}^n\setminus\mathbf{U}$ instead of $\tilde{u}(t,x)=\tilde{v}(t,x)=0, x\in\partial\mathbf{U}$. From [23] and the references therein, we know such a interpretation is consistent with the non-local nature of the fractional operator. Thus, let $H=\left\{(u,v)\in L^2\left(\mathbf{R}^n\right)\times L^2\left(\mathbf{R}^n\right): u=v=0 \text{ } a.e. \text{ on } \mathbf{R}^n\setminus\mathbf{U}\right\}$ and $V=\left\{(u,v)\in H^s\left(\mathbf{R}^n\right)\times H^s\left(\mathbf{R}^n\right): u=v=0 \text{ } a.e. \text{ on } \mathbf{R}^n\setminus\mathbf{U}\right\}$.

Then, we give the essential assumptions and define a continuous random dynamical system for the stochastic fractional FitzHugh-Nagumo equations in *H*. More precisely, we consider:

$$\begin{aligned}&\mathrm{d}\tilde{u}+\left((-\Delta)^s\tilde{u}+\lambda_1\tilde{u}+\alpha_1\tilde{v}\right)\mathrm{d}t\\&=f(x,\tilde{u})\mathrm{d}t+g(t,x)\mathrm{d}t+\phi_1(x)\mathrm{d}\omega_1,\ x\in\mathbf{U},t>\tau,\end{aligned}\tag{11}$$

$$\mathrm{d}\tilde{v}+\left(\lambda_2\tilde{v}-\alpha_2\tilde{u}\right)\mathrm{d}t=h(t,x)\mathrm{d}t+\phi_2(x)\mathrm{d}\omega_2,\ x\in\mathbf{U},t>\tau,\tag{12}$$

with initial conditions

$$\tilde{u}(\tau,x)=\tilde{u}_\tau(x),\ \tilde{v}(\tau,x)=\tilde{v}_\tau(x),\ x\in\mathbf{U},\tag{13}$$

and boundary conditions

$$\tilde{u}(t,x)=\tilde{v}(t,x)=0,\ x\in\mathbf{R}^n\setminus\mathbf{U},t>\tau.\tag{14}$$

Define $z(\omega):=\left(z_1(\omega_1),z_2(\omega_2)\right)$ by

$$z_1(\omega_1)=-\lambda_1\int_{-\infty}^0\mathrm{e}^{\lambda_1 s}\omega_1(s)\mathrm{d}s,$$

$$z_2(\omega_2)=-\lambda_2\int_{-\infty}^0\mathrm{e}^{\lambda_2 s}\omega_2(s)\mathrm{d}s.$$

It is well know that $y_i(t,\omega)=z_i(\theta_t\omega_i)(i=1,2)$ is the unique stationary solution of the following stochastic equations

$$\mathrm{d}y_1+\lambda_1y_1\mathrm{d}t=\mathrm{d}W_1(t),\tag{15}$$

$$\mathrm{d}y_2+\lambda_2y_2\mathrm{d}t=\mathrm{d}W_2(t).\tag{16}$$

In addition, assume that $\delta=\min\{\lambda_1,\lambda_2\}$, from [18], we have

$$\int_{-\infty}^0\mathrm{e}^{\delta s}\left(\left|z_1(\theta_s\omega_1)\right|^2+\left|z_1(\theta_s\omega_1)\right|^p+\left|z_1(\theta_s\omega_1)\right|^{2p-2}\right)\mathrm{d}s<\infty,\tag{17}$$

and

$$\int_{-\infty}^0\mathrm{e}^{\delta s}\left|z_2(\theta_s\omega_2)\right|^2\mathrm{d}s<\infty.\tag{18}$$

It follows from [10] that there exists a θ_t-invariant set of full measure (still denoted by Ω) such that $z_i(\theta_t\omega_i)(i=1,2)$ are both continuous in *t* for each $\omega\in\Omega$.

We now transform the stochastic Equations (11)-(14) into a pathwise deterministic one by using the random variable $z_i(i=1,2)$. Given $\tau\in\mathbf{R}, t\ge\tau, \omega\in\Omega$ and $(\tilde{u}_\tau,\tilde{v}_\tau)\in L^2(\mathbf{U})\times L^2(\mathbf{U})$, let $(\tilde{u},\tilde{v})=(\tilde{u}(t,\tau,\omega,\tilde{u}_\tau),\tilde{v}(t,\tau,\omega,\tilde{v}_\tau))$ is a solution of (11)-(14), introduce variables transformation:

$$u(t,\tau,\omega,u_\tau):=\tilde{u}(t,\tau,\omega,\tilde{u}_\tau)-\phi_1 z_1(\theta_t\omega_1) \text{ with } u_\tau=\tilde{u}_\tau-\phi_1 z_1(\theta_\tau\omega_1), \tag{19}$$

$$v(t,\tau,\omega,v_\tau):=\tilde{v}(t,\tau,\omega,\tilde{v}_\tau)-\phi_2 z_2(\theta_t\omega_2) \text{ with } v_\tau=\tilde{v}_\tau-\phi_2 z_2(\theta_\tau\omega_2). \tag{20}$$

By (11)-(14) and (15)-(16) we get

$$\begin{aligned}&\frac{\partial u}{\partial t}+(-\Delta)^s u+\lambda_1 u+\alpha_1 v\\ &=f(x,\tilde{u})+g(t,x)-(-\Delta)^s\phi_1 z_1(\theta_t\omega_1)-\alpha_1\phi_2 z_2(\theta_t\omega_2),\ x\in\mathbf{U}, t>\tau,\end{aligned} \tag{21}$$

$$\frac{\partial v}{\partial t}+\lambda_2 v-\alpha_2 u=h(t,x)+\alpha_2\phi_1 z_1(\theta_t\omega_1),\ x\in\mathbf{U}, t>\tau, \tag{22}$$

with initial conditions

$$u(\tau,x)=u_\tau(x), v(\tau,x)=v_\tau(x),\ x\in\mathbf{U}, \tag{23}$$

and boundary conditions

$$u(t,x)=v(t,x)=0,\ x\in\mathbf{R}^n\setminus\mathbf{U}, t>\tau. \tag{24}$$

Recall that H and V are Hilbert space. Note that the definition of H is consistent with conditions (23) and (24), since $(u,v)\in L^2(\mathbf{U})\times L^2(\mathbf{U})$ can be considered as an element of H by setting $u(x)=v(x)=0$ for $x\in\mathbf{R}^n\setminus\mathbf{U}$. It follows from the standard arguments as in [24] that under conditions (1)-(7), problem (21)-(24) is well-posed in H. The unique solution defines two cocycles $\Phi,\tilde{\Phi}:\mathbf{R}^+\times\mathbf{R}\times\Omega\times H\to H$ as follows: for $t\in\mathbf{R}^+, \tau\in\mathbf{R}, \omega\in\Omega$ and $(u_\tau,v_\tau)\in H$

$$\Phi(t,\tau,\omega,(u_\tau,v_\tau))=(u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau),v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau)), \tag{25}$$

$$\begin{aligned}\tilde{\Phi}(t,\tau,\omega,(\tilde{u}_\tau,\tilde{v}_\tau))&=(\tilde{u}(t+\tau,\tau,\theta_{-\tau}\omega,\tilde{u}_\tau),\tilde{v}(t+\tau,\tau,\theta_{-\tau}\omega,\tilde{v}_\tau))\\ &=\Phi(t,\tau,\omega,(u_\tau,v_\tau))+(\phi_1 z_1(\theta_t\omega_1),\phi_2 z_2(\theta_t\omega_2)),\end{aligned} \tag{26}$$

where $u_\tau=\tilde{u}_\tau-\phi_1 z_1(\omega_1), v_\tau=\tilde{v}_\tau-\phi_2 z_2(\omega_2)$. Both cocycles Φ and $\tilde{\Phi}$ are equivalent and so we will discuss only the cocycle Φ induced by Equations (21)-(24) in this paper. Let $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}$ be a family of bounded nonempty subsets of H. Such a family B is called tempered if for every $c>0, \tau\in\mathbf{R}$ and $\omega\in\Omega$,

$$\lim_{t\to+\infty}\mathrm{e}^{-ct}\|B(\tau-t,\theta_{-t}\omega)\|_H^2=0, \tag{27}$$

where the norm $\|B\|$ of set B in H is given by $\|B\|=\sup_{u\in B}\|u\|$. From now on, we will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of H:

$$\mathcal{D}=\{B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}: B \text{ is tempered in } H\}.$$

The functions g and h in (1)-(2) are satisfy, for every $\tau\in\mathbf{R}$,

$$\int_{-\infty}^0\mathrm{e}^{\delta s}\left(\|g(s+\tau,\cdot)\|^2+\|h(s+\tau,\cdot)\|^2\right)\mathrm{d}s<\infty, \tag{28}$$

$$\int_{-\infty}^{0} e^{\delta s} \left\| (-\Delta)^{\frac{s}{2}} h(s+\tau,\cdot) \right\|^2 ds < \infty. \tag{29}$$

and for every $c > 0$,

$$\lim_{r\to+\infty} e^{-cr} \int_{-\infty}^{0} e^{\delta s} \left(\|g(s-r,\cdot)\|^2 + \|h(s-r,\cdot)\|^2 \right) ds = 0. \tag{30}$$

Throughout this paper, C denotes a constant which may be different from the context.

3. Existence of Random Attractor

This section is devoted to uniform estimates of solutions for the problem (21)-(24), which are useful for constructing random pullback absorbing sets. We begin with the uniform estimates of solutions in *H*.

Lemma 3.1. *Under conditions* (5)-(8) *and* (28), *for every* $\sigma \in \mathbf{R}, \tau \in \mathbf{R}, \omega \in \Omega$ *and* $B = \{B(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,B,\sigma) > 0$ *such that for all* $t \geq T$, *the solution* (u,v) *of problem* (21)-(24) *satisfies*

$$\begin{aligned}
&\alpha_2 \left\| u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) \right\|^2 + \alpha_1 \left\| v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \right\|^2 \\
&+ \frac{\delta}{2}\alpha_1 \int_{-t}^{\sigma-\tau} e^{\delta(s+\tau-\sigma)} \left\| v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \right\|^2 ds \\
&+ C\int_{-t}^{\sigma-\tau} e^{\delta(s+\tau-\sigma)} \left\| u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) \right\|_{H^s(\mathrm{U})}^2 ds \\
&+ \alpha_2\beta_1 \int_{-t}^{\sigma-\tau} e^{\delta(s+\tau-\sigma)} \int_{\mathrm{U}} \left| \tilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t}) \right|^p dx ds \\
&\leq C + C\int_{-\infty}^{\sigma-\tau} e^{\delta(s+\tau-\sigma)} \left(\|g(s+\tau)\|^2 + \|h(s+\tau)\|^2 + 1 \right) ds.
\end{aligned}$$

where $(u_{\tau-t}, v_{\tau-t}) \in B(\tau-t,\theta_{-t}\omega)$.

Proof. It follows from (21) and (22) that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha_2}{2}\|u\|^2 + \frac{\alpha_1}{2}\|v\|^2 \right) + \lambda_1\alpha_2\|u\|^2 + \lambda_2\alpha_1\|v\|^2 + \alpha_2 \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2 \\
&= \alpha_2 \int_{\mathrm{U}} f(x,\tilde{u}) u dx + \alpha_2 \int_{\mathrm{U}} g(t,x) u dx + \alpha_1 \int_{\mathrm{U}} h(t,x) v dx \\
&\quad - \alpha_2 \left((-\Delta)^s \phi_1 z_1(\theta_t\omega_1), u \right) - \alpha_1\alpha_2 \left(\phi_2 z_2(\theta_t\omega_2), u \right) + \alpha_1\alpha_2 \left(\phi_1 z_1(\theta_t\omega_1), v \right).
\end{aligned} \tag{31}$$

We now estimate each term on the right-hand side of (31). For the first term, by (5) and (6) we obtain

$$\begin{aligned}
\alpha_2 \int_{\mathrm{U}} f(x,\tilde{u}) u dx &= \alpha_2 \int_{\mathrm{U}} f(x,\tilde{u}) \left(\tilde{u} - \phi_1 z_1(\theta_t\omega_1) \right) dx \\
&\leq -\alpha_2\beta_1 \int_{\mathrm{U}} \left| u + \phi_1 z_1(\theta_t\omega_1) \right|^p dx + \alpha_2 \int_{\mathrm{U}} \psi_1(x) dx \\
&\quad + \alpha_2\beta_2 \int_{\mathrm{U}} \left| u + \phi_1 z_1(\theta_t\omega_1) \right|^{p-1} \left| \phi_1 z_1(\theta_t\omega_1) \right| dx \\
&\quad + \alpha_2 \int_{\mathrm{U}} |\psi_2(x)| \left| \phi_1 z_1(\theta_t\omega_1) \right| dx \\
&\leq \frac{-\alpha_2\beta_1}{2} \int_{\mathrm{U}} \left| u + \phi_1 z_1(\theta_t\omega_1) \right|^p dx + C\left(\int_{\mathrm{U}} \psi_1(x) dx + \int_{\mathrm{U}} |\psi_2(x)|^q dx \right. \\
&\quad \left. + \left| z_1(\theta_t\omega_1) \right|^p \int_{\mathrm{U}} |\phi_1(x)|^p dx \right) \\
&\leq \frac{-\alpha_2\beta_1}{2} \int_{\mathrm{U}} \left| u + \phi_1 z_1(\theta_t\omega_1) \right|^p dx + C\left(1 + \left| z_1(\theta_t\omega_1) \right|^p \right),
\end{aligned} \tag{32}$$

where ψ_1,ψ_2 satisfies the assumed conditions.

By the Young inequality, we have the following estimates on the remaining terms on the right-hand side of (31)

$$\alpha_2\int_{\mathrm{U}} g(t,x)u\mathrm{d}x \le \frac{\lambda_1\alpha_2}{8}\|u\|^2+\frac{2\alpha_2}{\lambda_1}\|g(t)\|^2, \tag{33}$$

$$\alpha_1\int_{\mathrm{U}} h(t,x)v\mathrm{d}x \le \frac{\lambda_2\alpha_1}{8}\|v\|^2+\frac{2\alpha_1}{\lambda_2}\|h(t)\|^2, \tag{34}$$

$$\alpha_2\left|\left((-\Delta)^s\phi_1 z_1(\theta_t\omega_1),u\right)\right| \le \frac{\alpha_2}{2}\left|z_1(\theta_t\omega_1)\right|^2\left\|(-\Delta)^{\frac{s}{2}}\phi_1\right\|^2+\frac{\alpha_2}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2, \tag{35}$$

$$\alpha_1\alpha_2\left|\left(\phi_2 z_2(\theta_t\omega_2),u\right)\right| \le \frac{\lambda_1\alpha_2}{8}\|u\|^2+\frac{2}{\lambda_1}\alpha_1^2\alpha_2\left|z_2(\theta_t\omega_2)\right|^2\|\phi_2\|^2, \tag{36}$$

$$\alpha_1\alpha_2\left|\left(\phi_1 z_1(\theta_t\omega_1),v\right)\right| \le \frac{\lambda_2\alpha_1}{8}\|v\|^2+\frac{2}{\lambda_2}\alpha_1\alpha_2^2\left|z_1(\theta_t\omega_1)\right|^2\|\phi_1\|^2. \tag{37}$$

Since $\delta=\min\{\lambda_1,\lambda_2\}$, it follows from (31)-(37) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_2\|u\|^2+\alpha_1\|v\|^2\right)+\delta\left(\alpha_2\|u\|^2+\alpha_1\|v\|^2\right)+\frac{\delta}{2}\left(\alpha_2\|u\|^2+\alpha_1\|v\|^2\right)\\
&+\alpha_2\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2+\alpha_2\beta_1\int_{\mathrm{U}}|\tilde{u}|^p\,\mathrm{d}x\\
&\le\frac{4\alpha_2}{\lambda_1}\|g(t)\|^2+\frac{4\alpha_1}{\lambda_2}\|h(t)\|^2+C\left(1+\left|z_1(\theta_t\omega_1)\right|^2+\left|z_1(\theta_t\omega_1)\right|^p+\left|z_2(\theta_t\omega_2)\right|^2\right).
\end{aligned}\tag{38}$$

Multiplying (38) by $\mathrm{e}^{\delta t}$ and then integrating the above inequality on $(\tau-t,\sigma)$ with $\sigma>\tau-t$, we get

$$\begin{aligned}
&\alpha_2\left\|u(\sigma,\tau-t,\omega,u_{\tau-t})\right\|^2+\alpha_1\left\|v(\sigma,\tau-t,\omega,v_{\tau-t})\right\|^2\\
&+\frac{\delta}{2}\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\left(\alpha_2\left\|u(s,\tau-t,\omega,u_{\tau-t})\right\|^2+\alpha_1\left\|v(s,\tau-t,\omega,v_{\tau-t})\right\|^2\right)\mathrm{d}s\\
&+\alpha_2\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\left\|(-\Delta)^{\frac{s}{2}}u(s,\tau-t,\omega,u_{\tau-t})\right\|^2\mathrm{d}s\\
&+\alpha_2\beta_1\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\int_{\mathrm{U}}\left|\tilde{u}(s,\tau-t,\omega,\tilde{u}_{\tau-t})\right|^p\mathrm{d}x\mathrm{d}s\\
&\le\mathrm{e}^{\delta(\tau-t-\sigma)}\left(\alpha_2\|u_{\tau-t}\|^2+\alpha_1\|v_{\tau-t}\|^2\right)\\
&+\frac{4\alpha_2}{\lambda_1}\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\|g(s)\|^2\mathrm{d}s+\frac{4\alpha_1}{\lambda_2}\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\|h(s)\|^2\mathrm{d}s\\
&+C\int_{\tau-t}^{\sigma}\mathrm{e}^{\delta(s-\sigma)}\left(1+\left|z_1(\theta_s\omega_1)\right|^2+\left|z_1(\theta_s\omega_1)\right|^p+\left|z_2(\theta_s\omega_2)\right|^2\right)\mathrm{d}s.
\end{aligned}\tag{39}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above, after changes of variables, we obtain

$$\begin{aligned}
&\alpha_2\left\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2+\alpha_1\left\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\right\|^2\\
&+\frac{\delta}{2}\alpha_1\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\right\|^2\mathrm{d}s\\
&+\frac{\delta}{2}\alpha_2\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2\mathrm{d}s\\
&+\frac{\alpha_2}{2}C(n,s)\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|_{\dot{H}^s(\mathrm{U})}^2\mathrm{d}s\\
&+\alpha_2\beta_1\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\int_{\mathrm{U}}\left|\tilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\right|^p\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&\le \mathrm{e}^{\delta(\tau-t-\sigma)}\left(\alpha_2\|u_{\tau-t}\|^2+\alpha_1\|v_{\tau-t}\|^2\right)+\frac{4\alpha_2}{\lambda_1}\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|g(s+\tau)\|^2\,\mathrm{d}s\\&+\frac{4\alpha_1}{\lambda_2}\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|h(s+\tau)\|^2\,\mathrm{d}s\\&+C\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left(1+|z_1(\theta_s\omega_1)|^2+|z_1(\theta_s\omega_1)|^p+|z_2(\theta_s\omega_2)|^2\right)\mathrm{d}s.\end{aligned}\tag{40}$$

We now estimates the first term on the right-hand side of (40). Due to $(u_{\tau-t},v_{\tau-t})\in B(\tau-t,\theta_{-t}\omega)$ and $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}$ is tempered, we find

$$\begin{aligned}&\mathrm{e}^{\delta(\tau-t-\sigma)}\left(\alpha_2\|u_{\tau-t}\|^2+\alpha_1\|v_{\tau-t}\|^2\right)\\&=\mathrm{e}^{\delta(\tau-\sigma)}\mathrm{e}^{-\delta t}\left(\alpha_2\|u_{\tau-t}\|^2+\alpha_1\|v_{\tau-t}\|^2\right)\\&\le \mathrm{e}^{\delta(\tau-\sigma)}\mathrm{e}^{-\delta t}(\alpha_1+\alpha_2)\|B(\tau-t,\theta_{-t}\omega)\|^2\to 0\ \text{ as } t\to+\infty.\end{aligned}\tag{41}$$

Therefore, there exists $T=T(\tau,\omega,B,\sigma)>0$ such that for all $t\ge T$

$$\mathrm{e}^{\delta(\tau-t-\sigma)}\left(\alpha_2\|u_{\tau-t}\|^2+\alpha_1\|v_{\tau-t}\|^2\right)\le 1.\tag{42}$$

For the remaining terms on the right-hand side of (40), we have

$$\frac{4\alpha_2}{\lambda_1}\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|g(s+\tau)\|^2\,\mathrm{d}s\le\frac{4\alpha_2}{\lambda_1}\int_{-\infty}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|g(s+\tau)\|^2\,\mathrm{d}s,\tag{43}$$

$$\frac{4\alpha_1}{\lambda_2}\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|h(s+\tau)\|^2\,\mathrm{d}s\le\frac{4\alpha_1}{\lambda_2}\int_{-\infty}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|h(s+\tau)\|^2\,\mathrm{d}s,\tag{44}$$

$$\begin{aligned}&C\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left(1+|z_1(\theta_s\omega_1)|^2+|z_1(\theta_s\omega_1)|^p+|z_2(\theta_s\omega_2)|^2\right)\mathrm{d}s\\&\le C\int_{-\infty}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left(1+|z_1(\theta_s\omega_1)|^2+|z_1(\theta_s\omega_1)|^p+|z_2(\theta_s\omega_2)|^2\right)\mathrm{d}s.\end{aligned}\tag{45}$$

By (40)-(45) and (17) we obtain, for all $t\ge T$,

$$\begin{aligned}&\alpha_2\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+\alpha_1\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\\&+\frac{\delta}{2}\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\alpha_1\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\,\mathrm{d}s\\&+C\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s(\mathbf{U})}\,\mathrm{d}s\\&+\alpha_2\beta_1\int_{-t}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\int_{\mathbf{U}}|\tilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})|^p\,\mathrm{d}x\mathrm{d}s\\&\le C+C\int_{-\infty}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left(\|g(s+\tau)\|^2+\|h(s+\tau)\|^2+1\right)\mathrm{d}s.\end{aligned}\tag{46}$$

From (28), the desired estimates follow immediately.□

We now derive uniform estimates of u in $H^s(U)$.

Lemma 3.2. *Under conditions* (5)-(8) *and* (28), *for every* $\sigma\in\mathbf{R},\tau\in\mathbf{R},\omega\in\Omega$ *and* $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T(\tau,\omega,B,\sigma)\ge 2$ *such that for all* $t\ge T$ *and* $\sigma\in[\tau-1,\tau]$, *the solution u of problem* (21) *with* $(u_{\tau-t},v_{\tau-t})\in B(\tau-t,\theta_{-t}\omega)$ *satisfies*

$$\begin{aligned}&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{\dot{H}^s(\mathbf{U})}\\&\le C+C\int_{-\infty}^{\sigma-\tau}\mathrm{e}^{\delta(s+\tau-\sigma)}\left(\|g(s+\tau)\|^2+\|h(s+\tau)\|^2+1\right)\mathrm{d}s.\end{aligned}$$

Proof. Multiplying (21) by $(-\Delta)^s u$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2+\left\|(-\Delta)^s u\right\|^2+\lambda_1\int_{\mathbf{U}}u(-\Delta)^s u\mathrm{d}x+\alpha_1\int_{\mathbf{U}}v(-\Delta)^s u\mathrm{d}x\\
&=\int_{\mathbf{U}}f(x,\tilde{u})(-\Delta)^s u\mathrm{d}x+\int_{\mathbf{U}}g(t,x)(-\Delta)^s u\mathrm{d}x\\
&\quad-\int_{\mathbf{U}}(-\Delta)^s\phi_1 z_1(\theta_t\omega_1)(-\Delta)^s u\mathrm{d}x-\alpha_1\int_{\mathbf{U}}\phi_2 z_2(\theta_t\omega_2)(-\Delta)^s u\mathrm{d}x.
\end{aligned}\tag{47}$$

For the nonlinear term in (47), by (6)-(8) we have

$$\begin{aligned}
&2\int_{\mathbf{U}}f(x,\tilde{u})(-\Delta)^s u\mathrm{d}x\\
&=2\int_{\mathbf{U}}f(x,\tilde{u})(-\Delta)^s\tilde{u}\mathrm{d}x-2\int_{\mathbf{U}}f(x,\tilde{u})(-\Delta)^s\phi_1 z_1(\theta_t\omega_1)\mathrm{d}x\\
&\le C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left(f(x,\tilde{u}(x))-f(y,\tilde{u}(y))\right)\left(\tilde{u}(x)-\tilde{u}(y)\right)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\left|f(x,\tilde{u})\right|\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\le C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left(f(x,\tilde{u}(x))-f(y,\tilde{u}(x))\right)\left(\tilde{u}(x)-\tilde{u}(y)\right)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left(f(y,\tilde{u}(x))-f(y,\tilde{u}(y))\right)\left(\tilde{u}(x)-\tilde{u}(y)\right)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\left|f(x,\tilde{u})\right|\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\le C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left|\psi_4(x)-\psi_4(y)\right|\left|\tilde{u}(x)-\tilde{u}(y)\right|}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left|\psi_3(y)\right|\left|\tilde{u}(x)-\tilde{u}(y)\right|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\left|f(x,\tilde{u})\right|\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\le\frac{1}{2}C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left|\psi_4(x)-\psi_4(y)\right|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{1}{2}C(n,s)\int_{\mathbf{U}}\int_{\mathbf{U}}\frac{\left|\tilde{u}(x)-\tilde{u}(y)\right|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+C(n,s)\int_{\mathbf{U}}\left|\psi_3(y)\right|\int_{\mathbf{U}}\frac{\left|\tilde{u}(x)-\tilde{u}(y)\right|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\beta_2|\tilde{u}|^{p-1}\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\psi_2(x)\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\le\left\|\psi_4\right\|_{H^s(\mathbf{U})}^2+\left\|(-\Delta)^{\frac{s}{2}}\tilde{u}\right\|^2+2\left\|\psi_3\right\|_{L^\infty(\mathbf{U})}\left\|(-\Delta)^{\frac{s}{2}}\tilde{u}\right\|^2\\
&\quad+2\beta_2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}|\tilde{u}|^{p-1}\left|(-\Delta)^s\phi_1\right|\mathrm{d}x\\
&\quad+2\left|z_1(\theta_t\omega_1)\right|\int_{\mathbf{U}}\psi_2(x)\left|(-\Delta)^s\phi_1\right|\mathrm{d}x.
\end{aligned}\tag{48}$$

We now estimate the remaining terms on the right-hand side of (47). Using the Hölder inequality and Young inequality, we can get

$$\int_{\mathbf{U}} g(t,x)(-\Delta)^s u\mathrm{d}x \le \frac{1}{8}\left\|(-\Delta)^s u\right\|^2 + 2\left\|g(t)\right\|^2, \tag{49}$$

$$\left|\int_{\mathbf{U}} (-\Delta)^s \phi_1 z_1(\theta_t\omega_1)(-\Delta)^s u\mathrm{d}x\right| \le \frac{1}{8}\left\|(-\Delta)^s u\right\|^2 + 2\left|z_1(\theta_t\omega_1)\right|^2\left\|(-\Delta)^s \phi_1\right\|^2, \tag{50}$$

$$\left|\alpha_1\int_{\mathbf{U}} \phi_2 z_2(\theta_t\omega_2)(-\Delta)^s u\mathrm{d}x\right| \le \frac{1}{8}\left\|(-\Delta)^s u\right\|^2 + 2\alpha_1^2\left|z_2(\theta_t\omega_2)\right|^2\left\|\phi_2\right\|^2. \tag{51}$$

For the last term on the left-hand side of (47), we obtain

$$\left|\alpha_1\int_{\mathbf{U}} v(-\Delta)^s u\mathrm{d}x\right| \le \frac{1}{8}\left\|(-\Delta)^s u\right\|^2 + 2\alpha_1^2\left\|v\right\|^2. \tag{52}$$

It follows from (47)-(52) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 + 2\lambda_1\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 + \left\|(-\Delta)^s u\right\|^2 \\
&\le C\Bigg(\left\|(-\Delta)^{\frac{s}{2}}\tilde{u}\right\|^2 + \left\|\tilde{u}\right\|_{L^p(\mathbf{U})}^p + \left\|v\right\|^2 + \left|z_1(\theta_t\omega_1)\right|^p \\
&\quad + \left|z_1(\theta_t\omega_1)\right|^2 + \left|z_2(\theta_t\omega_2)\right|^2 + \left\|g(t)\right\|^2 + 1\Bigg).
\end{aligned} \tag{53}$$

Given $t \in \mathbf{R}^+, \tau \in \mathbf{R}$ and $\omega \in \Omega$, let $\sigma \in (\tau-1,\tau)$ and $r \in (\tau-2,\tau-1)$. Multiplying (53) by $\mathrm{e}^{\delta t}$, first integrating over (r,σ) and then integrating with respect to r on $(\tau-2,\tau-1)$, replacing ω by $\theta_{-\tau}\omega$ we have

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{s}{2}} u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2 \\
&\le \int_{\tau-2}^{\tau-1} \mathrm{e}^{\delta(r-\sigma)}\left\|(-\Delta)^{\frac{s}{2}} u(r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2 \mathrm{d}r \\
&\quad + C\int_{\tau-2}^{\tau-1}\int_r^{\sigma} \mathrm{e}^{\delta(\xi-\sigma)}\Bigg(\left\|(-\Delta)^{\frac{s}{2}}\tilde{u}(\xi,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\right\|^2 \\
&\quad + \left\|\tilde{u}(\xi,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\right\|_{L^p(\mathbf{U})}^p + \left\|v(\xi,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\right\|^2 \\
&\quad + \left|z_1(\theta_{\xi-\tau}\omega_1)\right|^p + \left|z_1(\theta_{\xi-\tau}\omega_1)\right|^2 + \left|z_2(\theta_{\xi-\tau}\omega_2)\right|^2 + \left\|g(\xi)\right\|^2 + 1\Bigg)\mathrm{d}\xi\mathrm{d}r \\
&\le \int_{-2}^{-1} \mathrm{e}^{\delta(r+\tau-\sigma)}\left\|(-\Delta)^{\frac{s}{2}} u(r+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2 \mathrm{d}r \\
&\quad + C\int_{-2}^{\sigma-\tau} \mathrm{e}^{\delta(\xi+\tau-\sigma)}\Bigg(\left\|(-\Delta)^{\frac{s}{2}}\tilde{u}(\xi+\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\right\|^2 \\
&\quad + \left\|\tilde{u}(\xi+\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\right\|_{L^p(\mathbf{U})}^p + \left\|v(\xi+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\right\|^2 \\
&\quad + \left|z_1(\theta_{\xi}\omega_1)\right|^p + \left|z_1(\theta_{\xi}\omega_1)\right|^2 + \left|z_2(\theta_{\xi}\omega_2)\right|^2 + \left\|g(\xi+\tau)\right\|^2 + 1\Bigg)\mathrm{d}\xi.
\end{aligned}$$

Let T be the constant in Lemma 4.1 and $T_0 = \max\{2,T\}$. By the fact that

$\tilde{u} = u + \phi_1 z_1(\theta_t \omega_1)$, from $\sigma \in (\tau - 1, \tau)$ and Lemma 4.1, for all $t \geq T_0$, we obtain

$$\begin{aligned} &\left\| (-\Delta)^{\frac{s}{2}} u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \right\|^2 \\ &\leq C \left(\int_{-2}^{\sigma-\tau} e^{\delta(\xi+\tau-\sigma)} \left\| (-\Delta)^{\frac{s}{2}} u(\xi + \tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \right\|^2 d\xi \right. \\ &\quad + \int_{-2}^{\sigma-\tau} e^{\delta(\xi+\tau-\sigma)} \left\| \tilde{u}(\xi + \tau, \tau - t, \theta_{-\tau}\omega, \tilde{u}_{\tau-t}) \right\|_{L^p(\mathrm{U})}^p d\xi \\ &\quad + \int_{-2}^{\sigma-\tau} e^{\delta(\xi+\tau-\sigma)} \left\| v(\xi + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) \right\|^2 d\xi \\ &\quad \left. + \int_{-2}^{\sigma-\tau} e^{\delta(\xi+\tau-\sigma)} \left(|z_1(\theta_\xi \omega_1)|^p + |z_1(\theta_\xi \omega_1)|^2 + |z_2(\theta_\xi \omega_2)|^2 + \|g(\xi + \tau)\|^2 + 1 \right) d\xi \right). \end{aligned} \tag{54}$$

From (17)-(18) and Lemma 4.1, we immediately concludes the proof.□

Note that Equation (22) has no any smoothing effect on the solutions. To overcome this difficulty, we must decompose the solution operator into two parts. Let v_1 and v_2 be the solution of the following problems, respectively,

$$\frac{dv_1}{dt} + \lambda_2 v_1 = 0, \tag{55}$$

$$v_1(\tau - t) = v(\tau - t, x), \tag{56}$$

$$\frac{dv_2}{dt} + \lambda_2 v_2 = \alpha_2 u + h(t, x) + \alpha_2 \phi_1 z_1(\theta_t \omega_1), \tag{57}$$

$$v_2(\tau - t) = 0. \tag{58}$$

Then $v = v_1 + v_2$. Multiplying (55) by v_1, we obtain

$$\left\| v_1(\tau, \tau - t, \theta_{-\tau}\omega, v_{1,\tau-t}) \right\|^2 = e^{-2\lambda_2 t} \left\| v_{1,\tau-t} \right\|^2. \tag{59}$$

Lemma 3.3. *Under conditions* (5)-(7) *and* (29), *for every* $\tau \in \mathbf{R}, \omega \in \Omega$ *and* $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T(\tau, \omega, B)$ *such that for all* $t \geq T$, *the solution* v_2 *of problem* (57)-(58) *satisfies*

$$\begin{aligned} &\left\| (-\Delta)^{\frac{s}{2}} v_2(\tau, \tau - t, \theta_{-\tau}\omega, 0) \right\|^2 \\ &\leq C + C \int_{-\infty}^0 e^{\delta r} \left(\|g(r + \tau)\|^2 + \|h(r + \tau)\|^2 + \left\| (-\Delta)^{\frac{s}{2}} h(r + \tau) \right\|^2 + 1 \right) dr. \end{aligned} \tag{60}$$

Proof. Multiplying (57) by $(-\Delta)^s v_2$, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| (-\Delta)^{\frac{s}{2}} v_2 \right\|^2 + \lambda_2 \left\| (-\Delta)^{\frac{s}{2}} v_2 \right\|^2 \\ &= \alpha_2 \int_{\mathrm{U}} u (-\Delta)^s v_2 dx + \int_{\mathrm{U}} h(t, x) (-\Delta)^s v_2 dx + \alpha_2 \int_{\mathrm{U}} \phi_1 z_1(\theta_t \omega_1) (-\Delta)^s v_2 dx. \end{aligned} \tag{61}$$

For the first term on the right-hand side of (61), we can get

$$\begin{aligned} \alpha_2 \int_{\mathrm{U}} u (-\Delta)^s v_2 dx &\leq \alpha_2 \left\| (-\Delta)^{\frac{s}{2}} v_2 \right\| \left\| (-\Delta)^{\frac{s}{2}} u \right\| \\ &\leq \frac{\lambda_2}{8} \left\| (-\Delta)^{\frac{s}{2}} v_2 \right\|^2 + \frac{2\alpha_2^2}{\lambda_2} \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2. \end{aligned} \tag{62}$$

For the second term on the right-hand side of (61), we have

$$\int_{\mathrm{U}} h(t,x)(-\Delta)^s v_2 \mathrm{d}x \le \frac{\lambda_2}{4}\left\|(-\Delta)^{\frac{s}{2}} v_2\right\|^2 + \frac{1}{\lambda_2}\left\|(-\Delta)^{\frac{s}{2}} h(t)\right\|^2. \tag{63}$$

For the last term, we have

$$\alpha_2 \int_{\mathrm{U}} \phi_1 z_1(\theta_t\omega_1)(-\Delta)^s v_2 \mathrm{d}x \le \frac{\lambda_2}{8}\left\|(-\Delta)^{\frac{s}{2}} v_2\right\|^2 + \frac{2\alpha_2^2}{\lambda_2}\left|z_1(\theta_t\omega_1)\right|^2\left\|(-\Delta)^{\frac{s}{2}} \phi_1\right\|^2. \tag{64}$$

It follows from (61)-(64) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|(-\Delta)^{\frac{s}{2}} v_2\right\|^2 + \delta\left\|(-\Delta)^{\frac{s}{2}} v_2\right\|^2\\ &\le C\left(\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 + \left\|(-\Delta)^{\frac{s}{2}} h(t)\right\|^2 + \left|z_1(\theta_t\omega_1)\right|^2\left\|(-\Delta)^{\frac{s}{2}} \phi_1\right\|^2\right).\end{aligned} \tag{65}$$

Multiplying (65) by $\mathrm{e}^{\delta t}$, integrating over $(\tau-t,\tau)$, and then replacing ω by $\theta_{-\tau}\omega$ we have

$$\begin{aligned}&\left\|(-\Delta)^{\frac{s}{2}} v_2(\tau,\tau-t,\theta_{-\tau}\omega,0)\right\|^2\\ &\le C\int_{-t}^{0}\mathrm{e}^{\delta r}\left\|(-\Delta)^{\frac{s}{2}} u(r+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\right\|^2 \mathrm{d}r\\ &\quad + C\int_{-\infty}^{0}\mathrm{e}^{\delta r}\left\|(-\Delta)^{\frac{s}{2}} h(r+\tau)\right\|^2 \mathrm{d}r + C\left\|(-\Delta)^{\frac{s}{2}} \phi_1\right\|^2\int_{-\infty}^{0}\mathrm{e}^{\delta r}\left|z_1(\theta_r\omega_1)\right|^2 \mathrm{d}r,\end{aligned} \tag{66}$$

which along with Lemma 4.1 and (17) conclude the proof.

In the following, we will prove the existence of $\mathcal{D}$-pullback random attractor for problem (21)-(24).

Lemma 3.4. *Suppose* (5)-(7) *and* (30) *hold. Then the continuous cocycle* Φ *of problem* (21)-(24) *has a closed measurable* $\mathcal{D}$ *-pullback absorbing set* $K=\{K(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}$ *which is given by*

$$K(\tau,\omega)=\left\{(u,v)\in H:\|u\|^2+\|v\|^2\le R(\tau,\omega)\right\}, \tag{67}$$

where $R(\tau,\omega)$ is defined by

$$R(\tau,\omega)=C+C\int_{-\infty}^{0}\mathrm{e}^{\delta s}\left(\|g(s+\tau)\|^2+\|h(s+\tau)\|^2+1\right)\mathrm{d}s. \tag{68}$$

Then for every $\tau\in\mathbf{R},\omega\in\Omega$ and $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T(\tau,\omega,B)>0$ such that the solution (u,v) of problem (21)-(24) with $(u_{\tau-t},v_{\tau-t})\in B(\tau-t,\theta_{-t}\omega)$ satisfies, for all $t\ge T$,

$$\left(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}),v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\right)\in K(\tau,\omega). \tag{69}$$

In addition, the random variable $R(\tau,\omega)$ as in (68) is tempered, *i.e.*, for any $c>0$,

$$\lim_{t\to+\infty}\mathrm{e}^{-ct}R(\tau-t,\theta_{-t}\omega)=0. \tag{70}$$

Proof. As a special case of Lemma 4.1 with $\sigma=\tau$, we obtain (69) immediately. Then we have

$$\Phi\left(t,\tau-t,\theta_{-t}\omega,B\left(\tau-t,\theta_{-t}\omega\right)\right)\subseteq K\left(\tau,\omega\right). \tag{71}$$

We now verify (70). By (68)

$$R\left(\tau-t,\theta_{-t}\omega\right)=C+C\int_{-\infty}^{0}\mathrm{e}^{\delta s}\left(\left\|g\left(s+\tau-t\right)\right\|^{2}+\left\|h\left(s+\tau-t\right)\right\|^{2}+1\right)\mathrm{d}s. \tag{72}$$

We have by (30)

$$\begin{aligned}
&\lim_{t\to+\infty}\mathrm{e}^{-ct}R\left(\tau-t,\theta_{-t}\omega\right)\\
&\le\lim_{t\to+\infty}\mathrm{e}^{-ct}\left(C+C\int_{-\infty}^{0}\mathrm{e}^{\delta s}\left(\left\|g\left(s+\tau-t\right)\right\|^{2}+\left\|h\left(s+\tau-t\right)\right\|^{2}+1\right)\mathrm{d}s\right)\\
&\le\lim_{r\to+\infty}\mathrm{e}^{-c\tau}\mathrm{e}^{-cr}\left(C+C\int_{-\infty}^{0}\mathrm{e}^{\delta s}\left(\left\|g\left(s-r\right)\right\|^{2}+\left\|h\left(s-r\right)\right\|^{2}+1\right)\mathrm{d}s\right)\\
&=0.
\end{aligned} \tag{73}$$

□

Next, we establish the $\mathcal{D}$-pullback asymptotic compactness of Φ in *H*, for this purpose, we need to split Φ as follows. Given $t\in\mathbf{R}^{+},\tau\in\mathbf{R},\omega\in\Omega$ and $\left(u_{\tau},v_{\tau}\right)\in H$, Let

$$\Phi_{1}\left(t,\tau,\omega,\left(u_{\tau},v_{\tau}\right)\right)=\left(0,v_{1}\left(t+\tau,\tau,\theta_{-\tau}\omega,v_{\tau}\right)\right), \tag{74}$$

and

$$\Phi_{2}\left(t,\tau,\omega,\left(u_{\tau},v_{\tau}\right)\right)=\left(u\left(t+\tau,\tau,\theta_{-\tau}\omega,u_{\tau}\right),v_{2}\left(t+\tau,\tau,\theta_{-\tau}\omega,0\right)\right). \tag{75}$$

where $v_{1}\left(t+\tau,\tau,\theta_{-\tau}\omega,v_{\tau}\right)$ is the solution of (55) with initial condition v_{τ} at initial time τ, and $v_{2}\left(t+\tau,\tau,\theta_{-\tau}\omega,0\right)$ is the solution of (57). By (25) we find that for every $t\in\mathbf{R}^{+},\tau\in\mathbf{R},\omega\in\Omega$, and $\left(u_{\tau},v_{\tau}\right)\in H$

$$\Phi\left(t,\tau,\omega,\left(u_{\tau},v_{\tau}\right)\right)=\Phi_{1}\left(t,\tau,\omega,\left(u_{\tau},v_{\tau}\right)\right)+\Phi_{2}\left(t,\tau,\omega,\left(u_{\tau},v_{\tau}\right)\right). \tag{76}$$

The $\mathcal{D}$-pullback asymptotic compactness of Φ is presented below.

Lemma 3.5. *Under conditions* (5)-(7), (27) *and* (30), *the continuous cocycle* Φ *associated with problem* (21)-(24) *is* $\mathcal{D}$*-pullback asymptotically compact in H, that is, for every* $\tau\in\mathbf{R},\omega\in\Omega$ *and* $B=\left\{B\left(\tau,\omega\right):\tau\in\mathbf{R},\omega\in\Omega\right\}\in\mathcal{D}$, *the sequence* $\left\{\Phi\left(t_{n},\tau-t_{n},\theta_{-t_{n}}\omega,\left(u_{0,n},v_{0,n}\right)\right)\right\}_{n=1}^{\infty}$ *has a convergent subsequence in H whenever* $t_{n}\to\infty$ *and* $\left(u_{0,n},v_{0,n}\right)\in B\left(\tau-t_{n},\theta_{-t_{n}}\omega\right)$.

Proof. Since $t_{n}\to\infty,B\in\mathcal{D}$ and $\left(u_{0,n},v_{0,n}\right)\in B\left(\tau-t_{n},\theta_{-t_{n}}\omega\right)$, by (59), (76) and Lemma 4.1 we find that $\left\{\Phi_{2}\left(t_{n},\tau-t_{n},\theta_{-t_{n}}\omega,\left(u_{0,n},v_{0,n}\right)\right)\right\}_{n=1}^{\infty}$ is bounded in *H*. Therefore, there exists $\left(\eta_{1},\eta_{2}\right)\in H$ such that, up to a subsequence

$$\Phi_{2}\left(t_{n},\tau-t_{n},\theta_{-t_{n}}\omega,\left(u_{0,n},v_{0,n}\right)\right)\to\left(\eta_{1},\eta_{2}\right)\text{weakly in }H. \tag{77}$$

On the other hand, by Lemma 4.2 and Lemma 4.3, there exist $C=C\left(\tau,\omega\right)>0$ and $N=N\left(\tau,\omega,B\right)$ such that for all $n\ge N$

$$\left\|\Phi_{2}\left(t_{n},\tau-t_{n},\theta_{-t_{n}}\omega,\left(u_{0,n},v_{0,n}\right)\right)\right\|_{V}^{2}\le C. \tag{78}$$

By the compactness of embedding $V \hookrightarrow H$ and (77) we have, up to a subsequence,

$$\Phi_2\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)\to\left(\eta_1,\eta_2\right) \text{ strongly in } H, \tag{79}$$

which implies that there exists $N_1=N_1\left(\tau,\omega,B,\varepsilon\right)>N$ such that for all $n\geq N_1$,

$$\left\|\Phi_2\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H^2\leq\epsilon. \tag{80}$$

By (59) and (76) we have

$$\begin{aligned}&\left\|\Phi\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H\\&\leq\left\|\Phi_2\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H+\left\|\Phi_1\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)\right\|_H\\&\leq\left\|\Phi_2\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H+\left\|v_1\left(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}\right)\right\|_{L^2(\mathbf{U})}\\&\leq\left\|\Phi_2\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H+\mathrm{e}^{-\delta t_n}\left\|v_{0,n}\right\|_{L^2(\mathbf{U})},\end{aligned} \tag{81}$$

which along with (80) implies

$$\left\|\Phi\left(t_n,\tau-t_n,\theta_{-t_n}\omega,\left(u_{0,n},v_{0,n}\right)\right)-\left(\eta_1,\eta_2\right)\right\|_H\to 0, t_n\to\infty, \tag{82}$$

as desired.□

Theorem 3.1. *Suppose* (5)-(7), (29) *and* (30) *hold. Then the continuous cocycle* Φ *associated with problem* (21)-(24) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}=\left\{\mathcal{A}\left(\tau,\omega\right):\tau\in\mathbf{R},\omega\in\Omega\right\}\in\mathcal{D}$ *in H.*

Proof. Note that Φ is $\mathcal{D}$-pullback asymptotically compact in H as demonstrated by Lemma 4.5 and has a closed measurable $\mathcal{D}$-pullback absorbing set by Lemma 4.4. Thus by proposition 2.1, we find that Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in *H*.□

In conclusion, we prove the existence of a pullback random attractor in *H* for the random dynamical system associated with the non-autonomous stochastic fractional FitzHugh-Nagumo system.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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