

On the Maximum Displacement and Static Buckling of a Circular Cylindrical Shell

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How to cite this paper: Ozoigbo, G.E. and Ette, A.M. (2019) On the Maximum Displacement and Static Buckling of a Circular Cylindrical Shell. *Journal of Applied Mathematics and Physics*, 7, 2868-2882.
<https://doi.org/10.4236/jamp.2019.711196>

Received: August 18, 2019

Accepted: November 18, 2019

Published: November 21, 2019

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Abstract

The static buckling load of an imperfect circular cylindrical shell is here determined asymptotically with the assumption that the normal displacement can be expanded in a double Fourier series. The buckling modes considered are the ones that are partly in the shape of imperfection, and partly in the shape of some higher buckling mode. Simply-supported boundary conditions are considered and the maximum displacement and the static buckling load are evaluated nontrivially. The results show, among other things, that generally the static buckling load, λ_s decreases with increased imperfection and that the displacement in the shape of imperfection gives rise to the least static buckling load.

Keywords

Static, Maximum Displacement, Circular, Cylindrical Shell

1. Introduction

Cylindrical shells have wide engineering applications such as in the construction and study of aircraft, spacecraft and nuclear reactor, tanks for liquid and gas storage and pressure vessels, etc. The analyses of the buckling of cylindrical shells under various loading conditions have been made in the past years and both theoretical and experimental studies have been considered just as in [1] and [2]. Earlier studies on the buckling of shells were done by [3] [4] [5] [6] and [7], while Amazigo and Frazer [8], studied the buckling under external pressure of cylindrical shells with dimple-shaped initial imperfection. It would be recalled that Budiansky and Amazigo [9], investigated the buckling of infinitely long imperfect cylindrical shells subjected to static loads and Ette and Onwuchekwa [10],

equally studied the static buckling of an externally pressurized finite circular cylindrical shell using asymptotic method. In this regard, mention must be made of Lockhart and Amazigo [11], who used perturbation method to investigate the dynamic buckling of finite circular cylindrical shells with small arbitrary geometric imperfections under external step-loading. In the same way, Bich *et al.* [12], by using analytical approach, investigated the nonlinear static and dynamic buckling behaviour of eccentrically shallow shells and circular cylindrical shells based on Donnell shell theory. Relevant studies on the buckling analysis were investigated in [13] [14] [15] and [16] and Ette [17] [18] [19] [20] and [21].

In this study, we consider a statically loaded imperfect finite circular cylindrical shell and aim at determining the maximum displacement and the static buckling load for the case where the displacement is partly in the shape of imperfection and partly in some other buckling mode. The analysis is purely on the use of asymptotic expansions and perturbation procedures.

This analysis is organised as follows. We shall first write down the governing equations as in Amazigo and Frazer [8] and Budiansky and Amazigo [9]. Using the techniques of regular perturbation and asymptotics, we shall analytically determine a uniformly valid expression of the displacement which is followed determining by the maximum displacement. Lastly, we shall reverse the series of maximum displacement and determine the static buckling load.

2. Formulation

As in [11], the general Karman-Donnell equation of motion and the compatibility equation governing the normal deflection $W(X, Y)$ and Airy stress function $F(X, Y)$ for cylindrical shell, of length L , radius R , thickness h , bending stiffness $D = \frac{Eh^3}{12(1-\nu^2)}$ (where E and ν are the Young's modulus and Poisson's ratio respectively), mass per unit area ρ , subjected to external pressure per unit area P , are

$$\frac{1}{Eh} \nabla^4 F - \frac{1}{R} W_{,xx} = -S \left(W, \frac{1}{2} W + \bar{W} \right) \quad (2.1)$$

$$D \nabla^4 W + \frac{1}{R} F_{,xx} + P \left[\frac{1}{2} \alpha (W + \bar{W})_{,xx} + (W + \bar{W})_{,yy} \right] = S(W + \bar{W}, F) \quad (2.2)$$

$$W = W_{,xx} = F = F_{,xx} = 0 \quad \text{at } X = 0, L, \quad 0 < X < \pi, \quad 0 < Y < 2\pi. \quad (2.3)$$

where, X and Y are the axial and circumferential coordinates respectively and $\bar{W}(X, Y)$, is a continuously differentiable stress-free and time independent imperfection. In this work, an alphabetic subscript placed after a comma indicates partial differentiation while S is the symmetric bi-linear operator in X and Y given by

$$S(P, Q) = P_{,xx} Q_{,yy} + P_{,yy} Q_{,xx} - 2P_{,xy} Q_{,xy} \quad (2.4a)$$

and ∇^4 , is the two-dimensional bi-harmonic operator defined by

$$\nabla^4 = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right)^2 \tag{2.4b}$$

here, we shall neglect both axial and circumferential inertia and shall similarly assume simply-supported boundary conditions and neglect boundary layer effect by assuming that the pre-buckling deflection is constant.

As in [11] and [22], we now introduce the following non-dimensional quantities.

$$x = \frac{X\pi}{L}, \quad H = \frac{h}{R}, \quad \varepsilon \bar{w} = \frac{\bar{W}}{h}, \quad \lambda = \frac{L^2 RP}{\pi^2 D}, \quad \xi = \frac{L^2}{\pi^2 R^2} \tag{2.5a}$$

$$y = \frac{Y}{R}, \quad w = \frac{W}{h}, \quad K(\xi) = -\frac{A^2}{(1+\xi)^2}, \quad A = \frac{L^2 \sqrt{12(1-\nu^2)}}{\pi^2 Rh} \tag{2.5b}$$

where, ν is Poisson’s ratio and ε is a small parameter which measures the amplitude of the imperfection while L is the length of the cylindrical shell which is simply-supported at $x = 0, \pi$.

We shall neglect boundary layer effect by assuming that the pre-buckling deflection is constant so that we let

$$F = -\frac{1}{2} PR \left(X^2 + \frac{1}{2} \alpha Y^2 \right) + \left(\frac{Eh^2 L^2}{\pi^2 R(1+\xi)^2} \right) f \tag{2.7a}$$

$$W = \frac{PR^2 \left(1 - \frac{1}{2} \alpha \nu \right)}{Eh} + hw \tag{2.7b}$$

where, P is the applied static load and λ is the non-dimensional load amplitude. The first terms on the right hand sides of (2.7a) and (2.7b), are pre-buckling approximations, while the parameter α , shall take the value $\alpha = 1$, if pressure contributes to axial stress through the ends, otherwise $\alpha = 0$, if pressure only acts laterally.

Substituting (2.7a) and (2.7b) into (2.1 and (2.2), using (2.5a) and (2.5b), and simplifying results to

$$\bar{\nabla}^4 f - (1+\xi)^2 w_{,xx} = -H(1+\xi)^2 s \left(w, \frac{1}{2} w + \varepsilon \bar{w} \right) \tag{2.8}$$

and

$$\bar{\nabla}^4 w - K(\xi) f_{,xx} + \lambda \left[\frac{1}{2} \alpha (w + \varepsilon \bar{w})_{,xx} + (w + \varepsilon \bar{w})_{,yy} \right] = -HK(\xi) s(w + \varepsilon \bar{w}, f) \tag{2.9}$$

$$w = w_{,xx} = f = f_{,xx} = 0 \text{ at } x = 0, \pi, \quad 0 < x < \pi, \quad 0 < y < 2\pi, \quad 0 < \varepsilon \ll 1, \tag{2.10}$$

where,

$$s(P, Q) = P_{,xx} Q_{,yy} + P_{,yy} Q_{,xx} - 2P_{,xy} Q_{,xy}, \quad \bar{\nabla}^4 = \left(\frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2} \right)^2 \tag{2.11}$$

3. Classical Buckling Load

The classical buckling load λ_c is the load that is required to buckle the asso-

ciated linear perfect structure and its equations, from (2.8) and (2.9) are

$$\bar{\nabla}^4 f - (1 + \xi)^2 w_{,xx} = 0 \quad (3.1)$$

and

$$\bar{\nabla}^4 w - K(\xi) f_{,xx} + \lambda \left[\frac{1}{2} \alpha w_{,xx} + \bar{w}_{,yy} \right] = 0 \quad (3.2)$$

$$w = w_{,xx} = 0, \quad f = f_{,xx} = 0 \quad \text{at } x = 0, \pi. \quad (3.3)$$

The solution to (3.1)-(3.3) is a superposition of the form

$$(w, f) = (a_{mk}, b_{mk}) \sin(ky + \phi_{mk}) \sin mx \quad (3.4)$$

where, $(a_{mk}, b_{mk}) \neq (0, 0)$ and ϕ_{mk} is an inconsequential phase.

On substituting (3.4) into (3.1), using (3.3) and after lengthy simplification, we get

$$b_{mk} = - \frac{(1 + \xi)^2 m^2 a_{mk}}{(m^2 + \xi k^2)^2} \quad (3.5)$$

Substituting (3.5) into (3.2) and simplifying, yields

$$\lambda = \frac{(m^2 + \xi k^2)^2 - \frac{K(\xi) m^4 (1 + \xi)^2}{(m^2 + \xi k^2)^2}}{\frac{1}{2} \alpha m^2 + \frac{k^2 m^4 (1 + \xi)^2 \xi}{(m^2 + \xi k^2)^2}} \quad (3.6)$$

Thus, if n is the critical value of k that minimizes λ , then, the value of λ at $k = n$ was taken as the classical buckling load λ_c . Thus, in this case, we get

$$\frac{d\lambda}{dk} = 0 \quad (3.7)$$

Therefore, corresponding to $k = n$, we see that (3.6) is now equivalent to

$$\lambda = \frac{(m^2 + \xi n^2)^2 - \frac{K(\xi) m^4 (1 + \xi)^2}{(m^2 + \xi n^2)^2}}{\frac{1}{2} \alpha m^2 + \frac{n^2 m^4 (1 + \xi)^2 \xi}{(m^2 + \xi n^2)^2}} \quad (3.8)$$

Usually, m and n take the values $m = 1, 2, 3, \dots$ and $n = 0, 1, 2, \dots$

We recall that [23] had assumed that k varies continuously, and so, minimized λ with respect to k . If $m = 1$ is the nontrivial values of m and we let $\zeta = \xi n^2$, then, we have

$$\lambda_c = \frac{(1 + \zeta)^2 + \frac{A^2}{(1 + \zeta)^2}}{\frac{1}{2} \alpha + \zeta} \quad (3.9)$$

The corresponding displacement and Airy Stress function are

$$(w, f) = \left[\left\{ 1, -\left(\frac{1+\xi}{1+\zeta} \right)^2 \right\} \right] a_{1n} \sin(ky + \phi_{1n}) \sin x \tag{3.10}$$

4. Static Theory

In this section, we shall derive the equations satisfied by the displacement and Airy stress functions when the static load is applied.

Similar to (2.8) and (2.9), the structure satisfies the following equations at static loading

$$\bar{\nabla}^4 f - (1 + \xi)^2 w_{,xx} = -H(1 + \xi)^2 s \left(w, \frac{1}{2} w + \varepsilon \bar{w} \right) \tag{4.1}$$

and

$$\bar{\nabla}^4 w - K(\xi) f_{,xx} + \lambda \left[\frac{1}{2} \alpha (w + \varepsilon \bar{w})_{,xx} + \xi (w + \varepsilon \bar{w})_{,yy} \right] = -HK(\xi) s(w + \varepsilon \bar{w}, f) \tag{4.2}$$

$$w = w_{,xx} = f = f_{,xx} = 0 \quad \text{at } x = 0, \pi \tag{4.3}$$

We now assume the following asymptotic expansions

$$\begin{pmatrix} w \\ f \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} w^{(i)} \\ f^{(i)} \end{pmatrix} \varepsilon^i \tag{4.4}$$

Substituting (4.4) into (4.1) and (4.2), and equating the coefficients of orders of $\varepsilon^i, i = 1, 2, 3, \dots$, the following equations are obtained

$$\mathcal{O}(\varepsilon): \begin{cases} \bar{\nabla}^4 f^{(1)} - (1 + \xi)^2 w_{,xx}^{(1)} = 0. \\ \bar{\nabla}^4 w^{(1)} - K(\xi) f_{,xx}^{(1)} + \lambda \left[\frac{1}{2} \alpha (w^{(1)} + \bar{w})_{,xx} + \xi (w^{(1)} + \bar{w})_{,yy} \right] = 0 \end{cases} \tag{4.5}$$

$$\mathcal{O}(\varepsilon^2): \begin{cases} \bar{\nabla}^4 f^{(2)} - (1 + \xi)^2 w_{,xx}^{(2)} = -H(1 + \xi)^2 \left[\frac{1}{2} s(w^{(1)}, w^{(1)}) + s(w^{(1)}, \bar{w}) \right] \\ \bar{\nabla}^4 w^{(2)} - K(\xi) f_{,xx}^{(2)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(2)} + \xi w_{,yy}^{(2)} \right] \\ = -HK(\xi) \left[s(w^{(1)}, f^{(1)}) + s(\bar{w}, w^{(1)}) \right] \end{cases} \tag{4.6}$$

$$\mathcal{O}(\varepsilon^3): \begin{cases} \bar{\nabla}^4 f^{(3)} - (1 + \xi)^2 w_{,xx}^{(3)} = -H(1 + \xi)^2 \left[s(w^{(1)}, w^{(2)}) + s(w^{(2)}, \bar{w}) \right] \\ \bar{\nabla}^4 w^{(3)} - K(\xi) f_{,xx}^{(3)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(3)} + \xi w_{,yy}^{(3)} \right] \\ = -HK(\xi) \left[s(w^{(1)}, f^{(2)}) + s(w^{(2)}, f^{(1)}) + s(\bar{w}, f^{(2)}) \right] \end{cases} \tag{4.7}$$

etc.

We seek solutions to (4.5)-(4.7) in the form

$$\begin{pmatrix} f^{(i)} \\ w^{(i)} \end{pmatrix} = \sum_{(k=1),(p=0)}^{\infty} \left[\begin{pmatrix} f_1^{(i)} \\ w_1^{(i)} \end{pmatrix} \cos py + \begin{pmatrix} f_2^{(i)} \\ w_2^{(i)} \end{pmatrix} \sin py \right] \sin kx \tag{4.8}$$

and now assume

$$\bar{w}(x, y) = \bar{a} \sin mx \sin ny \quad (4.9)$$

As earlier obtained, we shall need the following simplifications

$$\bar{\nabla}^4 = \left(\frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2} \right)^2 = \left(\frac{\partial^4}{\partial x^4} + 2\xi \frac{\partial^4}{\partial x^2 \partial y^2} + \xi^2 \frac{\partial^4}{\partial y^4} \right) \quad (4.10a)$$

so that, if

$$f^{(1)} = f_{\Gamma_1}^{(1)} \cos py \sin kx$$

then, we have

$$\bar{\nabla}^4 f^{(1)} = (k^2 + \xi p^2)^2 f_{\Gamma_1}^{(1)} \cos py \sin kx, \quad \Gamma_1 = 1, 2 \quad (4.10b)$$

and, if

$$f^{(2)} = f_{\Gamma_2}^{(2)} \sin py \sin kx$$

we get

$$\bar{\nabla}^4 f^{(2)} = (k^2 + \xi p^2)^2 f_{\Gamma_2}^{(2)} \sin py \sin kx, \quad \Gamma_2 = 1, 2 \quad (4.10c)$$

We shall use the fact that

$$K(\xi) = -\frac{A^2}{(1+\xi)^2} \quad (4.10d)$$

Solution of Equations of First Order Perturbation

The equations necessary here, from (4.5), are

$$\bar{\nabla}^4 f^{(1)} - (1+\xi)^2 w_{,xx}^{(1)} = 0 \quad (4.11)$$

and

$$\bar{\nabla}^4 w^{(1)} - K(\xi) f_{,xx}^{(1)} + \lambda \left[\frac{1}{2} \alpha (w^{(1)} + \bar{w})_{,xx} + \xi (w^{(1)} + \bar{w})_{,yy} \right] = 0 \quad (4.12)$$

Substituting (4.8) and (4.9) into (4.11), using (4.10a), (4.10b) and (4.10c), multiplying the resultant equation through by $\cos ny \sin mx$ and integrating with respect to y from 0 to 2π and with respect to x from 0 to π , we note that for $p = n, k = m$, we easily get

$$f_1^{(1)} = -\frac{m^2 (1+\xi)^2 w_1^{(1)}}{(m^2 + n^2 \xi)^2} \quad (4.13)$$

Similarly, by multiplying the resultant equation through by $\sin ny \sin mx$, and integrating with respect to y from 0 to 2π and with respect to x from 0 to π , and for $p = n, k = m$, we obtain

$$f_2^{(1)} = -\frac{m^2 (1+\xi)^2 w_2^{(1)}}{(m^2 + n^2 \xi)^2} \quad (4.14)$$

In the same manner, substituting (4.8) and (4.9) into (4.12), assuming (4.10a),

(4.10b) and (4.10c), thereafter multiplying the resultant equation by $\sin ny \sin mx$ and integrating with respect to x from 0 to π and y from 0 to 2π , and for $p = n, k = m$, we get

$$\left[(m^2 + n^2 \xi)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right) \right] w_2^{(1)} + K(\xi) m^2 V_2^{(1)} = \lambda \bar{a} \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right) \tag{4.15}$$

On substituting for $f_2^{(1)}$ from (4.14) and $K(\xi)$ from (4.10d) in (4.15) and simplifying, yields

$$w_2^{(1)} = B_0 \tag{4.16}$$

where,

$$B_0 = \frac{\lambda \bar{a} \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right)}{\varphi_0^2}, \tag{4.17}$$

$$\varphi_0^2 = (m^2 + n^2 \xi)^2 + \left(\frac{m^2 A}{m^2 + n^2 \xi} \right)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right)$$

Next, multiplying the resultant equation by $\cos ny \cos mx$ and integrating with respect to x and y from 0 to π and 0 to 2π , respectively for $p = n, k = m$, and simplifying, we get

$$w_1^{(1)} = 0 \tag{4.18}$$

We therefore expect from (4.8) that for $i = 1$

$$w^{(1)} = B_0 \sin mx \sin ny; \quad f^{(1)} = -\Phi_0 B_0 \sin mx \sin ny, \quad \Phi_0 = \frac{m^2 (1 + \xi)^2}{(m^2 + n^2 \xi)^2} \tag{4.19}$$

Solution of Equations of Second Order Perturbation

Equations of the second order to be solved are from (4.6), namely

$$\bar{\nabla}^4 f^{(2)} - (1 + \xi)^2 w_{,xx}^{(2)} = -H (1 + \xi)^2 \left[\frac{1}{2} s(w^{(1)}, w^{(1)}) + s(w^{(1)}, \bar{w}) \right] \tag{4.20}$$

and

$$\begin{aligned} &\bar{\nabla}^4 w^{(2)} - K(\xi) f_{,xx}^{(2)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(2)} + \xi w_{,yy}^{(2)} \right] \\ &= -HK(\xi) \left[s(w^{(1)}, w^{(1)}) + s(\bar{w}, w^{(1)}) \right] \end{aligned} \tag{4.21}$$

Evaluating the symmetric bi-linear functions on the right hand sides of (4.20) and (4.21), substituting the same and simplifying, we get (after simplifying trigonometric terms)

$$\bar{\nabla}^4 f^{(2)} - (1 + \xi)^2 w_{,xx}^{(2)} = -H (1 + \xi)^2 (mn)^2 \left[\left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) (\cos 2mx + \cos 2ny) \right] \tag{4.22}$$

and

$$\begin{aligned} &\bar{\nabla}^4 w^{(2)} - K(\xi) f_{,xx}^{(2)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(2)} + \xi w_{,yy}^{(2)} \right] \\ &= -HK(\xi) (mn)^2 \left[\Phi_0 B_0^2 + \bar{a} B_0 \right] (\cos 2mx + \cos 2ny) \end{aligned} \tag{4.23}$$

Next we substitute (4.8) and (4.9) into (4.22), assuming (4.10a), (4.10b) and (4.10c), for $i = 2$. Thereafter, we multiply the resultant equation through by $\cos 2ny \sin mx$ and integrate with respect to x and y , to get

$$f_1^{(2)} = \Phi_1 \left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) - \Phi_2 w_1^{(2)},$$

$$\Phi_1 = \frac{4Hm^2 (1+\xi)^2}{\pi(m^2 + 4n^2\xi)^2}, \quad \Phi_2 = \frac{m^2 (1+\xi)^2}{\pi(m^2 + 4n^2\xi)^2}, \quad m = \text{odd} \quad (4.24)$$

Similarly, we next multiply the resultant equation by $\sin nys \sin mx$ and integrate as usual, and for $p = n, k = m$, we get

$$f_2^{(2)} = -\frac{m^2 (1+\xi)^2 w_2^{(2)}}{(m^2 + n^2\xi)^2}. \quad (4.25)$$

Next, we substitute (4.8) and (4.9) into (4.23), using (4.10a), (4.10b) and (4.10c), for $i = 2$, and then multiply the resultant equation through by $\cos 2ny \sin mx$ and integrate, as usual for $p = 2n, k = m$ to get

$$w_1^{(2)} = \Phi_5, \quad \Phi_5 = \Phi_3 (B_0^2 \Phi_0 + B_0 \Phi_0 \bar{a}) + \Phi_4 \left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) \quad (4.26)$$

$$\Phi_3 = \frac{(mn)^2 HK(\xi)}{(m^2 + 4n^2\xi)^2 + \left(\frac{m^2 A}{m^2 + 4n^2\xi} \right)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + 4n^2\xi \right)}$$

$$\Phi_4 = \frac{4m^3 n^2 A^2}{(m^2 + 4n^2\xi)^2 \left[(m^2 + 4n^2\xi)^2 + \left(\frac{m^2 A}{m^2 + 4n^2\xi} \right)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + 4n^2\xi \right) \right]} \quad (4.27)$$

In the same manner, multiply the resultant equation by $\sin nys \sin mx$ and integrate with respect to x and y , for $p = n, k = m$, and simplify to get

$$w_2^{(2)} = 0, \quad f_2^{(2)} = 0 \quad (4.28)$$

Therefore, we observe from (4.8), and for $i = 2$, that

$$w^{(2)} = w_1^{(2)} \cos 2ny \sin mx; \quad f^{(2)} = f_1^{(2)} \cos 2ny \sin mx, \quad (4.29)$$

On substitution in (4.29) using (4.24) and in the first part of (4.26), we get

$$w^{(2)} = \Phi_5 \cos 2ny \sin mx; \quad f^{(2)} = \Phi_{10} \cos 2ny \sin mx,$$

$$\Phi_{10} = \left[\Phi_1 \left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) - \Phi_2 \Phi_5 \right] \quad (4.30)$$

Solution of Equations of Third Order Perturbation

The actual equations of the third order are from (4.7), namely

$$\bar{\nabla}^4 f^{(3)} - (1+\xi)^2 w_{,xx}^{(3)} = -H(1+\xi)^2 \left[s(w^{(1)}, w^{(2)}) + s(w^{(2)}, \bar{w}) \right] \quad (4.31)$$

and

$$\begin{aligned} & \bar{\nabla}^4 w^{(3)} - K(\xi) f_{,xx}^{(3)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(3)} + \xi w_{,yy}^{(3)} \right] \\ & = -HK(\xi) \left[s(w^{(1)}, f^{(2)}) + s(w^{(2)}, f^{(1)}) + s(\bar{w}, f^{(2)}) \right] \end{aligned} \tag{4.32}$$

Evaluating the symmetric bi-linear functions on the right sides of (4.31) and (4.32) and substituting the same and simplifying, yields

$$\begin{aligned} & \bar{\nabla}^4 f^{(3)} - (1 + \xi)^2 w_{,xx}^{(3)} \\ & = -\frac{1}{4} H (1 + \xi)^2 (mn)^2 (\Phi_5 B_0 + \Phi_5 \bar{a}) [9 \sin 3ny - \sin ny \\ & \quad - \cos 2mx \sin 3ny + 9 \cos 2mx \sin ny] \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} & \bar{\nabla}^4 w^{(3)} - K(\xi) f_{,xx}^{(3)} + \lambda \left[\frac{1}{2} \alpha w_{,xx}^{(3)} + \xi w_{,yy}^{(3)} \right] \\ & = -\frac{1}{4} HK(\xi) (mn)^2 \left[-\Phi_0 \Phi_5 B_0 + B_0 \left\{ \Phi_1 \left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) - \Phi_2 \Phi_5 \right\} \right. \\ & \quad \left. + \bar{a} \left\{ \Phi_1 \left(\frac{1}{2} B_0^2 + B_0 \bar{a} \right) - \Phi_2 \Phi_5 \right\} \right] [9 \sin 3ny - \sin ny \\ & \quad - \cos 2mx \sin 3ny + 9 \cos 2mx \sin ny] \end{aligned} \tag{4.34}$$

We observe from the simplifications on the right hand sides of (4.33) and (4.34) that there will be four buckling modes generated $w_{i(r,p)}^{(3)}$ with their respective Airy stress functions $f_{i(r,p)}^{(3)}$. These buckling modes correspond to the following terms on the right hand sides of (4.33) and (4.34) : $\sin 3nysin mx$, $\sin nysin mx$, $\cos 2mx \sin 3ny$ and $\cos 2mx \sin ny$.

However, of the four modes, it is only the mode in the shape of $\sin nysin mx$ that is in the shape of the imperfection. We shall consider this mode and the additional mode in the shape of $\sin 3nysin mx$.

We now substitute (4.8) and (4.9) into (4.33), using (4.10a), (4.10b) and (4.10c), for $i = 3$, and thereafter, multiply the resultant equation through by $\sin 3nysin mx$ and integrate and for $k = m$, $p = 3n$, to get

$$f_{2(m,3n)}^{(3)} = \frac{1}{(m^2 + 9n^2 \xi)^2} \left\{ \frac{9}{\pi} H (1 + \xi)^2 mn^2 A_0 - m^2 (1 + \xi)^2 w_{2(m,3n)}^{(3)} \right\} \tag{4.35}$$

In the same way, we multiply the resultant equation by $\sin mx \sin ny$, integrate and for $k = m$, $p = n$, to get

$$\begin{aligned} & f_{2(m,n)}^{(3)} = \frac{1}{(m^2 + n^2 \xi)^2} \left\{ \frac{1}{\pi} H (1 + \xi)^2 mn^2 A_0 - m^2 (1 + \xi)^2 w_{2(m,n)}^{(3)} \right\}, \\ & A_0 = B_0^3 l_0, \quad l_0 = \Phi_5 \left(\frac{1}{B_0^2} + \frac{1}{B_0^3} \bar{a} \right) \end{aligned} \tag{4.36}$$

We next substitute (4.8) and (4.9) into (4.34), using (4.10a), (4.10b) and (4.10c), for $i = 3$. Thereafter, we multiply the resultant equation by $\sin 3nysin mx$, integrate and note that for, $k = m$, $p = 3n$, we get

$$w_{2(m,3n)}^{(3)} = \frac{\Theta_0 A_0 - \Theta_1 A_{01}}{\left(m^2 + 9n^2 \xi\right)^2 + \left(\frac{m^2 A}{m^2 + 9n^2 \xi}\right)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + 9n^2 \xi\right)},$$

$$\Theta_0 = \frac{9}{\pi(m^2 + 9n^2 \xi)^2} H m^3 n^2 A^2, \quad \Theta_1 = \frac{9}{\pi} H K(\xi) m n^2, \quad (4.37)$$

$$A_{01} = B_0^3 l_{01}, \quad l_{01} = \frac{1}{2} \Phi_1 - \frac{1}{B_0^2} \Phi_0 \Phi_5 + \frac{1}{B_0} \bar{a} \Phi_1 - \frac{1}{B_0^2} \Phi_2 \Phi_5$$

Similarly, we multiply through by $\sin ny \sin mx$ and note that for $k = m, p = k$, we get

$$w_{2(m,n)}^{(3)} = \frac{\Theta_2 A_{01} - \Theta_3 A_0}{\left(m^2 + n^2 \xi\right)^2 + \left(\frac{m^2 A}{m^2 + n^2 \xi}\right)^2 - \lambda \left(\frac{1}{2} \alpha m^2 + n^2 \xi\right)},$$

$$\Theta_3 = \frac{1}{\pi(m^2 + n^2 \xi)^2} H m^3 n^2 A^2, \quad \Theta_2 = \frac{1}{\pi} H K(\xi) m n^2$$

Thus, of the four non-zero buckling modes of this order and their respective Airy stress function, the ones we shall consider are

$$w_{2(m,3n)}^{(3)} \sin 3ny \sin mx, \quad w_{2(m,n)}^{(3)} \sin ny \sin mx,$$

$$f_{2(m,3n)}^{(3)} \sin 3ny \sin mx \quad \text{and} \quad f_{2(m,n)}^{(3)} \sin ny \sin mx \quad (4.39)$$

As a summary so far, we can write the displacement and its respective Airy stress functions as

$$\begin{pmatrix} w \\ f \end{pmatrix} = \varepsilon \begin{pmatrix} w_{2(m,n)}^{(1)} \\ f_{2(m,n)}^{(1)} \end{pmatrix} \sin mx \sin ny + \varepsilon^2 \begin{pmatrix} w_{1(m,2n)}^{(2)} \\ f_{1(m,2n)}^{(2)} \end{pmatrix} \cos ny \sin mx$$

$$+ \varepsilon^3 \left[\begin{pmatrix} w_{2(m,3n)}^{(3)} \\ f_{2(m,3n)}^{(3)} \end{pmatrix} \sin mx \sin 3ny + \begin{pmatrix} w_{2(m,n)}^{(3)} \\ f_{2(m,n)}^{(3)} \end{pmatrix} \sin mx \sin ny \right] + \dots \quad (4.40)$$

Equation (4.40) determines the displacement and the corresponding Airy stress functions.

5. Values of Independent Variables at Maximum Displacement

The analysis henceforth will be concerned with the displacement components that are partly in the shape of the imperfection or partly in the shape of $\sin mx \sin 3ny$. In this respect, we neglect the displacements of order $\mathcal{O}(\varepsilon^2)$ in (4.40) so that the displacement becomes

$$w = \varepsilon w_{2(m,n)}^{(1)} \sin mx \sin ny + \varepsilon^3 \left\{ w_{2(m,n)}^{(3)} \sin mx \sin ny \right.$$

$$\left. + \Omega w_{2(m,3n)}^{(3)} \sin mx \sin 3ny \right\} + \mathcal{O}(\varepsilon^4) \quad (5.1)$$

where,

$$\Omega = 0 \text{ or } 1$$

When $\Omega = 0$, we get the exact displacement that is purely in the shape of imperfection, but when $\Omega = 1$, we get the resultant displacement incorporating the modes $\sin mx \sin 3ny$ and $\sin mx \sin ny$.

Since the displacement w in (5.1) depends on x and y then, the conditions for maximum displacement are as follows

$$w_{,x} = w_{,y} = 0 \quad (5.2)$$

We now let x_a and y_a be critical values of x and y respectively at maximum displacement.

From (5.1), using (5.2), we see that for maximum displacement,

$$x_a = \frac{\pi}{2m}; \quad y_a = \frac{\pi}{2n} \quad (5.3)$$

where (5.3) are the least nontrivial values of x_a and y_a .

6. Maximum Displacement

The maximum displacement is obtained from (5.1) at the critical values of x and y where w has a maximum value. Hence, the value of w at these values becomes

$$w_a = \varepsilon w_{2(m,n)}^{(1)}(\lambda) + \varepsilon^3 \left\{ w_{2(m,n)}^{(3)}(\lambda) - \Omega w_{2(m,3n)}^{(3)}(\lambda) \right\} + \dots \quad (6.1)$$

Meanwhile, (6.1) can be recast as

$$w_a = \varepsilon c_1 + \varepsilon^3 c_3 + \mathcal{O}(\varepsilon^4) \quad (6.2)$$

where,

$$c_1 = w_{2(m,n)}^{(1)}(\lambda), \quad c_3 = w_{2(m,n)}^{(3)}(\lambda) - \Omega w_{2(m,3n)}^{(3)}(\lambda) \quad (6.3)$$

7. Static Buckling Load

The static buckling load, λ_s according to [24] [25] and [26], is obtained from the maximization

$$\frac{d\lambda}{dw_a} = 0 \quad (7.1)$$

The usual procedure (as in [22] [26] and [27]), is to first reverse the series (6.2) in the form

$$\varepsilon = d_1 w_a + d_3 w_a^3 + \dots \quad (7.2)$$

By substituting (6.1) into (7.2) and equating the coefficients of powers of orders of ε , we get

$$d_1 = \frac{1}{c_1}, \quad d_3 = -\frac{c_3}{c_1^4} \quad (7.3)$$

The maximization in (7.1) easily follows from (7.2) where w_a is now being substituted for w to yield, after some simplifications,

$$\varepsilon = \frac{2}{3\sqrt{3}} \sqrt{\frac{c_1}{c_3}} \quad (7.4)$$

On substituting into (7.4), using (6.3) and simplifying, we get

$$\left\{ \left(m^2 + n^2 \xi \right)^2 + \left(\frac{m^2 A}{m^2 + n^2 \xi} \right)^2 - \lambda_s \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right) \right\}^{\frac{3}{2}} = \frac{3\sqrt{3}}{2} \lambda_s (\varepsilon \bar{a}) \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right) \Psi_0 \quad (7.5)$$

This determines the static buckling load λ_s of the circular cylindrical shell structure, and the determination is implicit in the load parameter λ_s , where,

$$\Psi_0 = (\Theta_2 l_{01} - \Theta_3 l_0) \sqrt{1 - \frac{\Omega Q_{02}}{Q_{01}}}, \quad Q_{01} = \frac{(\Theta_2 l_{01} - \Theta_3 l_0)}{\varphi_0^2},$$

$$Q_{02} = \frac{(\Theta_0 l_0 - \Theta_1 l_{01})}{\varphi_0^2}, \quad \varphi_0^2 = \left[\left(m^2 + n^2 \xi \right)^2 + \left(\frac{m^2 A}{m^2 + n^2 \xi} \right)^2 - \lambda_s \left(\frac{1}{2} \alpha m^2 + n^2 \xi \right) \right] \quad (7.6)$$

8. Results and Discussion

The result (7.5) is asymptotic in nature. The results of the classical buckling load λ_c , and that of the cylindrical shell structure are as seen in (3.9), whereas, the corresponding displacement and Airy Stress function of the structure are as in (3.10). Similarly, the static buckling load λ_s , is as shown in (7.5). A computer program in MATLAB gives the relationship between the static buckling load λ_s , and the imperfection parameter ε , at $\Omega = 0$ or $\Omega = 1$, and where we have fixed the following as $\alpha = 1$, $A = 0.2$, $H = 0.2$, $\bar{a} = 0.02$, $\bar{\xi} = 0.8$, $m = 1$ and $n = 1$ is as shown in **Table 1**.

A careful appraisal of the graph of **Figure 1**, shows that static buckling load λ_s , decreases with increased imperfection parameter ε . This is expected. In other words, static buckling load λ_s increases with less imperfection. The value of static buckling load λ_s is higher when the buckling mode is a combination of the modes in the shape of imperfection ($\sin mx \sin ny$) and shape of other geometric form ($\sin mx \sin 3ny$), *i.e.* ($\Omega = 1$) compared to the case when the buckling mode is in the shape of imperfection ($\sin mx \sin ny$) (*i.e.* $\Omega = 0$).

Table 1. Relationship between static buckling load, λ_s and imperfection parameter, ε for some values of Ω of the circular cylindrical shell structure using Equation (7.5).

ε	λ_s	
	$\Omega = 0$	$\Omega = 1$
0.0100	2.4945	2.4954
0.0200	2.4944	2.4953
0.0300	2.4943	2.4952
0.0400	2.4942	2.4951
0.0500	2.4941	2.4949
0.0600	2.4940	2.4948
0.0700	2.4939	2.4946
0.0800	2.4937	2.4944
0.0900	2.4936	2.4942
0.1000	2.4933	2.4938

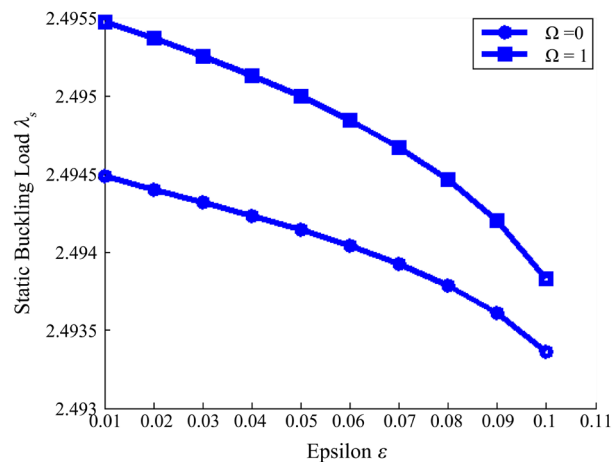


Figure 1. Graph of Static buckling load λ_s , as a function of imperfection parameter, ε for some values of Ω of the circular cylindrical shell structure using **Table 1**.

9. Conclusion

This analysis has analytically determined the maximum of the out-of-plane normal displacement of a finite imperfect cylindrical shell trapped by a static load. We have used the techniques of perturbation and asymptotics to derive an implicit formula for determining the static buckling load of the cylindrical shell investigated. The formulation contains a small parameter depicting the amplitude of the imperfection and on which all asymptotic series are expanded. Such an analytical approach can be duplicated for other structures including toroidal shell segments and plates.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

Acknowledgements

This paper is dedicated to the memory of late EZIKE OGBUEHI FELIX IKENNA OZOIGBO, who died on 20th day of July, 2019 and to be buried on 22nd day of November, 2019.

References

- [1] Xu, X.S., Ma, J.Q., Li, C.W. and Chu, H.J. (2009) Dynamic Local and Global Buckling of Cylindrical Shells under Axial Impact. *Engineering Structures*, **31**, 1132-1140. <https://doi.org/10.1016/j.engstruct.2009.01.009>
- [2] Fan, H.G., Chen, Z.P., Feng, W., Fan, Z. and Cao, G.W. (2015) Dynamic Buckling of Cylindrical Shells with Arbitrary Axisymmetric Thickness Variation under Time Dependent External Pressure. *International Journal of Structural Stability and Dynamics*, **15**, 21 p. <https://doi.org/10.1142/S0219455414500539>
- [3] Koiter, W.T., Elishakoff, I.Y.W. and Li, Y.W. (1994) Buckling of an Axially Compressed Cylindrical Shell of Variable Thickness. *International Journal of Solids Structures*, **31**, 797-805. [https://doi.org/10.1016/0020-7683\(94\)90078-7](https://doi.org/10.1016/0020-7683(94)90078-7)

- [4] Hutchinson, J.W. and Amazigo, J.C. (1967) Imperfect-Sensitivity of Eccentricity Stiffened Cylindrical Shells. *AIAA Journal*, **5**, 392-401. <https://doi.org/10.2514/3.3992>
- [5] Budiansky, B. and Roth, R.S. (1962) Axisymmetric Dynamic Buckling of Clamped Shallow Spherical Shells. Collected Papers on Stability of Shell Structures, NASA TN D-1510, Washington DC.
- [6] Budiansky, B. and Hutchinson, J.W. (1970) Buckling Progress and Challenge in Process of Trends in Solid Mechanics. University Press Delft, Delft, 93-116.
- [7] Roth, R. and Klosner, J. (1964) Nonlinear Response of Cylindrical Shells Subjected to Dynamic Axial Loads. *AIAA Journal*, **2**, 1788-1794. <https://doi.org/10.2514/3.2666>
- [8] Amazigo, J.C. and Frazer, W.B. (1971) Buckling under External Pressure of Cylindrical Shells with Dimple-Shaped Initial Imperfection. *International Journal of Solids Structures*, **7**, 883-900. [https://doi.org/10.1016/0020-7683\(71\)90070-9](https://doi.org/10.1016/0020-7683(71)90070-9)
- [9] Budiansky, B. and Amazigo, J.C. (1968) Initial Post-Buckling Behaviour of Cylindrical Shell under External Pressure. *Journal of Mathematical Physics*, **47**, 233-235. <https://doi.org/10.1002/sapm1968471223>
- [10] Ette, A.M. and Onwuchekwa, J.U. (2007) On the Static Buckling of an Externally Pressurized Finite Circular Cylindrical Shell. *Journal of the Nigerian Association of Mathematical Physics*, **11**, 323-332. <https://doi.org/10.4314/jonamp.v11i1.40226>
- [11] Lockhart, D. and Amazigo, J.C. (1975) Dynamic Buckling of Externally Pressurized Imperfect Cylindrical Shells. *Journal of Applied Mechanics*, **42**, 316-320. <https://doi.org/10.1115/1.3423574>
- [12] Bich, D.H., Dung, D.V., Nam, V.H. and Phuong, N.T. (2013) Nonlinear Static and Dynamic Buckling Analysis of Imperfect Eccentrically Stiffened Functionally Graded Circular Cylindrical Thin Shells under Axial Compression. *International Journal of Mechanical Science*, **74**, 190-200. <https://doi.org/10.1016/j.ijmecsci.2013.06.002>
- [13] Paimushin, V.N. (2016) Static and Dynamic Buckling Modes of Spherical Shells Subjected to External Pressure. *Russian Mathematics*, **60**, 37-46. <https://doi.org/10.3103/S1066369X1604006X>
- [14] Paimushin, V.N. (2017) Static and Dynamic Beam Buckling Shapes of Long Orthotropic Cylindrical Shells Undergoing External Pressure. *Journal of Applied Mathematics and Mechanics*, **72**, 1014-1027.
- [15] Lukankin, S.A. and Paimushin, V.N. (2014) Static and Dynamic Beam Buckling Shapes of Long Orthotropic Cylindrical Shells under External Pressure. *Izv. RAS. MTT*, No. 1, 108-128.
- [16] HuyBich, D., Dung, D.V., Namb, V.H. and Phuong, N.T. (2013) Nonlinear Static and Dynamic Buckling Analysis of Imperfect Eccentrically Stiffened Functionally Graded Circular Cylindrical Thin Shells under Axial Compression. *International Journal of Mechanical Sciences*, **74**, 190-200. <https://doi.org/10.1016/j.ijmecsci.2013.06.002>
- [17] Ette, A.M. (2003) Buckling of a Cylindrical Shell Pressurized by an Impulse. *Journal of the Nigerian Mathematical Society*, **22**, 82-110.
- [18] Ette, A.M. (2004) On the Dynamic Buckling of Stochastically Imperfect Finite Cylindrical Shells under Step Loading. *Journal of the Nigerian Association of Mathematical Physics*, **8**, 35-46. <https://doi.org/10.4314/jonamp.v8i1.39971>
- [19] Ette, A.M. (2006) On the Dynamic Buckling of Lightly Damped Cylindrical Shell Modulated by a Periodic Load. *Journal of the Nigerian Association of Mathematical Physics*, **10**, 327-344. <https://doi.org/10.4314/jonamp.v10i1.40141>

- [20] Ette, A.M. (2008) Perturbation Technique on the Dynamic Buckling of a Lightly Damped Cylindrical Shell Axially Stressed by an Impulse. *Journal of the Nigerian Association of Mathematical Physics*, **12**, 103-120. <https://doi.org/10.4314/jonamp.v12i1.45493>
- [21] Ette, A.M. (2008) Initial Post-Buckling of Toroidal Shell Segments Pressurized by External Load. *Journal of the Nigerian Association of Mathematical Physics*, **12**, 121-132. <https://doi.org/10.4314/jonamp.v12i1.45494>
- [22] Amazigo, J.C. (1974) Asymptotic Analysis of the Buckling of Externally Pressurized Cylinders with Random Imperfections. *Quarterly of Applied Mathematics*, **32**, 429-442. <https://doi.org/10.1090/qam/99693>
- [23] Batdorf, S.B. (1947) A Simplified Method of Elastic Stability Analysis for Thin Cylindrical Shells. NACA Report, 874.
- [24] Budiansky, B. (1966) Dynamic Buckling of Elastic Structures, Criteria and Estimation. In: *Dynamic Stabilities of Structures*, Hermann Pergamon, Oxford, 83-106.
- [25] Hutchinson, J.W. and Budiansky, B. (1966) Dynamic Buckling Estimates. *AIAA Journal*, **4**, 525-530. <https://doi.org/10.2514/3.3468>
- [26] Amazigo, J.C. and Ette, A.M. (1987) On a Two-Small Parameter Differential Equation with Application to Dynamic Buckling. *Journal of Nigerian Mathematical Society*, **6**, 90-102.
- [27] Amazigo, J.C. (1971) Buckling of Stochastically Imperfect Columns on Nonlinear Elastic Foundation. *Quarterly of Applied Mathematics*, **29**, 403-409. <https://doi.org/10.1090/qam/99755>