

# Generalized Fibonacci Sequence: Possible Template for the Constants of Nature

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## Abstract

The paper concerns the fundamental constants of physics. The fine structure constant is inferred first with the help of a generalized Fibonacci sequence. Next this generalized sequence is also implemented in order to show the interconnection between various fundamental constants. The analysis of an extended definition of Fibonacci sequence reveals that the fundamental constants take the meaning of archetypal templates that model the physical appearance of the observable Universe. Once more appears in this conceptual frame the natural and serendipitous link between quantum and relativistic theories.

## Keywords

Fibonacci Sequence, Fundamental Constants

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## 1. Introduction

The standard definition of Fibonacci sequence is

$$F = 1, 1, 2, 3, 5, 8, 13, \dots \quad (1.1)$$

quoted in the following as  $F = F(1.1)$  to emphasize the first two terms of the sequence. The first two numbers are crucial in determining all terms of the sequence, being by definition each term sum of the two preceding ones. This sequence was originally aimed to calculate the reproduction rate of rabbits under appropriate fertility hypotheses: the first two terms are required equal to one to signify the first couple of rabbits. If each couple becomes fertile at the end of the first month and gives birth to a new couple when the second month is completed, then the meaning of each term of the sequence is the number of fertile couples per month. From a mere mathematical point of view, it is well known why the Fibonacci numbers are related to the so-called “golden ratio”. In principle, how-

ever, no conceptual reason requires input values just equal to 1 if the sequence is not essentially restricted to the original purpose to which it was early aimed. In other words, nothing hinders to generalize this sequence in order to describe a wider range of natural phenomena simply changing the first two terms. Here is proposed the generalization of  $F(1.1)$  putting

$$F^* = F^*(f_0, f_1), \quad (1.2)$$

where  $f_0$  and  $f_1$  are two arbitrary numbers in principle not necessarily integers; this affects all successive terms. Clearly, the standard Fibonacci sequence  $F$  is uniquely defined; instead the mathematical implications of the modified sequence  $F^*$ , and thus their related physical meaning as well, just depending on how are defined the initial terms. Defining indeed

$$F^* = f_0, f_1, (f_1 + f_0), (2f_1 + f_0), (3f_1 + 2f_0), (5f_1 + 3f_0), (8f_1 + 5f_0), \dots \quad (1.3)$$

*i.e.*

$$F_1^* = f_0, \quad F_2^* = f_1, \quad F_4^* = 2f_1 + f_0, \quad F_n^* = F_n(f_0, f_1) = f_1 F_{n-1} + f_0 F_{n-2}, \quad (1.4)$$

the last equation evidences that now the standard values  $F_n$  still appear in  $F_n^*$  as a linear combination of  $F_{n-1}$  and  $F_{n-2}$  with coefficients  $f_0$  and  $f_1$ . As it actually means that the  $(n-1)$ -th and  $(n-2)$ -th terms of the sequence (1.2) are all multiplied by  $f_1$  and  $f_0$  respectively, (1.4) reads identically

$$F_n^* = F_{n-1}^{(1)} + F_{n-2}^{(0)}, \quad F_n^{(1)} = f_1 F_{n-1}, \quad F_n^{(0)} = f_0 F_{n-2}, \quad n > 2 \quad (1.5)$$

the inequality reminds that the standard coefficients of (1.3) appear indeed for  $n > 2$ , where are in fact calculable all  $F_n^*$  as sum of two previous  $F^{(0)}$  and  $F^{(1)}$  now defined by the arbitrary  $f_0$  and  $f_1$ . The recursive rule is therefore unchanged. Since the single terms of the new sequence  $F_n^*$  are actually linear combinations of two terms with different  $n$  of the standard sequence  $F_n$ , reasonably each addend of the former could have in principle its own physical meaning likewise as the number calculated in the latter. It is therefore reasonable the guess why the aforesaid ability of  $F$  in describing the natural phenomena should still hold and be even enhanced by the additional freedom degrees inherent  $F^*$ . This makes significant the appropriate definition of selected specific terms  $F_1^*$  and  $F_2^*$  suitable to provide information of interest about specific physical effects, likewise as each  $F_n$  yields the number of couples of rabbits reproduced in the  $n$ -th generations from the initial couple. To exemplify this point return back to the original Fibonacci sequence, where each term represents the result of all previous reproduction steps. There is no way to introduce in this (1.1) the chance that either partner generated from the initial couple, fertile by definition, could actually be sterile; in other words, all successive generations of rabbits are assumed healthy and fertile exactly as the former one. The common sense suggests however that in practice, for any biological reason, this is not realistic; is missing in the early Fibonacci calculation any reference to possible discontinuity caused by an external action affecting the reproduction regularity of rabbits. From a physical point of view this chance could instead take place for

example during the evolution of a system formed by an increasing number of particles from its initial configuration, e.g. early number of allowed states, to its current configuration after an appropriate number of intermediate steps; thus the evolutionary implication inherent the Fibonacci sequence can be advantageously modified and tailored to specific physical systems through the first two terms, as suggested by (1.4). In principle the deterministic character of (1.1) is inconsistent with the chance of describing quantum phenomena; also, the early  $F$  necessarily concerns phenomena describable via pure numbers. Instead the addends of  $F_n^*$ , as proposed here, skip this requirement, *i.e.* the various terms are required to be neither integers nor dimensionless only; is crucial in this respect the numerical and dimensional choice of the coefficients  $f_0$  and  $f_1$ . In practice, the first few terms of  $F^*$  read for example

$$\begin{aligned} F_3^* &= 2f_1 + f_0, & F_7^* &= 13f_1 + 8f_0, \\ F_{15}^* &= 610f_1 + 377f_0, & F_{20}^* &= 6765f_1 + 4181f_0; \end{aligned} \quad (1.6)$$

*i.e.* the coefficients that multiply  $f_1$  and  $f_2$  are just the respective numbers  $F_{n-1}$  and  $F_{n-2}$  of the standard sequence. So it seems worth investigating the chances opened by the kind of generalization prospected by (1.4), in particular the choice of the terms  $F_n^*$  pertinent to the specific physical problem of interest. For example, it is possible to multiply  $F_n^*$  by an arbitrary dimensional factor  $k_c$  in order that (1.6) reads

$$\begin{aligned} E_3^* &= (2f_1 + f_0)k_c, & E_7^* &= (13f_1 + 8f_0)k_c, \\ E_{15}^* &= (610f_1 + 377f_0)k_c, & E_{20}^* &= (6765f_1 + 4181f_0)k_c; \end{aligned} \quad (1.7)$$

the notation exemplifies the particular case where

$$k_c = \text{unit energy}. \quad (1.8)$$

So the terms of (1.6) and (1.7) are numerically identical although the latter represents a sequence of energies expressed in energy units corresponding to  $k_c$ , whereas the pure numbers of the standard Fibonacci sequence turn into energies directly related to the evolution of a physical system: the steps that describe the total number of the  $n$ -th generation of rabbits turn into the allowed energies of the system at various time steps. In the following we implement (1.6) for simplicity of notation, subtending however that when necessary the numerical terms of the generalized sequence  $F_n^*$  represent identically the physical meaning of the sequence

$$F_n^* \equiv \frac{E_n^*}{k_c}. \quad (1.9)$$

So (1.4) will be also implemented in a form immediately consequent putting

$$f_0 = \zeta'' f, \quad f_1 = \zeta' f; \quad (1.10)$$

owing to (1.5) it is possible to write (1.4) as exemplified in (1.6)

$$F_n^* = (\zeta' F_{n-1} + \zeta'' F_{n-2}) f, \quad f = f(\zeta), \quad (1.11)$$

where  $\zeta$  along with  $\zeta'$  and  $\zeta''$  are arbitrary parameters. Moreover, with-

out loss of generality, it is sensible the chance of regarding the  $n$ -th terms of  $F_n^*$  via the definitions

$$\begin{aligned} f(\zeta) &= \sigma_1 \alpha^\circ + \sigma_2 \alpha^{\circ 2} + \dots, \\ \zeta' &= 1 + \sigma_1' \alpha^\circ + \sigma_2' \alpha^{\circ 2} + \dots, \\ \zeta'' &= 1 + \sigma_1'' \alpha^\circ + \sigma_2'' \alpha^{\circ 2} + \dots, \end{aligned} \quad (1.12)$$

*i.e.* replacing the first two numbers of the standard sequence (1.1) with the series expansions (1.12) defining the parameters  $\zeta'$  and  $\zeta''$ , which just for this reason are both defined with 1 as zero order approximation value. The new information introduced via (1.12) is that now appears a unique parameter  $\alpha^\circ$  as a function of which is in principle calculable each  $F_n^*$  via suitable values of the various coefficients  $\sigma_j'$  and  $\sigma_j''$ , by definition uniquely fixed once for all and focused on the physical problem of interest. Note that if  $\sigma_j'$  and  $\sigma_j''$  are defined as

$$\sum_j \sigma_j' a^{\circ j} \geq 0, \quad \sum_j \sigma_j'' a^{\circ j} \geq 0, \quad \sum_j \sigma_j a^{\circ j} = k_c,$$

then, whatever the specific value of  $a^\circ$  might be, (1.11) reduces to either (1.6) or (1.7) depending on whether  $k_c = 1$  or for example  $k_c = \text{unit energy}$ ; clearly the latter chance implies regarding  $\sigma_j$  as dimensional factors with physical dimensions of energy according to (1.9).

So the positions (1.12) seem reasonable to define (1.11) consistently with particular cases of standard sequence (1.1) or extended sequence (1.6) or even dimensional extended sequence (1.7).

Also,  $\sigma_j = 0$  imply  $f(\zeta) = 0$  and  $F_n^* = 0$ , which means that for  $\alpha^\circ = 0$  the terms  $F_{n>2}^* = 0$  are identically null. This makes sense because (1.11) and (1.12) hold for  $n > 2$ ; so, with reference to the standard (1.1),  $F_1^* = 1$  and  $F_2^* = 1$  are enough to return back to the case of an initial couple of infertile rabbits.

Nevertheless, depending on the specific physical problem, it is possible that the general (1.11) admits useful simplifications; putting for example  $\zeta' = \zeta''$ , one finds

$$F_n^{**} = (F_{n-1} + F_{n-2}) \zeta' f. \quad (1.13)$$

This particular case is interesting because  $F_n^{***} \approx F_{n-1} + F_{n-2}$  is compliant just with the original Fibonacci intuition; indeed, recalling (1.9),  $\zeta' f$  is the general form of the factor  $k_c$ , which in turn can become itself mere dimensional factor. Furthermore, it is also possible that the series expansions (1.12) can be truncated to the respective first order approximation terms; if so, then (1.11)

$$F_n^{***} \approx (F_{n-1} + F_{n-2}) (1 + \sigma_1' \alpha^\circ + \sigma_2' \alpha^{\circ 2} + \dots) (\sigma_1 \alpha^\circ + \sigma_2 \alpha^{\circ 2} + \dots) \quad (1.14)$$

reduces to

$$F_n^{***} \approx (F_{n-1} + F_{n-2}) (1 + \sigma_1' \alpha^\circ) \sigma_1 \alpha^\circ. \quad (1.15)$$

Finally it is clear that just for this reason, despite all manipulations so far introduced, holds anyway

$$F_n^{***} > 0 \quad (1.16)$$

this condition is necessary in order that  $F_n^{***}$  and the initial  $F_n$  is conceptually consistent. It is not enough that both are defined by the same sum rule of successive terms, it is also necessary that all terms are in both cases still positive although numerically different. If so, then (1.16) can be legitimately regarded as generalized Fibonacci sequence even for  $\alpha^\circ < 0$ . Note that (1.13) is simply a particular case of (1.11) but in principle it does not contain any approximation, as instead (1.15) does. Thus these last equations must be motivated by and justified for the specific physical problem of interest case by case. Nevertheless both contain the essential feature that substantiates the aforesaid premise of the present paper, *i.e.*: 1) the physical dimensions of the parameter  $\alpha^\circ$ , and thus of  $\sigma'$  too, to fit the specific physical problem and 2) the actual chance of generalizing (1.1) through appropriate definitions of both parameters. Remains still evident the link between (1.15) and (1.1), which is identically reproduced putting in particular  $\alpha^\circ = 1$  and  $\sigma'_1 = \sigma_1 = 0$ . It is clear however that (1.15) does not require all higher order approximation coefficients  $\sigma'_{j>1} = 0$ , as it concerns the first order of approximation only. So, despite the mere numerical character of the approximation (1.15), any deviation from these boundary conditions is itself a sensible generalization of (1.1). If  $F_n^{***}$  is known for any  $n$ , (1.15) can be trivially solved with respect to  $\alpha^\circ$ ; *i.e.*

$$\alpha_\pm^\circ = \frac{-\sigma_1 \pm \sqrt{\sigma_1^2 + 4\sigma'_1\sigma_1 R}}{2\sigma'_1\sigma_1}, \quad R = \frac{F_n^{***}}{F_{n-1} + F_{n-2}}, \quad (1.17)$$

where  $\sigma_1 > 0$  and  $R > 0$  by definition whereas  $\sigma'_1$  can be in principle positive or negative.

The aim of the present paper is to show what has to do  $F^*$  with the fine structure constant  $\alpha$ , whose acknowledged numerical value is

$$\alpha = 0.007297352664. \quad (1.18)$$

As concerns this number, recall that implementing

$$e = 4.8025 \times 10^{-10} \text{ e.s.u.}, \quad G = 6.67430 \cdot 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$$

one finds that [1]

$$e = k_g \alpha G, \quad k_g = 0.986 \text{ g}^{3/2} \cdot \text{s} \cdot \text{cm}^{-3/2} \quad \text{i.e.} \quad \frac{eG}{\hbar c} \approx 1.01 \text{ cm}^{3/2} \cdot \text{g}^{-3/2} \cdot \text{s}^{-1}, \quad (1.19)$$

being  $k_g$  dimensional proportionality constant. The fact that this constant is numerically very close to 1 cannot be accidental, rather it shows that reasonably  $\alpha$  provides a direct physical link between  $e$  and  $G$ . The explanation and related implications have been concerned in [1]; however here it is worth emphasizing that even this relationship must be somehow inferable in a general conceptual frame aimed to understand profoundly the physical meaning of  $\alpha$  and to calculate its numerical value.

The standard Fibonacci sequence is, per se, a mere recursive list of pure numbers; nevertheless it actually appears in several events and objects occurring in

Nature [2] [3] [4] [5]. For example it is known that it describes the shape of nautilus shell and spirals during hurricane formation, undergoing cancer cell division and filament profile during spiral galaxy formation, chicken egg contour and curls of numerous insects [6] [7] [8], to name just a few. In principle nothing requires the correspondence between sequence of pure numbers and natural occurrences quantifiable through lengths, energies and momenta; as a matter of fact, however, the spirals consisting of arcs of circles inscribed into contiguous squares with size ratios progressively increasing in agreement with the Fibonacci numbers effectively overlap the observed shapes of galaxy arms and insect curls. The chance of examining these dimensionless ratios is just an example of how even pure numbers can take the worth of observable physical evidence. A huge literature is known about how and why this sequence is surprisingly suitable in describing natural phenomena [9]. The chance of acknowledging the existence of well-defined fingerprints while observing largest and smallest natural phenomena cannot be accidental; rather it suggests that, although surprisingly, a mere sequence of pure numbers becomes the key criterion underlying occurrences usually measured through dimensional quantities like lengths or times or velocities and explained through familiar concepts of physics like minimum energy or maximum probability.

In other words, here is the first leading idea of the present paper: it seems sensible to expect that the numbers (1.1) are actually a sort of core boundary condition that controls even the fundamental constants of Nature.

In this way, regardless of how these constants or their combinations govern fields or interactions, these numbers can be identified as templates to which conform from time to time any physical observable; the measure units and their physical dimensions become therefore simply factors to turn the experimental observation into quantifiable data and predicting ability of standard scientific research.

Here is thus the second leading idea of this paper: to demonstrate the existence of hidden templates to which Nature conforms through its fundamental constants.

In this respect the indirect correspondence between pure numbers and natural events seems a limiting restriction to the actual worth of (1.1) and its full implementability; nevertheless it suggests the realistic chance of describing observable phenomena directly through measurable parameters somehow related to (1.4), which requires in turn to convert the pure numbers of the sequence (1.1) into dimensional quantities like (1.9) with specific physical meaning. The basic step in this respect is the idea of extending the early Fibonacci sequence in order to calculate both dimensionless and dimensional constants of the Nature, like  $\alpha$  or proton to electron mass ratio  $r_{pe}$  or Avogadro number  $N_A$  on the one hand and electron charge  $e$  or Boltzmann constant  $K_B$  or  $G$  on the other hand.

In fact, this will be done implementing appropriately the coefficients  $f_0$  and  $f_1$ .

To understand this point in agreement with (1.7) and (1.9), note that according to (1.5)  $F^*$  is sum of two sequences obtained multiplying (1.1) a first time by  $f_0$  and a second time by  $f_1$  and next collecting the particular terms  $f_1 F_{n-1}$  and  $f_0 F_{n-2}$  as in (1.4). Analogously multiply all terms of  $F$  a first time by an arbitrary dimensional factor  $k_a f_1$ , so that all  $F_n$  turn into  $k_a f_1 F_n$ , and next by another arbitrary dimensional factor  $k_b f_0$ , so that all  $F_n$  turn now into  $k_b f_0 F_n$ ; the two sequences obtained in this way consist of terms with well defined physical dimensions, in general different when  $k_a \neq k_b$ . If for example the dimensional factor  $k_a$  is defined as  $\text{cm}^{-1} \cdot \text{g}^{-1} \cdot \text{s}^{-1}$ , then the  $f_1 k_a$  means  $f_1$  per unit length, unit mass and unit time. If more realistically  $k_a$  is unit energy and  $k_b$  unit momentum, then the sequence (1.1) early introduced to describe uniquely the population increment of rabbits now generates two separate sequences concerning possible time evolutions of energy and momentum of a physical system; note however that nothing changes from a mere numerical point of view, *i.e.* energy and momentum sequences are still defined by (1.1). Eventually, multiply again these sequences by  $f'_0$  and  $f'_1$  whose physical dimensions are defined by  $k_c/k_a$  and  $k_c/k_b$  as follows

$$\begin{aligned} F_n'^* &= f'_0 F_{n-2}' + f'_1 F_{n-1}', & f'_0 &= f_0 \frac{k_c}{k_a}, & f'_1 &= f_1 \frac{k_c}{k_b}, \\ F_{n-2}' &= F_{n-2} k_a, & F_{n-1}' &= F_{n-1} k_b \end{aligned} \quad (1.20)$$

or identically

$$\begin{aligned} F_n''^* &= f_0'' F_{n-2}'' + f_1'' F_{n-1}'', & f_0'' &= f_0 k_c k_a, & f_1'' &= f_1 k_c k_b, \\ F_{n-2}'' &= \frac{F_{n-2}}{k_a}, & F_{n-1}'' &= \frac{F_{n-1}}{k_b}, \end{aligned} \quad (1.21)$$

being  $k_c$  a third dimensional factor. So the addends  $f'_0 F_{n-2}'$  and  $f'_1 F_{n-1}'$  have both physical dimension  $k_c$  and therefore can be summed up likewise in (1.5), which yields indeed  $F_n'^* = k_c F_n^*$  in this example. Since all factors just introduced are in principle arbitrary, but actually defined here for convenience unitary with appropriate physical dimensions, in fact (1.20) and (1.21) keep identically the numerical features of (1.7) although describing through  $k_a$  and  $k_b$ , for example, unit energy and unit momentum. The primed notations emphasize that  $F_n'^*$  at the left hand side and right hand side of (1.20) are numerically identical to  $F_n^*$  of (1.4), being in turn  $f'_0$  and  $f'_1$  numerically identical to  $f_0$  and to  $f_1$  as well; so the numerical values of the early addends  $f_0 F_{n-2}$  and  $f_1 F_{n-1}$  of  $F_n^*$  are now directly referable to different physical quantities concerning the physical features of any system. The same considerations hold for (1.21).

In fact the current literature about the physical and biological implications of the standard sequence (1.1) concerns essentially contour profiling, shape matching and considerations on the importance of the golden ratio inherent its recursive rule only. All of this seems astonishing on the one hand, but also reductive on the other hand; the sequence of numbers (1.1) represents a preferential principle of Nature, quantitative formulas should be inferable likewise as

from any physical law. The present paper proposes a mathematical model to overcome the gap between quantitative physical data, *i.e.* the acknowledged values of some physical constants, and results predictable via (1.3) through direct and detailed calculations assigning specific physical meaning to  $f_0$  and  $f_1$ .

At this point however the true challenge is to demonstrate how far an appropriate choice of dimensional units really allows describing in fact a physical system; the goal is to implement explicitly  $F_n^{**}$  and  $F_n^{**}$  without changing the numerical value of  $F_n^*$ , anyway governed by the Fibonacci recursive rule. The chance of obtaining physical information must still hold while preserving the numerical values of two generalized Fibonacci sequences merged together.

The Section 2 introduces some preliminary hints about the fine structure  $\alpha$ ; this allows to calculate a first approximation numerical value of this constant in the Section 3. The Section 4 shows that with the help of the extended Fibonacci sequence the numerical value of  $\alpha$  can be calculated with improved approximation, one order of magnitude better. The Section 5 shows how to reveal via (1.3) with the help of (1.20) and (1.21) the subtle link between various fundamental constants, thus clarifying the concept of template itself. The importance of (1.20) and (1.21) will appear in the Section 5: owing to the numerical coincidence of  $f_0'$  and  $f_1'$  with  $f_0$  and  $f_1$ , and thus that of  $F_n^{**}$  with  $F_n^*$ , the calculations will be carried out implementing (1.4) although  $f_0'F_{n-2}'$  and  $f_1'F_{n-1}'$  will actually concern dimensional physical quantities. Accordingly, the paper consists in fact of two parts: the former one describes a “standard” approach based on (1.17) to infer  $\alpha$ , the latter one clarifies how actually the fundamental constants of Nature conform themselves to hidden templates compliant with a unique recursive rule, still that guessed by Fibonacci.

The text is organized with the main intention of making the exposition as self-contained as possible. For this reason also a few basic concepts of classical electromagnetism are shortly quoted below.

## 2. The Fine Structure Constant: Preliminary Considerations

Before obtaining an approximate numerical estimate of the dimensionless constant  $\alpha$  that characterizes the electromagnetic interaction, introduce first a few well-known concepts indicating how to start from first principles. Consider preliminarily the possibility of writing

$$\alpha = \frac{e^2}{\hbar c} = \xi \frac{\varepsilon \delta r}{\hbar c}, \quad (2.1)$$

where  $\xi/\hbar c$  is the proportionality constant linking  $\varepsilon \delta r$  to  $\alpha$ . Owing to this physical meaning of  $\alpha$ , try to identify an energy  $\varepsilon$  and length  $\delta r$  considering a bound system of charges. In particular it is sensible to think  $\varepsilon \delta r$  at the right hand side as product of hydrogen-like atom properties, whose analytical expressions are known and simple. As are useful short reminds about outcomes of elementary wave mechanics for a charged reduced mass  $m$  in a central field of nuclear charge  $Ze$ , introduce



$$\varepsilon = -\frac{Z^2 e^4 m}{2n^2 \hbar^2}, \quad \delta r = \frac{n^2 \hbar^2}{Z m e^2}, \quad m = \frac{m_e A}{m_e + A}, \quad (2.2)$$

where  $m_e$  and  $A$  are for example electron reduced and nucleus masses. In several papers, e.g. [10] [11], these formulas have been found without solving any wave function, but as straightforward consequence of the statistical formulation of the quantum uncertainty only

$$\delta p \delta x = n \hbar = \delta \varepsilon \delta t, \quad (2.3)$$

which in turn is a corollary itself of an operative definition  $\hbar G/c^2$  of space time [11] in the frame of an evolutionary quantum model of Universe [12]. Anyway, multiplying side by side the first two (2.2) one finds

$$\varepsilon \delta r = -\frac{(Ze)e}{2}, \quad (2.4)$$

whence  $|\xi| = 2/Z$  so that

$$\alpha = \frac{2}{Z} \frac{\varepsilon \delta r}{\hbar c}. \quad (2.5)$$

The remarkable simplicity of (2.2) is the reason of having introduced just the non-relativistic hydrogen-like atom, and not for example a complex many electron atoms: the latter implies complex many electron correlations and thus the lack of analytical formulas able to include various forms of interaction, the former is implemented as electromagnetic interaction between two charges only compliant with  $|(Ze)e|$  defining  $\alpha$ . This kind of interaction appears explicitly because it also follows that

$$|\varepsilon| = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^2 m c^2 = \frac{(Ze)e}{2 \delta r}, \quad (2.6)$$

$$\delta r = \frac{n \hbar}{m c} \left( \frac{n}{\alpha Z} \right) = n \lambda_c \left( \frac{n}{\alpha Z} \right), \quad \frac{\hbar}{m c} = \lambda_c;$$

the systematic presence of  $\alpha Z/n$  suggests that just  $\alpha Z$  is the key quantity to understand  $\alpha$ . Despite (2.2) include neither spin/orbit nor spin-spin electron/nucleus interaction or the Lamb shift, conceptually crucial even for  $Z=1$  although in fact numerically negligible, valuable information is in fact hidden in these equations; this suggests implementing (2.2) as an acceptable basis in defining both  $\varepsilon$  and  $\delta r$  of (2.1). The factor 1/2 is the fingerprint of the quantum uncertainty: although  $m$  and nucleus are  $\delta r$  apart, the Coulomb-like form requires the total uncertainty range of the charge  $-e$  with respect to  $Ze$  that indeed is  $2\delta r$  *i.e.* the total radial delocalization range of the electron around the nucleus. Of course  $\delta r$  is itself arbitrary and unknowable because  $n$  is arbitrary [1]; thus the physical content of (2.6) is well beyond that of the mere Coulomb law. Moreover it is easy to realize with the help of (2.3) that

$$\frac{\delta r}{\delta t} = \frac{n \lambda_c}{\delta t} \frac{n}{\alpha Z} = \frac{\lambda_c \delta \varepsilon}{\hbar} \frac{n}{\alpha Z} = \frac{\delta \varepsilon}{m c} \frac{n}{\alpha Z} = v_r, \quad v_r = \frac{\delta r}{\delta t}, \quad (2.7)$$

being  $v_r$  the average radial velocity of  $m$ . So, owing to (2.6),

$$\frac{\alpha Z}{n} = \frac{\delta\varepsilon/mv_r}{c} = \frac{\delta v_r}{c} \quad (2.8)$$

having defined  $\delta v_r$  as  $\delta\varepsilon/mv_r$  by dimensional reasons; indeed

$$\delta\varepsilon = mv_r \delta v_r = m\delta(v_r^2)/2 \quad \text{i.e.} \quad \varepsilon = mv_r^2/2 + \text{const} \quad (2.9)$$

is trivially the classical kinetic energy of a free particle of mass equal to the reduced mass  $m$  moving at radial velocity  $v_r$ . However the non-trivial consequence of this short analysis is that the early Bohr energy can be expressed not only via Coulomb energy, as it is obvious, but also via an expression not explicitly referred to charge interaction; rather it is simply related to the rest mass  $mc^2$  via  $\alpha Z/n$ , because now  $e^4$  of (2.2) becomes hidden in  $\alpha^2$  of (2.6). However, although  $\varepsilon$  is even expressed as classical kinetic energy of reduced mass  $m$  moving at velocity  $v_r$ , the quantization is still evident through (2.7).

At this point, before implementing these results to evaluate  $\alpha$ , note that just (2.6) suggest further chances to manipulate the early Bohr formulas to highlight explicitly the electromagnetic character of the Coulomb interaction via the electromagnetic field associated to a radial electromagnetic wave: if it is true that magnetic field  $\mathbf{H}$  is generated by moving charges, then this  $\mathbf{H}$  and the electric field  $\mathbf{E}$  itself should be somehow both hidden in (2.2). Also, the force  $\Phi$  acting on  $m$  due to these fields should also be contextually inferable. To show these points note first of all that trivial manipulations of (2.2) yield

$$\begin{aligned} |\varepsilon| &= \frac{Z^2 e^5 m^2 \hbar^2}{2n^2 \hbar^4 e m} = \frac{E \hbar^2 c}{2emc} = \frac{E \hbar \lambda_C c}{2e} = \frac{E \mu_B}{\alpha}, \\ E &= \frac{Z^2 e^5 m^2}{n^2 \hbar^4} = \left(\frac{\alpha Z}{n}\right)^2 \frac{e}{\lambda_C^2}, \quad \mu_B = \frac{e \lambda_C}{2} \end{aligned} \quad (2.10)$$

and also

$$\begin{aligned} \frac{e \lambda_C}{\mu_B} = 2, \quad |\varepsilon| &= \frac{\mu_B}{e \lambda_C} \left(\frac{\alpha Z}{n}\right)^2 mc^2 = \frac{(Ze)e}{2\delta r} = H \mu_B, \\ H &= \frac{mc^2}{e \lambda_C} \left(\frac{\alpha Z}{n}\right)^2 = \frac{Ze}{\lambda_C \delta r}, \end{aligned} \quad (2.11)$$

which define the moduli  $E$  and  $H$  of electric and magnetic fields inside the atom; contextually is even defined the Bohr magneton  $\mu_B$ . In fact (2.11) suggest what is well known, *i.e.* that charges randomly moving around the nucleus induce  $H$ , which however at the atom scale is quantized itself. Multiplying side by side (2.10) and (2.11), implement thus

$$\varepsilon^2 = \frac{EH \mu_B^2}{\alpha}. \quad (2.12)$$

Moreover, since

$$\mu_B^2 = \frac{e^2 \lambda_C^2}{4} = \frac{e^2 \lambda_C^3}{4 \lambda_C} = \frac{3}{16\pi} \frac{e^2}{\lambda_C} V_C = \frac{3}{16\pi} \eta V_C, \quad V_C = \frac{4\pi}{3} \lambda_C^3, \quad \eta = \frac{e^2}{\lambda_C}, \quad (2.13)$$

it follows then

$$\varepsilon^2 = \frac{3}{16\pi} \frac{EHV_C \eta}{\alpha}; \quad (2.14)$$

hence, according to the first (2.6),

$$\frac{\varepsilon^2}{\eta} = \frac{Z^2 e^4 \lambda_C}{4\delta r^2 e^2} = \frac{Z \lambda_C Z e^2}{2 \delta r 2\delta r} = \frac{Z \lambda_C}{2 \delta r} |\varepsilon|. \quad (2.15)$$

So, merging (2.14) and (2.15), from

$$\frac{\lambda_C}{\delta r} |\varepsilon| = \frac{3}{8\pi\alpha Z} EHV_C \quad (2.16)$$

and owing to (2.6)

$$\frac{\alpha Z}{n^2} = \frac{\lambda_C}{\delta r} \quad (2.17)$$

one finds

$$|\varepsilon| = \frac{3}{8\pi} \left( \frac{n}{\alpha Z} \right)^2 EHV_C, \quad \frac{EH}{|\varepsilon|} = \left( \frac{\alpha Z}{n} \right)^2 \frac{2}{\lambda_C^3}. \quad (2.18)$$

Note now that if the electric and magnetic fields of electromagnetic waves are orthogonal, then in fact numerically  $EH = |\mathbf{E} \times \mathbf{H}|$  while remaining true of course that the modulus at right hand side is related to cross vector product that points towards the propagation direction of the wave. In this sense a running plane wave is also described by and equivalent to the quantized energy in the volume  $V_C$  of hydrogenlike atom; *i.e.* photons generated by electron decay between discrete energy levels propagate as steady e.m. waves. In other words the energy density field  $HE$  of e.m. wave in  $V_C$  is quantized, whereas photons are just the energy quanta of field corresponding to

$$\delta|\varepsilon| = \delta|HEV_C| \quad (2.19)$$

between  $n$ -th allowed levels. Note in this respect that the second couple of (2.3) reads

$$\delta|\varepsilon| = n\hbar/\delta t = n\hbar\omega, \quad (2.20)$$

in agreement with Planck, whereas the first couple reads  $\delta p = nh/2\pi\delta x$ . Then

$$2\pi\delta x = n\lambda \quad \text{if } p = p_0 + \frac{h}{\lambda} \quad (2.21)$$

according to De Broglie wave definition of momentum with  $p_0$  arbitrary constant.

So the uncertainty requires on the one hand the corpuscular nature of the charge  $e$ , see Equations (2.9) and (2.2) itself, but also wavelike energy and momentum on the other hand, see oscillating fields  $\mathbf{E}$  and  $\mathbf{H}$  that describe an e.m. wave propagating within  $V_C$ . So the wave confined in this volume cannot be radiated without changing  $n$ , whereas photons can be however emitted/absorbed compatibly with the Pauli principle. Obviously analogous conclusion holds for any quantum system of charges bound with interaction strength constant  $\alpha$ : given  $|\varepsilon|$ , the greater the numerical value of  $\alpha$ , the greater the energy density

*EH*. In conclusion, at the left hand side of (2.19) still appears the well known quantized Bohr energy, at the right hand side appears the modulus *EH* of the Poynting vector  $\mathbf{E} \times \mathbf{H}$ ; the proportionality constant between the energies involves  $\alpha$ . As the last (2.11) shows that  $eH$  has physical dimensions of force, the question at this point is whether this force can be guessed and inferred itself straightforwardly from the basic Equations (2.2). While (2.2) are well known, they have marked the birth of the old quantum mechanics, nothing in principle excludes the chance of defining also a further equation alternative to (2.4); note indeed that appears at left hand side of (2.16) the ratio  $|\varepsilon|/\delta r$ . So, owing to the definitions of  $\lambda_c$  and  $\mu_B$ , write with the help of second (2.2)

$$\Phi_r = \frac{|\varepsilon|}{\delta r} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{m^2 c^3}{n \hbar} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{m c^2}{n \lambda_c} = m a_r, \quad a_r = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{c^2}{n \lambda_c}, \quad (2.22)$$

where  $\Phi_r$  has in fact physical dimensions of force; in effect this force appears reasonably expressed as  $m$  times a new amount  $a_r$  having physical dimensions of acceleration, quantized itself too. If so, then the force here acknowledged acting on  $m$  in the field of nucleus is the radial component of a more general quantized electromagnetic interaction force of modulus  $\Phi = |\Phi|$  between nucleus and electron at Bohr radial distance  $\delta r = |\delta r|$  around it, thus delocalized in an ideal sphere of diameter  $2\delta r$  centered on the nucleus. This explains why  $a_r$  results defined via  $c^2$  over  $n$  times the length  $2\lambda_c$  compliant with the quantum uncertainty. Replacing in (2.16), one finds therefore owing to (2.17)

$$\Phi_r \delta r = \frac{3}{8\pi\alpha^2} E H V_c. \quad (2.23)$$

The fact that at the left hand side appears  $\Phi_r \delta r$  suggests putting

$$\Phi \cdot \delta \mathbf{r} = (\Phi \delta r) \cos \varphi, \quad \Phi_r = |\Phi| \cos \varphi, \quad (2.24)$$

being  $\varphi$  an appropriate angle. Before concerning in some more detail (2.23) and  $\varphi$  of (2.24), note that the definitions (2.10) and (2.11) allow inferring immediately from the radial force component  $\Phi_r$  further well-known concepts that confirm the validity of the present considerations. Trivial manipulations of (2.22) yield with the help of (2.10)

$$\begin{aligned} \Phi_r &= \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{m c^2 e}{2 n \mu_B} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{m^2 c^2 e}{2 m n \mu_B} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{(m c)^2 c \gamma}{n \mu_B} \\ &= \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{\hbar^2 c \gamma}{n \lambda_c^2 \mu_B}, \end{aligned} \quad (2.25)$$

$$\gamma = \frac{e}{2 m c},$$

whence

$$\begin{aligned} \Phi_r &= \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{(m c)^2 \gamma c}{n \mu_B} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{M^2 c}{n \lambda_c^2 j(j+1) \mu_B} \gamma, \\ M^2 &= \hbar^2 j(j+1), \quad \frac{e \lambda_c}{\mu_B} = 2 \end{aligned} \quad (2.26)$$

once having introduced the square angular momentum  $M^2$  of  $m$  around the nucleus as a function of  $j = l \pm s$ ; then it is possible to write

$$\Phi_r = \left(\frac{\alpha Z}{n}\right)^3 \frac{|M|^3 c \gamma^2}{nj(j+1)\lambda_C^2 \mu_B^2}, \quad \frac{1}{2} = \gamma \frac{|M|}{\mu_B} \quad (2.27)$$

and thus, recalling the last (2.11) and that  $\mu_B/\gamma = \hbar$ ,

$$\Phi_r = \left(\frac{\alpha Z}{n}\right)^3 \frac{\hbar c \sqrt{j(j+1)}}{n\lambda_C^2}. \quad (2.28)$$

As concerns the second (2.27), especially interesting is the particular case where  $|M| = |L + S|$  reduces to the angular momentum of spin  $|S|$  of  $m$  only, in which case the second (2.27) yields the so-called Landé factor of  $m$ ; this equation reads indeed

$$2 = \frac{1}{\gamma} \frac{\mu_B}{|S|}, \quad \frac{1}{\gamma} = \frac{2mc}{e}, \quad (2.29)$$

Note however that usually  $\gamma$  and  $\mu_B$  are expressed through the rest mass  $m_e$  corresponding to  $m$  rather than through the reduced  $m$  itself. For example, in the case of a hydrogen-like atom with one electron, (2.10) and (2.25) yield

$$\frac{1}{\gamma} = \frac{2mc}{e} = \frac{2(m/m_e)m_e c}{e} = \frac{m}{m_e} \frac{1}{\gamma_e},$$

$$\mu_B = \frac{e\hbar}{2mc} = \frac{e\hbar m_e}{2m_e mc} = \mu_{eB} \frac{m_e}{m}, \quad \frac{m_e}{m} = \frac{m_e + A}{A},$$

hence replacing  $\gamma$  and  $\mu_B$  so far defined with  $\gamma_e$  and  $\mu_{eB}$ , (2.29) turns into

$$2 = \frac{1}{\gamma} \frac{\mu_B}{|S|} = \frac{m}{m_e} \frac{1}{\gamma_e} \frac{m_e}{m} \frac{\mu_{eB}}{|S|}$$

so that

$$2 = \frac{1}{\gamma_e} \frac{\mu_{eB}}{|S|}, \quad \frac{1}{\gamma_e} = \frac{2m_e c}{e},$$

as expected in non-relativistic approach. Moreover, since from (2.6)

$$\frac{1}{\lambda_C} \left(\frac{\alpha Z}{n}\right)^2 = \frac{He}{mc^2},$$

it also follows

$$\frac{mc^2}{\lambda_C H} \left(\frac{\alpha Z}{n}\right)^2 = e, \quad \gamma = \frac{1}{2} \frac{c}{\lambda_C H} \left(\frac{\alpha Z}{n}\right)^2 = \frac{2\pi\nu}{H}, \quad \nu = \frac{c}{2\lambda_C} \left(\frac{\alpha Z}{n}\right)^2, \quad (2.30)$$

being clearly  $\nu$  a frequency. Thus, results defined

$$\omega = H\gamma = \frac{eH}{2mc},$$

being  $\nu = H\gamma/2\pi$  the Larmor frequency.

Furthermore, comparing last (2.30) and first (2.6) one finds  $|\varepsilon| = \nu\lambda_C mc = (\nu h)(\lambda_C/h)mc$ , where  $\nu\lambda_C$  has physical dimensions of velocity;

so being  $\lambda_C = h/mc$  it follows that by definition  $|\varepsilon| = h\nu$  whereas  $\lambda_C/h = p$  by dimensional reasons, which are the old Planck and De Broglie relationships already identified while commenting (2.18). Note that all relations are quantized, being inferred from early (2.2) only, *i.e.* without any consideration about the classical definitions of angular and magnetic momenta. Moreover, dividing side by side (2.22) and (2.11), one finds

$$\frac{\Phi_r}{H} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right) \frac{e}{n}; \quad (2.31)$$

so according to (2.30)

$$\frac{n}{\alpha Z} \frac{v_\lambda}{c} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right), \quad v_\lambda = v\lambda_C$$

and thus replacing in (2.31)

$$\Phi_r = H \frac{v_\lambda e}{c}, \quad v_\lambda = \frac{\lambda_C \nu}{\alpha Z},$$

being  $v_\lambda$  the modulus of velocity  $\mathbf{v}_\lambda$  calculated via Compton length of  $m$  times the frequency defined in (2.30). The vector equation corresponding to this scalar relationship can be nothing else but

$$\Phi = e \frac{\mathbf{v}_\lambda}{c} \times \mathbf{H}$$

as stated before about (2.18), the magnetic field  $\mathbf{H}$  of the wave along with the electric field  $\mathbf{E}$ , must be orthogonal to its direction propagation velocity  $\mathbf{v}_\lambda$ . Hence,  $\mathbf{v}_\lambda \times \mathbf{H} = v_\lambda H \mathbf{n}$ , with the unit vector  $\mathbf{n}$  that defines the propagation direction of a transversal e.m. plane wave made by orthogonal magnetic and electric oscillating fields, whose quantization implies the concept of photon.

Nevertheless this elementary approach has a further interesting implication. Define the function

$$G\Phi_r = \frac{1}{2} G \left( \frac{\alpha Z}{n} \right)^3 \frac{m^2 c^3}{n\hbar} = Xc^4, \quad (2.32)$$

where  $G$  is the gravity constant and thus  $X$  a dimensionless function to be found; indeed the physical dimensions of the product  $G$  times *force* are *velocity*<sup>4</sup>. Since

$$X = G \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{m^2}{n\hbar c},$$

then

$$X = G \frac{m^2}{\epsilon_0 \lambda}, \quad \epsilon_0 \lambda = n \frac{\hbar c}{\frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3}$$

being  $\lambda$  an arbitrary length and  $\epsilon_0$  an energy defined as

$$\epsilon_0 = \frac{n\hbar\omega_0}{\frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3}, \quad \omega_0 = \frac{c}{\lambda}.$$

So

$$\epsilon_0 = G \frac{m_1 m_2}{\lambda}, \quad m_1 m_2 = \frac{m^2}{X}. \quad (2.33)$$

Note that being  $\lambda$  arbitrary, are also arbitrary  $m_1 m_2$  and thus  $m_1$  and  $m_2$  themselves. The interesting fact is that even the gravitational law appears to be nested in a natural way in the present elementary approach.

This result supports the validity of (1.19). Eventually calculate also

$$\frac{G}{\Phi_r} = \frac{G}{m a_r} = \left( \frac{\ell}{m} \right)^2,$$

where  $\ell$  is another arbitrary length; the last equality is inferred by dimensional reasons. Thus, owing to the second (2.22),

$$\ell = \left( \frac{mG}{a_r} \right)^{1/2}, \quad a_r = \frac{mG}{\ell^2} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{c^2}{n \lambda_c}$$

that defines  $\ell$ . Consider now the particular value  $\ell_{bh}$  of the length  $\ell$  defined as follows

$$\ell_{bh} = \frac{2mG}{c^2}, \quad (2.34)$$

which implies

$$\frac{2mG}{c^2} = \left( \frac{mG}{a_r} \right)^{1/2}, \quad \frac{4(mG)^2}{c^4} = \frac{mG}{a_r}, \quad a_r^{(bh)} = \frac{c^4}{4mG} = \frac{1}{2} \frac{c^2}{\ell_{bh}};$$

the form of the particular acceleration  $a_r^{(bh)}$  is the same as that reported in (2.22) for  $a_r$ . Note that

$$a_r^{(bh)} \frac{\ell}{c^2} = \frac{\ell c^2}{4mG} = \frac{1}{\delta\phi} \quad (2.35)$$

the last equality is obvious, thinking that in general  $v^2/r$  is the classical centripetal force acting on a point mass traveling along a circular path at tangential speed  $v$ . So  $\ell \delta\phi$  is the length of an arc of circumference defined by the angle  $\delta\phi$  traveled by the point mass. The fact that here appears in particular  $v \equiv c$  means that this elementary interpretation holds for a photon. Thus

$$\delta\phi = \frac{4mG}{\ell c^2} \quad (2.36)$$

In effect, this result is nothing else than the deflection angle  $\delta\phi$  of a photon moving in the gravity field of  $m$  at distance  $\ell$ , which corresponds to the famous light deflection of light beam traveling in a curved space time; the first order approximation of Einstein approach corresponds here to having approximated via (2.35) the actual photon path just along a circular arc.

This short and elementary discussion has shown that basic concepts of classical electromagnetism are hidden in the simple Bohr results (2.2), in turn inferable as straightforward and serendipitous corollaries of (2.3). Moreover this result shows once more what has been emphasized several times in previous papers [10] [11] [12] [13], *i.e.* the intimate and natural merging of quantum phys-

ics and relativity.

It is useful therefore to show shortly here how all results hitherto inferred are related to classical physics on the one hand and relativistic physics on the other hand; of course here this last aspect of the problem is shortened as much as possible, it has been more thoroughly examined in [11] [12]. Nevertheless, simple and short final remarks allow extending these non-relativistic concepts simply implied by the early Bohr approach. Define indeed more in general  $\Phi_r$  acting on the electron in the field of nucleus as  $\Phi$  is no longer required to be radial component as prospected by the second (2.24), whereas the electron delocalization around the nucleus is described by  $\delta r \cos \varphi$ . Then assuming  $\varphi = \varphi(\alpha)$ , as it is natural owing to (2.24) and (2.22), and expanding in series  $\cos \varphi$  around  $\varphi = 0$  one finds recalling (2.22)

$$\mathcal{E} = \varepsilon_B (1 + \zeta_1 \alpha + \zeta_2 \alpha^2 + \dots) = \varepsilon_B + \dots, \quad \varepsilon_B = \Phi \delta r, \quad \zeta_j = \zeta_j(Z, A, m, N), \quad (2.37)$$

being clearly  $\zeta_j$  the coefficients of series expansion; in this way the early Bohr energy  $\varepsilon_B$  introduced in (2.2) and so far implemented results to be in fact the first order term only of a more complex energy  $\mathcal{E}$  in which are hidden further forms of interaction. This latter statement can be explicitly evidenced writing

$$\mathcal{E} = \varepsilon_B + (\zeta_1 \alpha + \zeta_2 \alpha^2 + \zeta_3' \alpha^3 + \dots) \varepsilon_B + \zeta_3'' \alpha^3 \varepsilon_B + \dots, \quad \zeta_3 = \zeta_3' + \zeta_3'', \quad (2.38)$$

in this way the first addend  $\varepsilon_B$  of  $\mathcal{E}$  is the Bohr zero order term, the second addend yields the electron correlation terms as a function of  $\alpha$  for a number of electrons  $N \geq 1$ , which is certainly possible in principle determining appropriately the series coefficients  $\zeta_j$  of (2.37). In particular the third addend  $\propto \alpha^5$  is the fingerprint of the Lamb energy due to the "vacuum" polarization. Even for  $N = 1$  appear thus expectable in principle the electron spin-orbit and spin-spin interactions missing in (QQQ). These higher order effects have been considered in [12] [13] simply starting from the statistical formulation of quantum uncertainty, but will not be considered here for brevity. However it is enough noticing that rewriting (2.3) as  $\delta \varepsilon = v \delta p$ , with velocity modulus defined by  $v = \delta x / \delta t$ , and multiplying both sides by  $\varepsilon$ , one finds  $\varepsilon \delta \varepsilon = \varepsilon v \delta p$ . To shorten as much as possible this discussion, put now by definition  $p = \sigma \varepsilon v$ , where  $\sigma$  is a proportionality constant; this position is purposeful to obtain  $\varepsilon \delta \varepsilon = \sigma \varepsilon v \delta(\varepsilon v)$  that reads then

$$\frac{1}{2} \delta(\varepsilon^2) = \frac{1}{2} \sigma \delta(\varepsilon v)^2, \quad \sigma = \text{constant}.$$

Hence  $\delta(\varepsilon^2) = \delta \sigma (\varepsilon v)^2$  yields identically

$$\delta(\varepsilon^2) = \delta(\sigma (\varepsilon v)^2 + \text{const}), \quad (2.39)$$

because of course  $\delta \text{const} = 0$  by definition. So the last equation reads

$$\varepsilon^2 = \sigma (p/\sigma)^2 + \text{const}.$$

Since by dimensional reasons  $\sigma = \text{velocity}^{-2}$ , this constant velocity can be nothing else but  $\sigma = c^{-2}$ . So



$$\varepsilon^2 = (pc)^2 + \text{const}, \quad p = \varepsilon \frac{v}{c^2}. \quad (2.40)$$

Moreover implement the boundary condition for  $v \rightarrow 0$  to find *const*, which has physical dimensions of square energy: as

$$\lim_{v \rightarrow 0} \frac{p}{v} = m = \frac{\varepsilon_0}{c^2},$$

then (2.39) reads

$$\varepsilon_0^2 = (mc^2)^2, \quad \lim_{v \rightarrow 0} \varepsilon^2 = \text{const} = \varepsilon_0^2 = (mc^2)^2. \quad (2.41)$$

Clearly (2.41) and (2.40) imply the Lorentz factor  $\sqrt{1-v^2/c^2}$ . Calculate with the help of (2.2) and (2.40) via (2.3), then

$$(\delta\varepsilon)^2 - (\delta pc)^2 = \frac{n^2 \hbar^2}{(\delta t)^2} - \frac{n^2 \hbar^2 c^2}{\delta x^2} = (n\hbar)^2 \frac{(\delta x)^2 - (\delta ct)^2}{(\delta x \delta t)^2};$$

if one implements arbitrary ranges  $\delta\varepsilon = \varepsilon_2 - \varepsilon_1$  and  $\delta p = p_2 - p_1$  at the left hand side with  $\varepsilon$  and  $p$  fulfilling (2.40), it follows that the ratio at the right hand side must be invariant. This means

$$(\delta x)^2 - (c\delta t)^2 = \text{inv}, \quad \delta x \delta t = \text{inv}.$$

Of course the same holds considering  $(\delta pc)^2 - (\delta\varepsilon)^2$  that would yield  $(c\delta t)^2 - (\delta x)^2 = \text{inv}$ . These invariant results are actually the signature of the quantum uncertainty; they hold not only for a free particle, but also for a particle of reduced mass  $m$  subjected to e.m. interaction. It is known that the 4D invariant interval, here symbolized by  $\delta x^2$  only instead of  $\delta x_i^2, i=1,2,3$  for 3 space coordinates, is the conceptual basis of special relativity [14].

Finally implement Equation (2.8) dividing both sides by  $c^2 \delta t$  and write according to (2.7)

$$\frac{\delta\varepsilon/mc^2}{v_r \delta t} = -\frac{\delta\phi/c^2}{\delta r} = -\frac{\delta v_r}{\delta t}, \quad \phi = \frac{\varepsilon}{m}, \quad (2.42)$$

so that, replacing  $\varepsilon = \hbar\omega$  to describe in particular a photon in the field  $\phi$ , one finds the relationship  $\delta(\hbar\omega/mc^2) = \delta\phi/c^2$ ; thus the respective energy changes yield

$$\frac{\delta\omega}{\omega_0} = \frac{\delta\phi}{c^2}, \quad \omega_0 = \frac{mc^2}{\hbar}. \quad (2.43)$$

Being  $\delta\omega$  arbitrary, define in general  $\delta\omega = \omega_1 - \omega_0$ . Note that (2.42) has the form of one dimensional space gradient component  $\delta\phi/\delta r$  of the function  $\phi$  equal to velocity component change rate  $\dot{v}_r = \delta v_r/\delta t$ , i.e. in the usual 3D form  $\nabla\phi = -\dot{v}$ , typical of the energy potential [14]. So, regarding the left hand side of (2.43) as the energy change of a photon in the field gradient  $\phi$ , the result

$$\frac{\delta\omega}{\omega_0} = \frac{\phi_s - \phi_o}{c^2}$$

concerns energy loss  $\delta\omega < 0$  of a photon moving against the field strength gra-

dient; the subscripts stand for source and observer respectively. This equation is in particular the red shift of a photon beam moving from the emission point of a source,  $\phi_s$ , towards the observation point,  $\phi_o$ , with decreased energy potential  $\phi_s < \phi_o$ .

These short considerations have evidenced the importance of the constant  $\alpha$  in determining the physical properties of hydrogenlike atoms and the existence of quantized electromagnetic field required by the elementary Bohr formulas. Mostly important, this elementary analysis of the e.m. interaction shows that there is no conflict between QM and relativity; rather, trivial algebraic steps are enough to bridge both theories.

With this introductory background in mind, it is possible to tackle self-consistently in the next sections the problem of determining the numerical value of  $\alpha$ .

### 3. First Approximate Calculation of $\alpha$

After having implemented (2.2) to emphasize a few basic concepts of electromagnetism, is reasonable the attempt to implement also  $\Phi_r$  to get a preliminary estimate of  $\alpha$ . Note that the Equations (2.22) contain explicitly  $m$ , whose value determines whether these equations describe specifically hydrogenlike atoms or mesic atoms or even a proton/antiproton system; thus it is convenient in principle to rewrite (2.22) in dimensionless form

$$\Phi_r^* = \Phi_r \frac{n\hbar}{m^2 c^3} = \frac{\Phi_r}{2\pi} \frac{nh}{m^2 c^3} = \frac{n\Phi_r}{\pi} \frac{\lambda_c}{2mc^2} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \quad (3.1)$$

in order that  $\Phi_r^*$  holds for any bound system governed by electromagnetic interaction with central charge  $Ze$  regardless of the particular reduced mass  $m$  of the charge  $e$ . First of all (3.1) suggests identifying at the left hand side of the last equality the amount  $(\Phi_r/2)(\hbar/m^2 c^3)n$  such that

$$\frac{n\Phi_r}{ma_r} = \frac{n\Phi_r}{\Phi_r} = n = \frac{\pi}{2} \left( \frac{\alpha Z}{n} \right)^3, \quad \Phi_r = ma_r, \quad a_r = \frac{2c^2}{\lambda_c}, \quad (3.2)$$

where clearly  $a_r$  is the radial acceleration of  $m$  due to the force  $\Phi_r$ . So comparing with (2.22)

$$a_r = \frac{2c^2}{\lambda_c} = \frac{1}{2} \left( \frac{\alpha Z}{n} \right)^3 \frac{c^2}{n\lambda_c} = \pi \left( \frac{\alpha Z}{n} \right)^3 \frac{c^2}{\lambda_c}$$

*i.e.*

$$n^4 \frac{2}{\pi} = (\alpha Z)^3, \quad \alpha = \alpha(Z, n).$$

In this way,  $\alpha$  is a function of  $n$  and  $Z$ . To define uniquely the value of  $\alpha Z$  is interesting to calculate this result in particular for the fundamental state  $n=1$  and  $Z=Z_{\max}$ , which actually calculates the minimum value allowed for  $\alpha$ . Introducing thus the boundary condition  $n=1$  and  $Z=Z_{\max}$  that defines  $\Phi_{\max}^* = \Phi_r^*(n=1, Z=Z_{\max})$ , the last equation reads

$$\alpha = \left(\frac{2}{\pi}\right)^{1/3} \frac{1}{Z_{\max}}. \quad (3.3)$$

This result can be calculated recalling that in [15] it has been found that the upper limit of high atomic number heavy nuclei is  $Z_{\max} = 118$ . Then

$$\alpha = \left(\frac{2}{\pi}\right)^{1/3} \frac{1}{118} = 0.0072902883, \quad dev = -0.0968\%. \quad (3.4)$$

The deviation of  $\alpha$  calculated in this way from the true value (1.18) is due to the non-relativistic approximation of the Equations (2.2); even so, however, the approach hitherto followed appears basically correct.

The agreement provided by this kind of preliminary approach, although carried out through classical concepts of acceleration and force, explains the stability of the quantum system of interacting charges. The definition of acceleration in (3.2) reminds formally that  $v^2/r$  of the circular motion of  $m$  tethered around a fixed point, which however before the birth of QM has posed the problem of explaining why there is no radiation loss in the case of a bound system of a charge  $m$  around a fixed nucleus. The reason is that classically, being in general  $v = v(t)$  and  $r = r(t)$ , the emission of radiation is allowed to occur contextually to the spiral motion of the charge  $m$  approaching closer and closer to the nucleus till to the collapse of the system. However the definition (3.2) of  $a_r$  prevents such behavior, because both  $c$  and  $\lambda_c$  are constant quantities: here instead no change of  $c$  and  $\lambda_c$ , and then no collapse. Hence the atomic system does not radiate because  $m$  cannot spiral towards the nucleus; the Bohr postulate is actually the constancy of light speed in vacuum. Nevertheless the system can still emit discrete amounts of energy, we call them photons, changing  $n$  by integer amounts.

Anyway, a much more effective approach to obtain a better value of  $\alpha$  is next carried out implementing the Fibonacci sequence.

#### 4. The Generalized Fibonacci Sequence

Often an approximation is useful to check immediately and easily whether or not any reasoning or algebraic steps point to the right direction in calculating something, provided that however the basic requirements of the calculation model are still valid; for this reason this section starts just from (1.16) and (1.17), where  $\sigma_1 > 0$  and  $R > 0$  by definition whereas  $\sigma'_1$  can be in principle positive or negative.

When switching from (1.1) to (1.3), two considerations arise about  $f_0$  and  $f_1$  replacing 1. The first problem concerns the chances  $f_0 > 1$  and  $f_1 > 1$  or  $0 < f_0 < 1$  and  $0 < f_1 < 1$ , both possible in principle. The second problem is that just the most general meaning of (1.3) requires determining two initial unknowns, a sort of boundary condition without which the sequence is in fact undefined. These requirements are compliant with (1.10) regarding either  $\sigma'_j > 0$  and  $\sigma''_j > 0$  or  $\sigma'_j < 0$  and  $\sigma''_j < 0$  in the definitions (1.12). Actually both

requirements are fulfilled in the simplified form (1.15) at the first order only, where  $\sigma_1' = \sigma_1''$ ; even neglecting the higher order coefficients, it is enough to regard either  $\sigma_1' > 0$  or  $\sigma_1' < 0$ , in the following indicated with the notation  $-|\sigma_1'|$ . For  $\sigma_1' > 0$  holds the upper sign only, *i.e.* the first (1.17) reads

$$\alpha_+^o = \frac{-1 + \sqrt{1 + 4\sigma_1'R/\sigma_1}}{2\sigma_1'}, \quad (4.1)$$

whereas for  $\sigma_1' < 0$  both signs of the solutions are admissible but with the following condition

$$\alpha_{\pm}^o = \frac{-\sigma_1 \pm \sqrt{\sigma_1^2 - 4|\sigma_1'|\sigma_1 R}}{-2|\sigma_1'|\sigma_1} = \frac{1 \mp \sqrt{1 - 4R|\sigma_1'|/\sigma_1}}{2|\sigma_1'|}, \quad 4|\sigma_1'|\sigma_1 R \leq 1. \quad (4.2)$$

This last case is interesting as

$$\alpha_+^o + \alpha_-^o = \frac{1}{|\sigma_1'|} \quad (4.3)$$

*i.e.* if  $|\sigma_1'| = 1$ , then  $\alpha_+^o$  and  $\alpha_-^o$  take probabilistic meaning whose sum yields the certainty. This means that  $F_n^{***}$  has probabilistic meaning, as it can take here two values depending on whether  $\alpha^o$  takes either value  $\alpha_+^o$  or  $\alpha_-^o$ . Clearly (1.17) admits in general several values of  $\alpha^o$  depending on the number of series expansion terms in (1.14); yet it is reasonable to think that a probabilistic equation analogous to (4.3) still holds for the sum of all real solutions  $\alpha_j^o$ . Anyway, since  $|\sigma_1'| = 1$  with  $\sigma_1' < 0$  means  $\sigma_1' = -1$ , now (1.15) reads

$$\frac{F_n^{***}}{\sigma_1} = (F_{n-1} + F_{n-2})(1 - \alpha^o)\alpha^o. \quad (4.4)$$

As (3.4) implies that  $\alpha$  must depend on  $Z_{\max}^{-1}$ , it is reasonable to expect also now

$$\alpha^o = \frac{\text{const}}{Z_{\max}}, \quad \frac{F_n^{***}}{\sigma_1} \approx Z_{\max} \quad (4.5)$$

so that the solution of (4.4) with respect to  $\alpha^o$  reads

$$\alpha_{\pm}^o = \frac{1 \mp \sqrt{1 - 4Z_{\max}/(F_{n-1} + F_{n-2})}}{2}, \quad 4Z_{\max} \leq F_{n-1} + F_{n-2}. \quad (4.6)$$

Since by definition two known addends  $F_{n-1} + F_{n-2}$  concur to the  $n$ -th term of the standard (1.1), which is thus known, this result is calculable as a function of  $Z_{\max}$ . If the present reasoning is correct, it should be true that

$$\alpha = \frac{\alpha_{\pm}^o}{Z_{\max}} = \frac{1 \mp \sqrt{1 - 4Z_{\max}/(F_{n-1} + F_{n-2})}}{2Z_{\max}}; \quad (4.7)$$

in other words,  $\alpha_{\pm}^o$  replace the factor  $(2/\pi)^{1/3}$  of (3.4). Yet the interesting fact is that now not only  $\alpha_{\pm}^o$  depends itself upon  $Z_{\max}$  but also that it can take two values. To assess this conclusion according to the condition (4.6), it is enough to put  $Z_{\max} = 118$ , *i.e.*  $4Z_{\max} = 472$ , and check which  $n$  of (1.1) fulfills

$$\frac{1 \pm \sqrt{1 - 472 / (F_{n-1} + F_{n-2})}}{2} \approx \left(\frac{2}{\pi}\right)^{1/3}, \quad F_{n-1} + F_{n-2} \geq 472 \quad (4.8)$$

if the reasoning is correct, then an appropriate value of  $n$  must exist such that either sign should provide a better approximation to the true value of  $\alpha$ . A few results for selected values of  $n$  are reported in following **Table 1** that for convenience of comparison also reminds the reference value (4.8) early introduced in (3.3) to calculate (3.4).

$$\left(\frac{2}{\pi}\right)^{1/3} = 0.86025\dots$$

It appears that in effect for  $n = 16$  the expression (4.8) with plus sign fulfills unambiguously the given similarity condition  $\alpha_+^o \approx (2/\pi)^{1/3}$ , which hopefully should provide the value of  $\alpha_+^o Z_{\max}$  of interest to calculate according to (4.7) a better value of  $\alpha$ . Indeed

$$\alpha_{calc} = \frac{\alpha_+^o|_{n=16}}{Z_{\max}} = 0.007298074622, \quad dev = 0.0098\% \quad (4.9)$$

now the deviation from (1.18) is one order of magnitude better than that of (3.4). Apart from the enhanced numerical agreement of this result, a crucial question arises now: is it possible to improve further the accuracy of the calculations? From the mere numerical point of view, certainly the implementation of (1.14) instead of (1.15) affects the final approximation inherent the value (4.9); so, it is in principle reasonable to expect that the calculation of  $\alpha_{\pm}^o$  including in (1.17) higher order terms prospected by (1.14) would bring to a result even better than (4.9).

The calculations proposed in this section aimed merely to provide a first answer to the challenge proposed at the end of the Section 1, *i.e.*: the Fibonacci sequence appears in fact adequate to calculate an acceptable value of  $\alpha$  if appropriately implemented. Nevertheless the next section will describe in this respect a much better and far reaching approach by following a completely different strategy.

## 5. Calculation of Fundamental Constants via (1.4)

Consider again (1.4) with the purpose of correlating the fundamental constants of nature, which are now assumed all known: this section aims indeed to find the possible interconnection between these constants.

**Table 1.** Calculation of  $\alpha_+^o$  and  $\alpha_-^o$  according to the Equation (4.7) for various values of  $n$ . The solutions for  $n < 15$  are imaginary.

$n$	$\alpha_+^o$	$\alpha_-^o$
15	0.7378179493	0.2621820507
16	0.8611728054	0.1388271946
17	0.9196563582	0.0803436418
18	0.9520335888	0.0479664112

To this purpose implement in particular Avogadro's number  $N_A$  and ratio  $r_{pe}$  of proton to electron rest masses  $m_p/m_e$ , here reported for completeness along with  $\alpha$  itself quoted in (1.18)

$$r_{pe} = \frac{m_p}{m_e} = 1836.15267343, \quad N_A = 6.02214076 \times 10^{23}. \quad (5.1)$$

It is possible in principle to calculate  $f_0$  and  $f_1$  for various test values of  $n$  by defining appropriately  $F_n^*$ , for example through the positions

$$\frac{f_1 - f_0}{n} = \alpha, \quad \frac{F_n^*}{N_A} = 3, \quad F_n^* = f_1 F_{n-1} + f_0 F_{n-2}, \quad (5.2)$$

and solving this system of two equations with respect to  $f_0$  and  $f_1$ . The result of this calculation that specifies the respective  $F_n^*(n)$  is reported in **Table 2** for selected  $n$  only, the ones of interest for the next considerations.

Regarding  $n$  as arbitrary numerical parameter, this calculation has actual physical worth only if it somehow identifies among the various  $f_0(n)$  and  $f_1(n)$  a particular value  $n^*$  uniquely related to some specific quantity already introduced, like for example  $\alpha_+^o$  and  $\alpha_-^o$  of **Table 1**; if so, then (5.2) not only define a relationship linking the values of  $r_{pe}$  and  $N_A$  to  $\alpha$  but also reveal how this  $n^*$  is in turn related itself via  $f_0 = f_0(n^*)$  and  $f_1 = f_1(n^*)$  to another fundamental constant, *i.e.* to  $Z_{\max}$ .

This premise outlines the strategy of the present approach: instead of improving further (3.3) and (4.9) to calculate  $\alpha$  only, which would be conceptually restricted and futile, are now implemented two further constants of Nature,  $N_A$  and  $r_{pe}$ , to understand how their mutual correlation defines uniquely  $F_n^*$  and how even other constants are possibly hidden in this  $F_n^*$  along with  $\alpha$  too.

Regarding in fact (5.2) as boundary conditions for  $\alpha$  and  $N_A$ , it follows that the index  $n$  is upper bound in a natural way by  $n_{\max} = 118$ , beyond which appear negative values of  $f_0$ . But these negative values are to be excluded, as they would imply  $F_n^* = f_1 F_{n-1} - |f_0| F_{n-2}$ , *i.e.*  $F_n^*$  would no longer correspond to the sum of two previous values  $f_1 F_{n-1}$  and  $f_0 F_{n-2}$ . To comply with the original definition of sequence (1.3), therefore, identify  $n^* \equiv n_{\max}$  as upper limit of  $n$  fulfilling (5.2) and (3.4), and thus  $n_{\max} \equiv Z_{\max}$ . In other words, the upper boundary of  $Z$  allowed to the heaviest nucleus of hydrogen-like atom is related to the meaning of  $n_{\max}$ -th term for which the generalized sequence (1.4) is self-consistently satisfied. So, if this conclusion is true, the numbers calculable through  $f_0$  and  $f_1$  of **Table 2** corresponding to  $n = Z_{\max}$  should have their own physical meaning; indeed, accepting the positions (5.2),  $Z_{\max}$  enters in a natural way into the model via the upper boundary allowed to  $n$ .

Consider separately the addends  $f_0(n)/n$  and  $f_1(n)/n$  of the first (5.2) that by definition concur to calculate  $\alpha$ . This means considering in particular the ratios

$$\frac{f_1}{Z_{\max}} = 0.007410575579, \quad \frac{f_0}{Z_{\max}} = 0.0001132, \quad n = n_{\max} = Z_{\max} \quad (5.3)$$

**Table 2.** Calculation of  $f_0$  and  $f_1$  solutions of (5.2) for selected values of  $n$ .

$n$	$f_1$	$f_0$
115	2.631495991	1.792300446
116	1.751578284	0.9050853867
117	1.208823904	0.3550336533
118	0.8744479183	0.01336031542
119	0.6688568636	-0.1995280918

that extend further the approximate (3.3) and (4.7) successfully implemented together to calculate  $\alpha$ , here assumed instead exactly known. These initial positions are introduced thinking that in fact nothing excludes their own physical meaning even though regarded separately. To highlight the possible links of (5.3) with other fundamental constants of nature, exploit the quantities of **Table 2** recalling also both (1.20) and (1.21)

$$f_1' = f_1 \frac{k_c}{k_b}, \quad f_1'' = f_1 k_c k_b, \quad f_0' = f_0 \frac{k_c}{k_b}, \quad f_0'' = f_0 k_c k_b$$

since  $k_a$  and  $k_b$  and  $k_c$  are dimensional unit factors, anyway the numerical values of **Table 2** remain in fact unchanged. According (1.20) these multiplicative factors simply switch  $f_0$  and  $f_1$  to  $f_0'$  and  $f_1'$  and thus assign physical dimensions to  $F_{n-1}$  and  $F_{n-2}$  that switch to  $F_{n-1}'$  and  $F_{n-2}'$ . Although  $F_n^*$  takes by consequence physical dimensions  $F_n^{**}$ , nothing changes from a mere numerical point of view as concerns **Table 2** once considering (1.20) instead of (1.4). Analogous reasoning holds for the corresponding double primed quantities of (1.21).

To highlight the worth of these considerations about (5.2), let us examine the implications of **Table 2**.

1) The first implication is that just mentioned, *i.e.*  $n_{\max} = Z_{\max}$  indicates the maximum atomic number  $Z_{\max} = 118$  expectable for the reasons previously exposed in the Section 3.

2) Define the following ratios directly inferred from (5.3).

$$\begin{aligned} r_{1p} &= \frac{m_p}{m_e} \frac{f_1'}{Z_{\max}} = 13.60694816 \frac{k_c}{k_b}, \\ r_{1d} &= \frac{m_p}{m_e} \frac{1}{\frac{f_1''}{Z_{\max}}} = \frac{251619.0161}{k_c k_b} \end{aligned} \quad (5.4)$$

and analogously

$$\begin{aligned} r_{0p} &= \frac{m_p}{m_e} \frac{f_0'}{Z_{\max}} = 0.2078947362 \frac{k_c}{k_a}, \\ r_{0d} &= \frac{m_p}{m_e} \frac{1}{\frac{f_0''}{Z_{\max}}} = \frac{1.621713325 \times 10^7}{k_c k_a} \end{aligned} \quad (5.5)$$

indeed there is no reason to exclude that the analytical form guessed in (5.4) for  $f_1$  should be also allowed as in (5.5) for  $f_0$ . The ratio  $r_{pe} = m_p/m_e$  is regarded in both cases as proportionality constant, purposely introduced to include and link in the following pattern of calculations also the proton to electron mass ratio. Clearly the idea is to implement fundamental constants only, and not arbitrary proportionality constants possibly regarded as additional best fit values.

Eventually examine also

$$\begin{aligned}
 f_0 f_1 \frac{f'_1}{Z_{\max}} &= 0.865770134 \times 10^{-4} \frac{k_c}{k_b}, \\
 f_0 f_1 \frac{f'_0}{Z_{\max}} &= 1.322773129 \times 10^{-6} k_c k_a,
 \end{aligned}
 \tag{5.6}$$

where  $f_0 f_1 = f_0 f_1(Z_{\max})$  is another proportionality constant compliant with the leading ideas of (5.4) and (5.5). Eventually (5.6) have been also added to the list as further examples of a different proportionality factor,  $f_1(n_{\max})f_0(n_{\max})$ , appropriately replacing  $m_p/m_e$  for a further check of results.

So the positions (5.4), (5.5) and (5.6) that simply add a proportionality factor to (5.3) according to (1.20) and (1.21), should be reasonably admissible taking into account the idea just exposed: *i.e.* any expression involving  $f_1(n_{\max})$  cannot exclude  $f_0(n_{\max})$  of **Table 2** too.

3) Start from (5.4); a glance to the first numerical value suggests putting purposely  $k_c/k_b \rightarrow \text{unit energy}$  and  $(k_b k_c)^{-1} \rightarrow \text{unit energy}$  too, as it is in fact possible. Multiplying side by side these correspondences one finds  $1/k_b^2 = (\text{unit energy})^2$  *i.e.*

$$k_b^{-1} = \pm \text{unit energy} \tag{5.7}$$

whatever  $k_c$  might be.

Put then  $k_c = 1$  in order that both  $r_{1p}$  and  $r_{1d}$  take the meaning of energies whose numerical values coincide of course with  $(m_p/m_e)(f_1/Z_{\max})$  and  $(m_p/m_e)/(f_1/Z_{\max})$ . In other words (5.4), although calculated as pure numbers solving (5.2), with this particular choice of unit dimensional factors correspond in fact to two different values of energy allowed by the generalized Fibonacci sequence.

The importance of this conclusion appear recalling the Dirac equation of hydrogen like atoms

$$\begin{aligned}
 \frac{E_D}{m_e c^2} &= \frac{1}{\sqrt{1 + \frac{\alpha^2 Z^2}{\left(n - (j + 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2 Z^2}\right)^2}}} - 1, \\
 m_e c^2 &= 0.51099895 \times 10^6 \text{ eV};
 \end{aligned}
 \tag{5.8}$$

once calculated for  $n = 1, j = l + s = 1/2$ , this expression yields for  $Z = 1$  and  $Z = 118$  respectively

$$E_D(Z = 1) = 13.60601134 \text{ eV}, \quad E_D(Z_{\max}) = 251178.148 \text{ eV}. \tag{5.9}$$



Now it is possible to compare (5.4) and (5.9): it is immediate to acknowledge that the energy  $E_D(Z=1)$  corresponds to the template value  $(m_p/m_e)(f_1'/Z_{\max})$  reported in the first (5.4) with numerical deviation 0.007% only. If this idea is sensible, analogous conclusion should hold also for  $E_D(Z=Z_{\max})$ : *i.e.* the upper boundary of numerical value of Dirac energy should fit its template given by the second (5.4). In effect the second (5.4) results fairly consistent with the idea that  $(m_p/m_e)(f_1''/Z_{\max})^{-1}$  is the energy to which corresponds  $E_D(Z=Z_{\max})$ ; now however the Dirac value deviates from its numerical template by 1.4%. It must be reminded in this respect that the early Dirac equation quoted here is inherently approximate, as it does not account for the Lamb shift, which increases with  $Z$ ; in effect this term appears in a natural way in a modified Dirac equation inferred in [12]. So it is not surprising that the deviation of  $E_D(Z=1)$  from  $(m_p/m_e)(f_1'/Z_{\max})^{-1}$  is better than that of  $E_D(Z=Z_{\max})$  from  $(m_p/m_e)(f_1''/Z_{\max})$ . Hence, regardless of the physical background that brings up to (5.8) starting from first principles, it seems that both boundary values (5.9) are coherent with and in fact already inherent (5.4).

It is also worth noticing in this respect that the double sign allowed in (5.7) is consistent with the negative energy states of Dirac too.

As anticipated in Section 1, it appears here that that  $F_n^*(n_{\max})$  is a sort of numerical template, such that the dimensionless values from it calculated via  $f_0$  and  $f_1$  represent a constrain to the actual fundamental constants governing observable physical properties of Nature like the generalized Dirac energy.

Let us verify this idea to remove the suspect that the prospective Fibonacci templates of energy fit accidentally only these favorable checks looking for further correspondences similarly as that just proposed. In other words, let us check whether or not characteristic energies or lengths or times describing natural events or properties of matter are actually allowed values relatable to and in line with distinctive pure numbers inferred from the generalized Fibonacci sequence.

To identify significant templates that govern physical constants and/or their combinations is in principle a difficult task. The fact that the values calculable via (5.4), (5.5) and (5.6) are actually abstract numbers, does not indicate “a priori” which specific physical properties they refer to; only “a posteriori” one acknowledges, by comparison like that of (5.4) and (5.9), which allowed physical constants and properties tentatively found are really correlatable. Moreover are crucial in this respect also the possible physical corollaries like (5.7) contextual to the numerical comparisons. Additional matches are shown in the following.

Consider the next template numbers (5.5) to find their possible corresponding fundamental constants and note that

$$\frac{3}{ec} = 0.2083384819 \frac{\text{s}^2}{\text{cm}^{5/2} \cdot \text{g}}, \quad \frac{G}{h} = 1.613785378 \times 10^7 \frac{\text{cm/s}}{\text{g}^2} \quad (5.10)$$

having expressed  $h$  as eVs. The dimensional unit factors  $k_c/k_a$  and  $k_c k_a$ , now appropriately regarded in (5.5) in order to match the ones in (5.10), allow comparing the respective numerical values. It is immediate to observe that these ac-

known values deviate from the corresponding two (5.5) by 0.2% and 0.5% respectively. It is worth noticing that the numerical agreement of the first (5.10) with its own numerical template (5.5) requires  $c$  and  $e/3$ .

Consider eventually, by analogy with (5.4) and (5.5), the positions (5.6) involving as further proportionality constant  $f_0 f_1$  alternative to  $r_{pe}$  already tested. Is crucial also now the fact that the physical dimensions of  $k_c$  and  $k_a$  separately allow defining new dimensions  $k_c/k_a$  and  $k_c k_a$  consistent with further that of fundamental constants of Nature. Quote then the Boltzmann constant with energy expressed in eV as well

$$k_B = 0.8617333262145 \times 10^{-4} \text{ eV/K} \quad (5.11)$$

then  $k_B$  identifies now the dimensional ratio  $k_c/k_a$ , whereas the numerical deviation of  $k_B$  from the first template (5.6) is 0.5% only. Moreover calculate

$$\frac{3er_{pe}}{2} = \frac{\alpha r_{pe} \hbar c}{2e/3} = 1.322912454 \times 10^{-6} \frac{\text{g}^{1/2} \cdot \text{cm}^{3/2}}{\text{s}} \quad (5.12)$$

this value differs from the second template (5.6) by 0.01% only, once having assumed the physical dimensions of  $k_c k_a$  consistent with that at the right hand side of (5.12). This agreement requires now  $2e/3$ .

Note now that the first Equation (1.19) can be better understood and enhanced comparing once more the link of  $G$  to  $e$  ratio via  $\alpha$  through its pertinent template

$$\frac{G\alpha}{e} = 1.013971342 \frac{\text{cm}^{3/2}}{\text{g}^{3/2} \cdot \text{s}}, \quad \frac{f'_1}{\alpha n_{\max}} = \frac{f_1}{\alpha Z_{\max}} \frac{k_c}{k_b} = 1.015515629 \frac{k_c}{k_b}; \quad (5.13)$$

the numerical deviation of the former from its own template is 0.15% only. It is significant that one calculates here just the numerical proportionality factor tentatively estimated in (1.19) with the known values of  $G$  and  $e/\alpha$ , whereas the dimensional matching is fulfilled once more by  $k_c/k_b$ .

Emphasize also now the physical implications of these numerical comparisons.

On the one hand the last results (5.10), (5.12) and (5.13) provide a list of “effective charges” allowed by the respective templates

$$e, \quad \frac{2}{3}e, \quad \frac{1}{3}e. \quad (5.14)$$

On the other hand, the link between  $G$  and  $e$  via  $\alpha$  intuitively guessed in (1.19) appears actually to be the further result of a wider conceptual frame that according to (5.2) assigns also to  $f_1/n\alpha = f_1/(f_1 - f_0)$  the physical meaning of template value, thus extending the Fibonacci driven list of hidden links that in fact represent the constants of Nature.

In summary: the templates (5.4), (5.5) and (5.6) has been related respectively to (5.9), (5.5) and (5.11), (5.12).

## 6. Discussion

For sake of clarity the present paper has deliberately proposed a stepwise exposi-

tion about the fundamental constants of Nature; the purpose of this strategy was to show first in the Sections 3 and 4 that (1.12) and (1.6) account decently for the numerical definition of  $\alpha$  and prospect in principle even better results simply depending on the number of higher order terms of series expansion that define the parameters of (1.11).

Next the considerations of Section 5 have explored the idea that the visible look of the Universe is controlled by a *set* of fundamental constants that organize its observable features analogously to the formal aspect of a typewritten sheet subjected to a pattern of rules coded in a basic template. This holds if in fact exists a subtle link underlying the numerical values of the various constants that take physical meaning via their physical dimensions; *i.e.* physical dimensions and numerical values cannot be regarded as separate properties. In other words, the gravity constant or the electric charge have the value  $G$  and  $e$  we know coherently with the respective physical dimensions; the link (1.19) confirmed in (5.13) must be regarded in this way, which justifies the dimensional factor  $k_g$  in a natural way and a wider conceptual frame. In effect just the dimensional requirements allow the correspondences between physical constants and their own templates via the unit factors  $k_a$  and  $k_b$ : as shown by (1.20) and (1.21), the sequence (1.4) switches from (1.6) to  $k_c F_n^*$  likewise as shown for example in (1.9).

The numerical agreements of Section 5 can be hardly interpreted as mere accidental coincidences.

In other words: considering “exact” by definition the known values of  $N_A$  and  $\alpha$  implemented to calculate the coefficients  $f_0$  and  $f_1$  of the generalized Fibonacci sequence (1.3) as a function of  $n$ , the fundamental constant  $m_p/m_e$  results to be itself the proportionality factor enabling not only to calculate reasonably the ground states (5.9) of the Dirac hydrogen-like atom at  $Z=1$  and  $Z=Z_{\max}$  but also to obtain the further correspondences (5.10) with (5.5). It is significant the fact that after having introduced (5.3) the proportionality factor defining (5.4) and (5.5) is just one among the fundamental constants of Nature and not an arbitrary numerical parameter that would appear inevitably as a best fit value to match purposely some known physical constant. The way to define (5.4) and (5.5), instead, excludes any best fit attempt; rather all correspondences found in Section 5 are self contained, in that the template model plugs physical constants into the extended Fibonacci sequence via (5.3) only. Indeed the same holds identically for the positions (5.6) that implement the factor  $f_0 f_1$  and not some superfluous proportionality parameter to be intentionally defined.

For this reason has been omitted the possible explanation about why just the factor  $m_p/m_e$  links reasonably  $f_1/Z_{\max}$  to  $E_D(Z=1)$  and  $E_D(Z=Z_{\max})$ : if so, then the boundary conditions at  $Z=Z_{\max}$  and  $Z=1$  of the generalized Fibonacci sequence are accordingly related even regardless of the theoretical approach leading to the Dirac equation.

Anyway, apart from the chance of justifying rationally why the energies (5.9) conform to their templates (5.4), if it is correct the present idea of regarding (1.2) at  $n = n_{\max}$  as numerical template to which the properties of Nature conform likewise a boundary condition, the approach just described links and correlates in a unique frame the ratio  $m_p/m_e$ , the upper and lower limits of Dirac relativistic ground energy, the Avogadro number and  $\alpha$  itself along with the value of  $Z_{\max}$ ; indeed the latter two have defined  $f_1$  at  $n = Z_{\max}$ . It is as if the generalized Fibonacci sequence “would know” in advance that the standard Dirac equation, as such, is not fully adequate to describe the reality although being an analytical expression very close to the reality; in other words (5.4), despite its mathematical abstractness, seems suggesting that something is missing in  $E_D$  to fit correctly the numerical template of Nature.

The last section has shown that selecting appropriately case by case the dimensional unit factors introduced in (1.20) and (1.21), the numerical values calculable through (1.4) take physical meaning of further fundamental constants actually governing the Nature.

The practical help of the results in Section 5 is not limited to the determination of some fundamental constants, rather it implies the knowledge of further values having the rank of fundamental constants themselves. A typical example is the second (5.9): even without knowing the result inferred in [12], the significant deviation of  $E_D(Z_{\max})$  from its template value required by the second (5.4) reveals that something is missing in the Dirac equation. In this specific case, the discrepancy has a known origin, the vacuum polarization causing the Lamb shift; in a more general case, however, the values calculated via the extended sequence (1.3) set end points of calculations useful to refine and adjust the theoretical models.

Last but not the least: the few remarks in Section 2, although very shortly sketched, are enough to show once again in a straightforward and elementary way the intimate link between relativity and quantum physics.

## 7. Conclusion

The sequence (1.1) is defined by its own sum rule of terms only, the sequence (1.3) instead does not because are also required the initial values; it needs defining also the initial values of two arbitrary parameters  $f_0$  and  $f_1$ . In this sense the sequence  $F_n^*$  (1.4) is closer to the rational formulation of physical problems, whose solution requires indeed specifying appropriate boundary conditions. Thus it is not surprising the fact that the abstractness of pure numbers (1.1) points via (1.4) directly to the reality of natural events, regardless of how the fundamental constants enter into specific formulas calculable and verifiable by comparison with the experience. This assertion is probably the best way to summarize the concept of template hitherto concerned, because it is equivalent to say: the fundamental constants shape the observable Universe regardless of how the theoretical models infer specific formulas likewise as a template controls

how a typewritten sheet appears regardless of its specific text content.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix A

The purpose of this section is to sketch how (2.3) allow inferring (2.2) along with  $j$  of (2.26), thus making really self-contained the results of present paper.

1) Angular momentum component. Since by definition  $\mathbf{M} = \mathbf{r} \times \mathbf{p}$ , write its classical form and its radial component as  $\delta \mathbf{M} = \delta \mathbf{r} \times \delta \mathbf{p}$  and  $\delta M_k = \delta \mathbf{r} \times \delta \mathbf{p} \cdot \mathbf{k}$ : the left hand sides emphasize the range sizes resulting from the range of values of  $\mathbf{r}$  and  $\mathbf{p}$  within the respective range sizes,  $\mathbf{k}$  is an arbitrary unit vector. Trivial manipulations yield  $\delta M_k = \mathbf{k} \times \delta \mathbf{r} \cdot \delta \mathbf{p}$ , so that calling  $\delta \mathbf{r}_\perp = \mathbf{k} \times \delta \mathbf{r}$  one finds  $\delta M_k = \delta \mathbf{r}_\perp \cdot \delta \mathbf{p}$ . Hence with  $\delta p_\perp = \delta \mathbf{r}_\perp \cdot \delta \mathbf{p} / |\delta \mathbf{r}_\perp|$  the result is  $\delta M_k = \delta r_\perp \delta p_\perp$ . Therefore (2.3), which must be anyway fulfilled whatever the concerned ranges might specifically be, requires  $\delta M_k = l\hbar$ ; with usual notation  $l$  is the integer including 0 corresponding to the product of ranges  $\delta r_\perp \delta p_\perp$  whose sizes, whatever they might be, must be anyway regarded according to the initial (2.2). Now  $\delta M_k$  is the uncertainty range of values of angular momentum components  $M_k = l\hbar$  corresponding to the arbitrary values of  $l$ .

2) Angular momentum. Of course it is evident that one component only of  $M_k$  is definable, usually called  $M_z$ : changing  $\mathbf{k}$  does not add new physical information to  $\delta M_k$ . Yet it is shown in [1] that it is possible to calculate

$$M^2 = \langle M_x^2 \rangle + \langle M_y^2 \rangle + \langle M_z^2 \rangle = \hbar^2 \sum_{i=1}^3 \frac{l_i^2}{2L+1} = L(L+1)\hbar^2, \quad -L \leq l_i \leq L$$

and that the allowed  $l_i$  concurring to the quantum angular number  $L$  are just the values of magnetic quantum number usually denoted with  $m$ .

3) Energy. The classical energy  $\varepsilon$  of hydrogen like atom  $\varepsilon = \varepsilon_{cm} + p_r^2/2m_e + M^2/2m_e r^2 - Ze^2/r$ , where  $\varepsilon_{cm}$  is the center of mass energy of the atom as a whole is rewritten in order to emphasize that actually  $r$  must be intended as  $\delta r$ , *i.e.* a range of distances of the electron from the nucleus rather than a deterministic classical distance. It is immediate to find that  $\delta \varepsilon = \varepsilon_{cm} + \delta p_r^2/2m_e + \delta M^2/2m_e \delta r^2 - Ze^2/\delta r$  yields just both (2.2) with the help of (2.3) only; the key step is that here  $\delta p_r = n\hbar/\delta r$ , whereas  $n$  appearing explicitly itself in the final form of  $\varepsilon$  is the principal quantum number.

4) Spin. Note the identity  $L(L+1) \equiv (L+s)^2 - s^2$  with  $s = 1/2$ ; so  $\mathcal{M}^2 = (L+s)^2 \hbar^2 - \hbar^2 s^2$  defines a new angular momentum  $\mathcal{M}^2 = M^2 + \hbar^2 s^2 + \hbar^2 (L+s) = \hbar^2 (L+s)^2 + \hbar^2 (L+s)$  *i.e.*  $M^2 = \hbar^2 j(j+1)$ , with  $j = L+s$ . So, the quantization implies itself the chance of the spin. Actually the existence of such half-integer angular momentum is shown starting from first principles in other papers, while being in fact  $j = L \pm s$  [12] [16].

These short remarks aim merely to sketch the chance of justifying (2.2) and (5.8) via (2.3) only, in order to make the present paper as self contained as possible.