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Simple Singular Perturbation Problems with Turning Points

Na Wang

Department of Applied Mathematics, Shanghai Institute of Technology, Shanghai, China
Email: wangna1621@126.com

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Abstract

The paper considers the asymptotic solution of two-point boundary value problems $\varepsilon y'' + A(x)y' = 0$, $0 \le x \le 1$, when $0 < \varepsilon \ll 1$, $A(x)$ is smooth with isolated zeros, $y(0) = 0$ and $y(1) = 1$. By using perturbation method, the limit asymptotic solutions of various cases are obtained. We provide a reliable and direct method for solving similar problems. The limiting solutions are constants in this paper, except in narrow boundary and interior layers of nonuniform convergence. These provide simple examples of boundary layer resonance.

Keywords

Singular Perturbations, Asymptotic Methods, Boundary Value Problems, Turning Points, Boundary and Interior Layers, Boundary Layer Resonance

1. Introduction

A typical turning point problem consists of the linear differential equation

$$\varepsilon y'' - xy' + ny = 0, \tag{1}$$

for a nonnegative integer n on $-1 \le x \le 1$ with prescribed boundary values $y(\pm 1)$ and a small positive parameter ε, *i.e.*, $0 < \varepsilon \ll 1$. Limiting solutions, away from narrow so-called boundary and interior shock layers of rapid change, take the form

$$Y_0(x) = x^n C \tag{2}$$

as $\varepsilon \to 0$ for constants *C*, so satisfy the limiting reduced equation

$$xY_0' = nY_0. \tag{3}$$

Determining constants C and the location of layers is a nontrivial task, the subject of boundary layer resonance [1]. It involves detailed asymptotic analysis and often uses special functions. The classical techniques of matched asymptotic expansions [2] [3] and the boundary function method of Vasil'eva *et al.* [4] may break down, though the newer composite asymptotic expansions [5] seem to apply. Many experts have studied such problems over the last fifty years [6] [7] [8] for surveys. An important application to stochastic differential equations in described in the 2017 SIAM von Neumann lecture by Matkowsky [9]. Computing solutions to such problems remains a challenge, although Trefethen *et al.* [10] succeed for some examples using the program Chebfun.

A simple, but still rich, the related problem concerns the asymptotic solution of the two-point problem [11]

$$\varepsilon y'' + A(x) y' = 0,\ 0 \le x \le 1, \tag{4}$$

with the special boundary values

$$y(0) = 0 \text{ and } y(1) = 1 \tag{5}$$

and a smooth coefficient $A(x)$. Its unique exact solution is

$$y(x,\varepsilon) = \frac{\int_0^x \mathrm{e}^{-\frac{1}{\varepsilon}\int_0^s A(t)\mathrm{d}t}\mathrm{d}s}{\int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}\int_0^s A(t)\mathrm{d}t}\mathrm{d}s}. \tag{6}$$

Since

$$y'(x,\varepsilon) = \frac{\mathrm{e}^{-\frac{1}{\varepsilon}\int_0^x A(t)\mathrm{d}t}}{\int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}\int_0^s A(t)\mathrm{d}t}\mathrm{d}s} > 0, \tag{7}$$

the solution y will increase monotonically with x. The asymptotic value of $\int_0^x I(s,\varepsilon)\mathrm{d}s$ in (6) is the area under the curve

$$I(s,\varepsilon) \equiv \mathrm{e}^{-\frac{1}{\varepsilon}\int_0^s A(t)\mathrm{d}t} \tag{8}$$

for $\varepsilon \to 0$. Sophisticated techniques to obtain the asymptotic evaluation of integrals can be found in Olver [12], Wong [13] and elsewhere. Simple arguments often provide the limiting ratio (6), often after rescaling I.

The variety of limiting behaviors to singularly perturbed linear two-point boundary value problems with turning points has not been clearly described. The first papers by Pearson in 1968 stressed a numerical approach. In the intervening fifty years, software has improved tremendously, though finding the limiting solution is extremely ill-conditioned as Trefethen recently observed. Due to the serious instability of direct numerical methods, the examples found in scattered literature are usually less detailed. Inspired by this, in this paper we consider the asymptotic solution of two-point boundary value problems (4)-(5). In our examples, we'll find the constant "outer" limits 0, 0.5, and 1.

Case 1: $A(x) > 0$.

Here, $I(s,\varepsilon)$ decays exponentially as $\varepsilon \to 0$, so for any fixed $x > 0$, the numerator and denominator of (6) are both $O(\varepsilon)$ and the ratio (6) is asymptotically one. Since $y(0,\varepsilon) = 0$, there is an initial boundary layer region of $O(\varepsilon)$ thickness involving nonuniform convergence of *y*. Here, we're using the big O order symbol.

As an example, take $A(x) \equiv 1$, $\varepsilon = 0.1, 0.01$ and 0.001, and plot the solution (6). One gets

$$y(x,\varepsilon) = \frac{\frac{1}{\varepsilon}\int_0^x e^{-\frac{s}{\varepsilon}} ds}{\frac{1}{\varepsilon}\int_0^1 e^{-\frac{s}{\varepsilon}} ds} = \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}. \tag{9}$$

The constant limiting solution $Y_0(x) = 1$ for $x > 0$ as $\varepsilon \to 0$, satisfies the reduced equation $Y_0' = 0$ away from $x = 0$. We plot the solution for three small ε values in **Figure 1**.

The limit of $y(x,\varepsilon)$ is discontinuous at $x = 0$, signaling nonuniform convergence.

Case 2: $A(x) < 0$.

Now $I(x,\varepsilon)$ grows exponentially large as $\varepsilon \to 0$. This causes y to be asymptotically zero for any $x < 1$ and a terminal boundary layer of nonuniform convergence to occur near $x = 1$.

As an example, take $A(x) \equiv -1$ and plot

$$y(x,\varepsilon) = \frac{\frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\int_0^x e^{\frac{s}{\varepsilon}} ds}{\frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\int_0^1 e^{\frac{s}{\varepsilon}} ds} = \frac{e^{\frac{x-1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}. \tag{10}$$

For $x < 1$, the limiting solution $e^{\frac{x-1}{\varepsilon}}$ as $\varepsilon \to 0$ is trivial. The limiting terminal layer will have $O(\varepsilon)$ thickness.

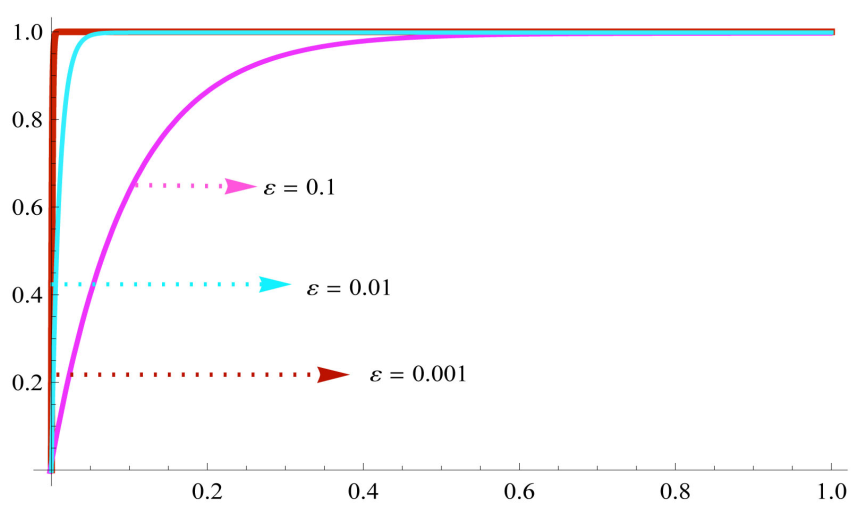

Figure 1. $y(x,\varepsilon)$ from (9).

2. Turning Points

Case 3: $A(x)=x-0.5$.

Now there's a simple turning point at $x=0.5$ and

$$I(s,\varepsilon)=\mathrm{e}^{-\frac{1}{2\varepsilon}\left[\left(s-\frac{1}{2}\right)^2-\frac{1}{4}\right]}=\mathrm{e}^{\frac{1}{8\varepsilon}}\mathrm{e}^{-\frac{1}{2\varepsilon}\left(s-\frac{1}{2}\right)^2}.$$

We write the ratio (6) as

$$y(x,\varepsilon)=\frac{\int_0^x I(s,\varepsilon)\mathrm{d}s}{\int_0^1 I(s,\varepsilon)\mathrm{d}s}. \tag{11}$$

The exact solution is

$$y(x,\varepsilon)=\frac{1}{2}\left(1+\frac{erf\left(\frac{x-\frac{1}{2}}{\sqrt{2\varepsilon}}\right)}{erf\left(\frac{1}{2\sqrt{2\varepsilon}}\right)}\right), \tag{12}$$

where $erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\mathrm{d}t$ is the error function [14]. It satisfies $z''+xz'=0$, it is odd, it increases monotonically, and it tends to ± 1 as $x\to\pm\infty$.

Since the integrands of (11) peak at the turning point and are asymptotically negligible elsewhere, we will have

$$y(x,\varepsilon)\sim\begin{cases}0, & \text{for } x<\frac{1}{2}\\ 1, & \text{for } x>\frac{1}{2}.\end{cases} \tag{13}$$

The numerator and denominator of (11) are both $O\left(\sqrt{\varepsilon}\right)$. Clearly, $y\left(\frac{1}{2},\varepsilon\right)=\frac{1}{2}$, there's antisymmetry about $s=\frac{1}{2}$, and an $O\left(\sqrt{\varepsilon}\right)$ thick region of nonuniform convergence about the midpoint.

Plotting the solution (12) for $\varepsilon=10^{-3}$, we get Figure 2.

Case 4: $A(x)=x-\alpha, 0<\alpha<1$.

We rescale I to get

$$\mathrm{e}^{-\frac{\alpha^2}{2\varepsilon}}I(s,\varepsilon)=\mathrm{e}^{-\frac{1}{2\varepsilon}(s-\alpha)^2},$$

a function that peaks in an $O\left(\sqrt{\varepsilon}\right)$ interval about $s=\alpha$ and is asymptotically negligible elsewhere. This implies that a shock layer of nonuniform convergence occurs about the turning point. The exact solution (6) is

$$y(x,\varepsilon)=\frac{erf\left(\frac{x-\alpha}{\sqrt{2\varepsilon}}\right)+erf\left(\frac{\alpha}{\sqrt{2\varepsilon}}\right)}{erf\left(\frac{1-\alpha}{\sqrt{2\varepsilon}}\right)+erf\left(\frac{\alpha}{\sqrt{2\varepsilon}}\right)}. \tag{14}$$

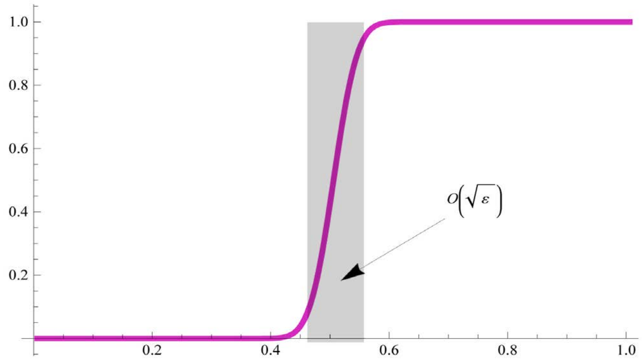

Figure 2. $y(x,10^{-3})$ from (12).

For $\alpha = \frac{1}{3}$ and $\varepsilon = 10^{-3}$, we get **Figure 3**.

Not surprisingly, the asymptotic solution is essentially a translation of that for $\alpha = 0.5$. For $\alpha = 0$,

$$y(x,\varepsilon) = \frac{\int_0^x e^{-\frac{s^2}{2\varepsilon}} ds}{\int_0^1 e^{-\frac{s^2}{2\varepsilon}} ds}, \tag{15}$$

so we again get an $O(\sqrt{\varepsilon})$ initial layer (see **Figure 4(a)** and **Figure 4(b)**).

For $\alpha = 1$, there is an analogous terminal layer. For $\alpha < 0$ or $\alpha > 1$, the boundary layer is $O(\varepsilon)$, *i.e.* thinner.

Case 5: $A(x) = (x-\alpha)^3, 0 < \alpha < 1$.

We have a third order turning point at $x = \alpha$. Again, the rescaled integral $I(s,\varepsilon)$ peaks at $s = \alpha$, causing y to jump there. The shock layer is now $O\left(\varepsilon^{\frac{1}{4}}\right)$ thick. The exact solution is

$$y(x,\varepsilon) = \frac{\Gamma\left(\frac{1}{4}, \frac{(x-\alpha)^4}{4\varepsilon}\right) - \Gamma\left(\frac{1}{4}, \frac{\alpha^4}{4\varepsilon}\right)}{\Gamma\left(\frac{1}{4}, \frac{(1-\alpha)^4}{4\varepsilon}\right) - \Gamma\left(\frac{1}{4}, \frac{\alpha^4}{4\varepsilon}\right)}, \tag{16}$$

where $\Gamma(a,z) = \int_z^{+\infty} u^{a-1} e^{-u} du$ is the incomplete gamma function.

Evaluating $y(x,\varepsilon)$ for $\alpha = \frac{1}{3}$ and $\varepsilon = 10^{-3}$, we get **Figure 5**.

To steepen the shock layer, we must take ε much smaller.

We change the sign of A for the next three examples.

Case 6: $A(x) = \frac{1}{2} - x$.

Rewriting (6) as

$$y(x,\varepsilon) = \frac{\int_0^x e^{\frac{s}{2\varepsilon}(s-1)} ds}{\int_0^1 e^{\frac{s}{2\varepsilon}(s-1)} ds}, \tag{17}$$

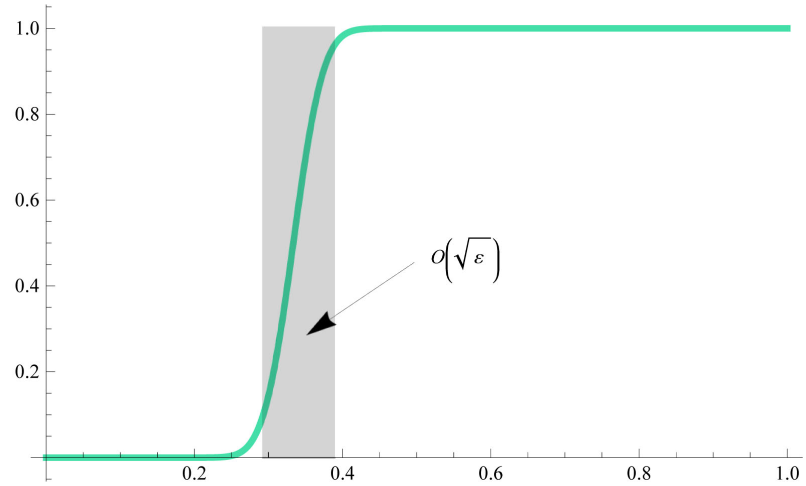

Figure 3. $y(x, 10^{-3})$ from (14) with $\alpha = \frac{1}{3}$.

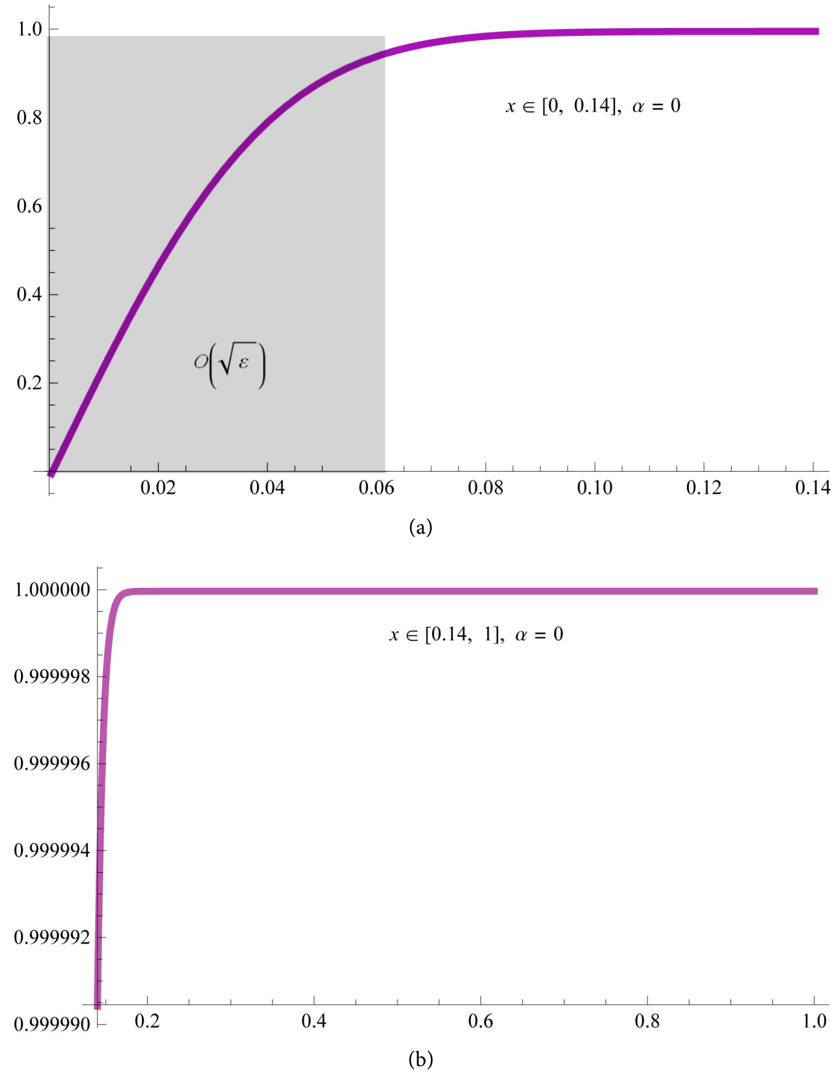

Figure 4. (a) $y(x, 10^{-3})$ from (15) in the initial layer; (b) $y(x, 10^{-3})$ from (15).

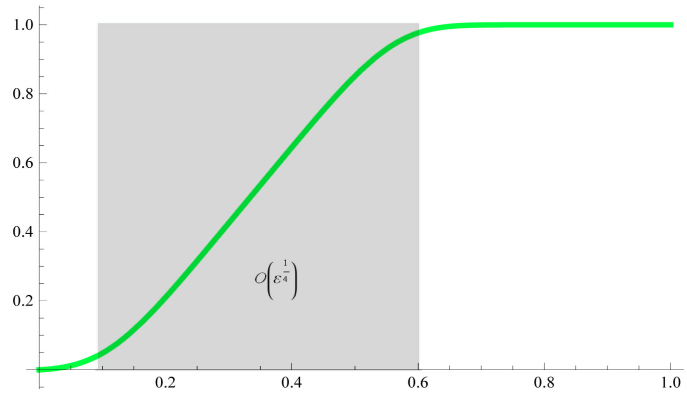

Figure 5. $y\left(x,10^{-3}\right)$ from (16) for $\alpha=\frac{1}{3}$.

the exact solution is

$$y(x,\varepsilon)=\frac{1}{2}\left(1+\frac{\int_0^{\frac{x-\frac{1}{2}}{\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t}{\int_0^{\frac{1}{2\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t}\right). \tag{18}$$

We note that the solution could be expressed in terms of Dawson's integral $F(z)=\mathrm{e}^{-z^2}\int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t$. The integrands in (17) peak symmetrically at $s=0$ and 1, being asymptotically negligible elsewhere. Moreover, $y\left(\frac{1}{2}\right)=\frac{1}{2}$. Indeed $y(x)\sim\frac{1}{2}$ for $0<x<1$, and twin $O(\varepsilon)$ boundary layers occur near both end-points. For $\varepsilon=10^{-3}$, we have Figure 6.

Case 7: For $A(x)=\alpha-x, 0<\alpha<\frac{1}{2}$, we have

$$y(x,\varepsilon)=\frac{\int_0^x \mathrm{e}^{\frac{s}{2\varepsilon}(s-2\alpha)}\,\mathrm{d}s}{\int_0^1 \mathrm{e}^{\frac{s}{2\varepsilon}(s-2\alpha)}\,\mathrm{d}s}. \tag{19}$$

Its exact solution is

$$y(x,\varepsilon)=\frac{\int_0^{\frac{x-\alpha}{\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t+\int_0^{\frac{\alpha}{\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t}{\int_0^{\frac{1-\alpha}{\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t+\int_0^{\frac{\alpha}{\sqrt{2\varepsilon}}}\mathrm{e}^{t^2}\,\mathrm{d}t}. \tag{20}$$

The integrand of (19) is asymptotically negligible for $s<2\alpha$, but asymptotically large for $s>2\alpha$. This implies that

$$y(x)\sim 0 \text{ for } x<1,$$

so there is as $O(\varepsilon)$-thick terminal layer.

As an example, consider

$$10^{-3}y''+\left(\frac{1}{3}-x\right)y'=0.$$

We have Figure 7 for picture of $y\left(x,10^{-3}\right)$.

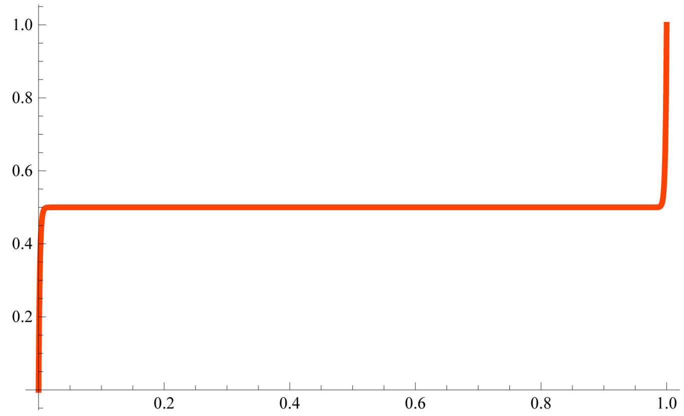

Figure 6. $y\left(x,10^{-3}\right)$ from (18).

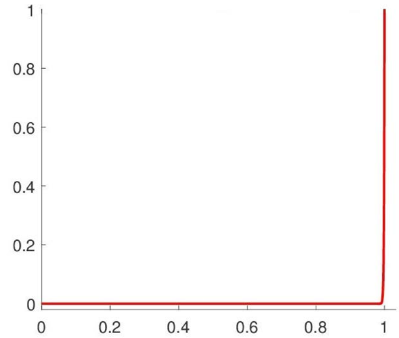

Figure 7. $y\left(x,10^{-3}\right)$ for $\alpha=\frac{1}{3}$.

Case 8: For $A(x)=\alpha-x, \alpha>\frac{1}{2}$, the integrand in (19) decays for $0<s<2\alpha$. Thus

$$y(x)\sim 1 \text{ for } x>0$$

and there is an $O(\varepsilon)$ thick initial layer (see **Figure 8**).

For appropriate $\alpha\sim\frac{1}{2}+o(1)$, we'd expect that the shock layer moves across the interval. We're now using the little o order symbol, which admittedly isn't very explicit.

Case 9: For $A((x)=\left(x-\frac{1}{4}\right)\left(x-\frac{3}{4}\right)$, we have simple turning points at $\frac{1}{4}$ and $\frac{3}{4}$. Moreover, $I(x,\varepsilon)=\mathrm{e}^{-\frac{x}{3\varepsilon}\left(x-\frac{3}{4}\right)^2}$ peaks at $x=0$ and $\frac{3}{4}$ and is asymptotically negligible elsewhere. The sizes of the contributions to the integral differ, however. The area under I near $x=0$ is $O(\varepsilon)$, but that near $x=\frac{3}{4}$ is $O\left(\sqrt{\varepsilon}\right)$, *i.e.*, larger. Thus, the ratio (6) is $O\left(\sqrt{\varepsilon}\right)$ for $0<x<\frac{3}{4}$ and $O(1)$ for $\frac{3}{4}<x<1$.

Computing for $\varepsilon = 10^{-3}$, we get **Figure 9**.

This relies on the following figures. We've increased ε in **Figure 10** to show the relative contributions. Normalizing to get $y(1) = 1$, we get the solution in **Figure 9**. And **Figure 11** shows the picture of integral for $I(x,\varepsilon)$ with $\varepsilon = 10^{-3}$.

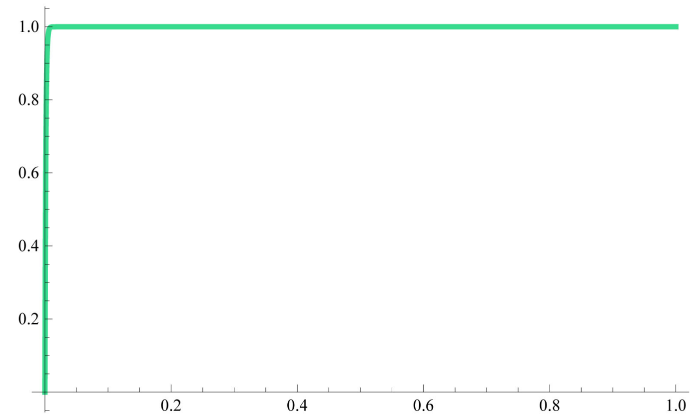

Figure 8. $y(x,10^{-3})$ for $\alpha = \frac{3}{4}$.

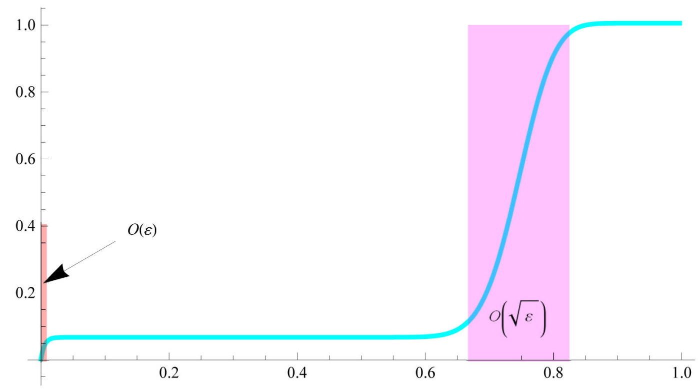

Figure 9. $y(x,10^{-3})$.

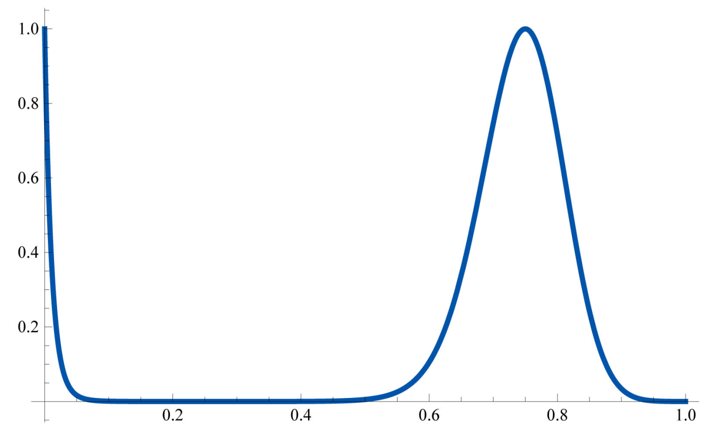

Figure 10. $I(x,0.002)$.

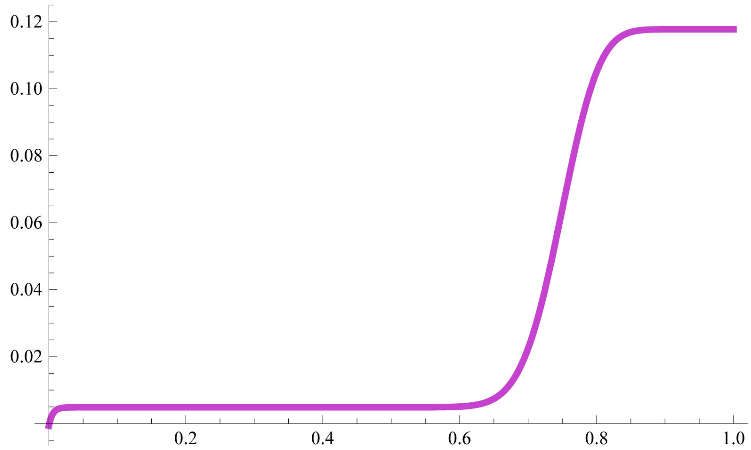

Figure 11. $\int_0^x I\left(s,10^{-3}\right)\mathrm{d}s$.

3. Conclusion

We have not been exhaustive, but we have certainly demonstrated a wide variety of asymptotic solutions to turning point problems of the form (4) - (5). They mimic the asymptotics of the more general boundary layer resonance problem. When the problem of turning points becomes complicated, numerical methods will become unreliable. Finding the limiting solution is extremely ill conditioned as Trefethen recently observed. Due to the serious instability of direct numerical methods, the examples found in scattered literature are usually less detailed. In this paper, we only give asymptotic solutions for a class of singularly perturbed with a turning point. Indeed, the techniques developed here might be expected to apply to that problem. Readers are encouraged to study other limiting possibilities for (4) - (5).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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