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Higher-Order Expansions of Sample Range from the Skew-t-Normal Distribution

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Abstract

For an independent and identically distributed skew-t-normal random sequence, this paper establishes the limit distribution of normalized sample range $M_n - m_n$. Based on the optimal normalized constants, the higher-order asymptotic expansion of the distribution function of sample range $M_n - m_n$ is further derived, and its convergence rate is given. In addition, the approximate accuracy between the empirical value and the asymptotic theoretical value is systematically compared by numerical simulation.

Keywords

Higher-Order Expansion, Skew-t-Normal Distribution, Rate of Convergence, Sample Range

1. Introduction

A random variable X is said to have a skew-t-normal distribution [1] with scale $\sigma^2(\sigma > 0)$, degrees of freedom $v > 0$ and shape parameter $\lambda \in \mathbb{R}$ (written as $X \sim STN(\sigma^2, \lambda, v)$) if its probability density function (pdf) is

$$f_\lambda(x;\sigma^2,\lambda,v) = \frac{2\Gamma\left(\frac{v+1}{2}\right)}{\sigma\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{\sigma^2 v}\right)^{-\frac{v+1}{2}}\Phi\left(\frac{\lambda}{\sigma}x\right), \quad -\infty < x < +\infty \tag{1}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf) with $\Gamma(\cdot)$ denoting the Gamma function. The skew-t-normal distribution is a probabilistic model capable of capturing both heavy-tailedness and pronounced skewness. It exhibits a substantially wider range of skewness compared to the skew-normal distribution [2] and related families, thereby offering greater flexibility in modeling data that deviates significantly from normality. These characteristics make this distribution deeply studied and widely applicable in practical applications, for example, the study of Vanadium pollution in Shadegan Wetland [3] and automated flow cytometry analysis [4]. Further practical applications of the skew-t-normal distribution can be located in [5]-[7].

Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with common marginal distribution function F_λ following the skew-t-normal distribution, and let $M_n = \max\{X_n, n \ge 1\}$ and $m_n = \min\{X_n, n \ge 1\}$ denote the partial maximum and minimum. If G is an extreme value distribution, we say a distribution F is in the domain of attraction of G (written $F \in D(G)$) if there exists $a_n \ge 0$, $b_n \in \mathbb{R}$, $n \ge 1$ such that

$$F^n(a_n x + b_n) \to G(x) \tag{2}$$

weakly as $n \to \infty$ where G is of the type of one of the following classes:

$$\begin{aligned}
&\text{Type I}: \quad \Lambda(x) = \exp\left(-\mathrm{e}^{-x}\right), \quad -\infty < x < \infty, \\
&\text{Type II}: \quad \Phi_\alpha(x) = \begin{cases} 0, & x \le 0, \\ \exp\left(-x^{-\alpha}\right), & x > 0, \end{cases} \\
&\text{Type III}: \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\}, & x \le 0, \\ 1, & x > 0, \end{cases}
\end{aligned} \tag{3}$$

for some $\alpha > 0$ (cf. [8]). In extreme value analysis, accurately determining the domain of attraction to which a distribution belongs is a prerequisite for establishing its limiting distribution and even higher-order expansions under appropriate normalization.

Yang and Hu (2025)[1] proved that under different values of λ, the skew-t-normal distribution belongs to the domain of attraction of different extreme value distribution with different normalizing constants, *i.e.*:

$$\begin{cases} \lim_{n\to\infty} P\left(M_n \le u_n(x)\right) = \Lambda(x), & \lambda < 0; \\ \lim_{n\to\infty} P\left(M_n \le \tilde{u}_n(x)\right) = \Phi_v(x), & \lambda > 0, \end{cases} \tag{4}$$

with

$$u_n(x) = a_n x + b_n,\ \lambda < 0; \quad \tilde{u}_n(x) = b_n x,\ \lambda > 0, \tag{5}$$

and

$$\begin{cases} \lim_{n\to\infty} P\left(m_n \le v_n(y)\right) = 1 - \Phi_v(-y), & \lambda < 0; \\ \lim_{n\to\infty} P\left(m_n \le \tilde{v}_n(y)\right) = 1 - \Lambda(-y), & \lambda > 0, \end{cases} \tag{6}$$

with

[1]The study by Yang, W. and Hu, S. ("Higher-order expansions of sample extremes from the skew-t-normal distribution") has been submitted to Communications in Statistics-Theory and Methods. The manuscript is currently under reconsideration by the journal, with reviewers evaluating the revised version. It is not listed in the bibliography to prevent circular citation. A copy of the manuscript can be provided by the corresponding author, should it be required.

$$v_n(y)=\bar{b}_n y,\ \lambda<0;\quad \tilde{v}_n(y)=\bar{a}_n y-\bar{b}_n,\ \lambda>0, \tag{7}$$

where the norming constants a_n, b_n, $\bar{a}_n$ and $\bar{b}_n$ satisfy

$$a_n=\frac{\sigma^2}{\lambda^2}b_n^{-1},\ 1-F_\lambda(b_n)=n^{-1};\quad \bar{a}_n=\frac{\sigma^2}{\lambda^2}\bar{b}_n^{-1},\ 1-F_{-\lambda}(\bar{b}_n)=n^{-1}, \tag{8}$$

and for fixed λ, b_n and $\bar{b}_n$ have a special relationship: $F_\lambda(b_n)=1-F_\lambda(-\bar{b}_n)$.

For independent and identically distributed (i.i.d.) random sequences, in addition to the asymptotic behavior of the normalized partial maxima and minima and their joint distribution, the asymptotic behavior of the sample range has also been studied. The weak convergence of the range was first discussed by De Haan (1974) [9]. Subsequently, Barakat and Nigm (1990) [10] extended this work by studying the weak convergence of the sample range, as well as the case where the sample size is a positive integer random variable. Later, Tan *et al.* (2007) [11] proved the almost sure convergence of the sample range, and Zang (2014) [12] derived an almost inevitable local central limit theorem for it under mild assumptions. Furthermore, Matula and Adler (2022) [13] proved the law of large numbers for the sample range of the Pareto distribution. More recently, Zhang and Lu (2023) [14] studied the higher-order expansion of the sample range for the skew normal distribution, while Lu *et al.* (2023) [15] established the distributional expansions of the normalized sample range from the general error distribution.

Shifting focus to the skew-t-normal distribution, Yang and Hu (2025) focused on the skew-t-normal distribution, discussing the convergence rates of its maximum and minimum distribution functions, as well as their joint distribution and density functions. However, the asymptotic theory for the range of skew-t-normal samples remains unexplored. Therefore, to fill this gap in the literature, this paper will systematically derive the higher-order asymptotic expansion of the range distribution for skew-t-normal samples. The significance of this work resides in its capacity to utilize high-order expansions for improving the precision and convergence of the limiting distribution approximation, furnishing a more dependable theoretical framework for statistical inference concerning extreme events.

The rest of this article is arranged as follows. Section 2 presents the main results. Section 3 compares the accuracy between the actual values and the approximate values. Section 4 gives some auxiliary lemmas and the proofs of the main results are given in Section 5. Section 6 concludes with a summary.

2. Main Results

In this section, we provide the limiting distribution and convergence rates of the distribution of M_n-m_n and its higher-order distributional expansion under different value of λ with different normalized constants.

Theorem 1. Let $\{X_i, 1\le i\le n\}$ be a sequence of independent random variables with common distribution $F_\lambda(x)$. Denote $M_n=\max\{X_i,1\le i\le n\}$ and $\min=\{X_i,1\le i\le n\}$ and $\tilde{u}_n(x)$ and $v_n(y)$ given by (5) and (7). Then, we have

(i). for $\lambda<0$,

$$I_{n,\lambda}(y) := P\left(M_n - m_n \le v_n(y)\right) - \Phi_v(y) \to 0, \tag{9}$$

and

$$\begin{aligned}&\lim_{n\to\infty} b_n^2\left[b_n^2\left(I_{n,\lambda}(y) - \int_{-\infty}^{+\infty} \mathrm{e}^{-x}\Lambda(x)\Phi_v(y)s_\lambda(x)\mathrm{d}x\right)\right]\\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-x}\Lambda(x)\Phi_v(y)r_\lambda(x)\mathrm{d}x,\end{aligned} \tag{10}$$

where

$$s_\lambda(x) = \frac{(v+3)\sigma^2}{\lambda^2} - \frac{\sigma^2}{2\lambda^2}x^2 - \frac{(v+2)\sigma^2}{\lambda^2}x + \kappa_\lambda(x) \tag{11}$$

and

$$\begin{aligned}r_\lambda(x) = {}& \omega_\lambda(x) + \frac{\kappa_\lambda^2(x)}{2} + \left[\frac{(v+3)\sigma^2}{\lambda^2} - \frac{\sigma^2}{2\lambda^2}x^2 - \frac{(v+2)\sigma^2}{\lambda^2}x\right]\kappa_\lambda(x)\\ &+\left(\frac{v(v+1)\sigma^4}{\lambda^2} + \frac{2\sigma^4}{\lambda^4} - \frac{(v+3)(v+2)\sigma^4}{\lambda^4}\right)x - \frac{2(v+4)\sigma^4}{\lambda^4}\\ &-\frac{v(v+1)\sigma^4}{\lambda^2} + \frac{(v+1)(v+3)\sigma^4}{2\lambda^4}x^2 + \frac{(v+2)\sigma^4}{2\lambda^4}x^3 + \frac{\sigma^4}{8\lambda^4}x^4\end{aligned} \tag{12}$$

with

$$\kappa_\lambda(x) = \left[\frac{1}{2}x^2 + (v+3)x\right]\frac{\sigma^2}{\lambda^2}\mathrm{e}^{-x} \tag{13}$$

and

$$\begin{aligned}\omega_\lambda(x) = -\Bigg[&\frac{2\sigma^2}{\lambda^2}\left(\frac{v(v+1)\sigma^2}{2} + \frac{(v+4)\sigma^2}{\lambda^2}\right)x + \frac{(v+3)\sigma^4}{2\lambda^4}x^2\\ &+\frac{1}{2}\left(\frac{\sigma^2}{2\lambda^2}x^2 + \frac{(v+3)\sigma^2}{\lambda^2}x\right)^2\Bigg]\mathrm{e}^{-x};\end{aligned} \tag{14}$$

(ii). for $\lambda > 0$,

$$\tilde{I}_{n,\lambda}(x) := P\left(M_n - m_n \le \tilde{u}_n(x)\right) - \Phi_v(x) \to 0, \tag{15}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \bar{b}_n^2\left[\bar{b}_n^2\left(\tilde{I}_{n,\lambda}(x) - \int_{-\infty}^{+\infty} \mathrm{e}^{y}\Lambda(-y)\Phi_v(x)s_{-\lambda}(-y)\mathrm{d}y\right)\right]\\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{y}\Lambda(-y)\Phi_v(x)r_{-\lambda}(-y)\mathrm{d}y\end{aligned} \tag{16}$$

where $s_{-\lambda}(-y)$ and $r_{-\lambda}(-y)$ are obtained from $s_\lambda(-y)$ and $r_\lambda(-y)$ by replacing λ with $-\lambda$.

Remark 1. For normalized sample range with norming sequence $\tilde{u}_n^*(x) = \eta_n x$ and $v_n^*(y) = \eta_n y$, where

$$\eta_n = \left(\frac{2\Gamma\left(\frac{v+1}{2}\right)\sigma^v v^{\frac{v}{2}-1}}{\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}\right)^{\frac{1}{v}} n^{\frac{1}{v}}.$$

is given by Yang and Hu (2025), its limit distribution of the sample range

$M_n - m_n$ is consistent with the conclusion of Theorem 1, and the corresponding higher-order expansions of sample range can also be obtained, that is,

(i). for $\lambda < 0$,

$$I_{n,\lambda}^*(y) := P\left(M_n - m_n \le v_n^*(y)\right) - \Phi_v(y) \to 0, \tag{17}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \log\log n\left[\frac{\log n}{(\log\log n)^2}\left(I_{n,\lambda}^*(y) - \int_{-\infty}^{+\infty} \mathrm{e}^{-x}\Lambda(x)\Phi_v(y)s_\lambda^*(x)\mathrm{d}x\right)\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-x}\Lambda(x)\Phi_v(y)r_\lambda^*(x)\mathrm{d}x,\end{aligned} \tag{18}$$

where

$$s_\lambda^*(x) = \kappa_\lambda^*(x) - \frac{(v+3)^2}{16} \tag{19}$$

and

$$r_\lambda^*(x) = \omega_\lambda^*(x) + \frac{(v+3)\left[v+2+x+\log\left(\frac{\sqrt{2}\Gamma\left(\frac{v+1}{2}\right)(-\lambda)^v v^{\frac{v}{2}}}{\pi\Gamma\left(\frac{v}{2}\right)}\right)\right]}{4} \tag{20}$$

with

$$\kappa_\lambda^*(x) = \frac{(v+3)^2}{16}\mathrm{e}^{-x} \tag{21}$$

and

$$\omega_\lambda^*(x) = -\frac{(v+3)\left(x+v+3+\log\frac{\sqrt{2}\Gamma\left(\frac{v+1}{2}\right)(-\lambda)^v v^{\frac{v}{2}}}{\pi\Gamma\left(\frac{v}{2}\right)}\right)}{4}\mathrm{e}^{-x}; \tag{22}$$

(ii). for $\lambda > 0$,

$$\tilde{I}_{n,\lambda}^*(x) := P\left(M_n - m_n \le \tilde{u}_n^*(x)\right) - \Phi_v(x) \to 0, \tag{23}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \log\log n\left[\frac{\log n}{(\log\log n)^2}\left(\tilde{I}_{n,\lambda}^*(x) - \int_{-\infty}^{+\infty} \mathrm{e}^{y}\Lambda(-y)\Phi_v(x)s_{-\lambda}^*(-y)\mathrm{d}y\right)\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{y}\Lambda(-y)\Phi_v(x)r_{-\lambda}^*(-y)\mathrm{d}y.\end{aligned} \tag{24}$$

where $s_\lambda^*(\cdot)$ and $r_\lambda^*(\cdot)$ are obtained by replacing λ with $-\lambda$ from $s_\lambda^*(\cdot)$ and $r_\lambda^*(\cdot)$ given in (19) and (20).

Note that $b_n = O\left(\sqrt{\log n}\right)$ and $\bar{b}_n = O\left(\sqrt{\log n}\right)$. From Theorem 1 and Remark 1, the convergence rate of the distribution of sample range $M_n - m_n$ is

$O\left(\frac{1}{\log n}\right)$ under normalized constants b_n and $\bar{b}_n$, while it is $O\left(\frac{(\log\log n)^2}{\log n}\right)$ under normalized constant η_n.

3. Numerical Analysis

In this section, numerical studies are presented to illustrate the accuracy of higher-order expansions of the cdf of the normalized $M_n - m_n$. Let $L_i(y)$ and $U_i(x)$, $i = 1, 2, 3$, denote the first-order, the second-order and the third-order asymptotics of the cdf of the normalized $M_n - m_n$. From Theorem 1, we have

Table 1. The absolute errors between actual values and its approximations of the cdf of the normalized $M_n - m_n$ with $(n, \sigma) = (100, 1)$ for $\lambda < 0$.

(v, λ, y)	$\lvert L - L_1\rvert$	$\lvert L - L_2\rvert$	$\lvert L - L_3\rvert$
$(2, -3, 0.5)$	0.009350061961	0.009350061959	0.009350061941
$(2, -3, 1)$	0.037794456846	0.037794456846	0.037794456481
$(2, -3, 2.5)$	0.007291688814	0.007291688815	0.007291687969
$(2, -3, 4)$	0.002310197299	0.002310197299	0.002310196367
$(2, -4, 0.5)$	0.006727253763	0.006721065231	0.006721065216
$(2, -4, 1)$	0.027098930697	0.027098930654	0.027098930363
$(2, -4, 2.5)$	0.005664154550	0.005664154550	0.005664153874
$(2, -4, 4)$	0.001877767574	0.001877767574	0.001877766829
$(1.5, -3, 0.5)$	0.013532735646	0.013532735640	0.013532735576
$(1.5, -3, 1)$	0.016635733516	0.016635733516	0.016635733121
$(1.5, -3, 2.5)$	0.003804335347	0.003804335347	0.003804334512
$(1.5, -3, 4)$	0.001415812591	0.001415812591	0.001415811643
$(1.5, -4, 0.5)$	0.010045832712	0.010045832705	0.010045832641
$(1.5, -4, 1)$	0.012265049126	0.012265049085	0.012265048684
$(1.5, -4, 2.5)$	0.002886327334	0.002886327334	0.002886326488
$(1.5, -4, 4)$	0.001106975657	0.001106975657	0.001106974696

Table 2. The absolute errors between actual values and its approximations of the cdf of the normalized $M_n - m_n$ with $(n, \sigma) = (1000, 1)$ for $\lambda < 0$.

(v, λ, y)	$\lvert L - L_1\rvert$	$\lvert L - L_2\rvert$	$\lvert L - L_3\rvert$
$(2, -3, 0.5)$	0.005999173742	0.005999173741	0.005999173739
$(2, -3, 1)$	0.017938460495	0.017938460495	0.017938460452
$(2, -3, 2.5)$	0.002808046188	0.002808046188	0.002808046088
$(2, -3, 4)$	0.001017960224	0.001017960224	0.001017960113
$(2, -4, 0.5)$	0.004541237239	0.004539342334	0.004539342333
$(2, -4, 1)$	0.013260117764	0.013260117751	0.013260117723
$(2, -4, 2.5)$	0.002125050279	0.002125050279	0.002125050216

Continued

$(2,-4,4)$	0.000849008491	0.000849008491	0.000849008421
$(1.5,-3,0.5)$	0.004593492428	0.004593492426	0.004593492418
$(1.5,-3,1)$	0.004988635638	0.004988635638	0.004988635593
$(1.5,-3,2.5)$	0.000721496906	0.000721496906	0.000721496810
$(1.5,-3,4)$	0.000000628940	0.000000628940	0.000000628951
$(1.5,-4,0.5)$	0.003477885996	0.003477885994	0.003477885989
$(1.5,-4,1)$	0.003877077425	0.003877077413	0.003877077377
$(1.5,-4,2.5)$	0.000861162434	0.000861162434	0.000861162358
$(1.5,-4,4)$	0.000371306232	0.000371306233	0.000371306146

Table 3. The absolute errors between actual values and its approximations of the cdf of the normalized $M_n - m_n$ with $(n,\sigma)=(100,1)$ for $\lambda > 0$.

(v,λ,x)	$\lvert U-U_1\rvert$	$\lvert U-U_2\rvert$	$\lvert U-U_3\rvert$
$(2,3,0.5)$	0.009350061808	0.009350061806	0.009350061788
$(2,3,1)$	0.037794449719	0.037794449719	0.037794449354
$(2,3,2.5)$	0.007291668033	0.007291668033	0.007291667187
$(2,3,4)$	0.002310175903	0.002310175903	0.002310174971
$(2,4,0.5)$	0.006727254051	0.006721065519	0.006721065504
$(2,4,1)$	0.027098980537	0.027098980494	0.027098980202
$(2,4,2.5)$	0.005664175501	0.005664175501	0.005664174826
$(2,4,4)$	0.001877789343	0.001877789343	0.001877788599
$(1.5,3,0.5)$	0.013533517090	0.013533517084	0.013533517020
$(1.5,3,1)$	0.016642050457	0.016642050457	0.016642050061
$(1.5,3,2.5)$	0.003818218828	0.003818218828	0.003818217994
$(1.5,3,4)$	0.001430121982	0.001430121982	0.001430121034
$(1.5,4,0.5)$	0.010045834757	0.010045834751	0.010045834686
$(1.5,4,1)$	0.012265089309	0.012265089267	0.012265088867
$(1.5,4,2.5)$	0.002886297768	0.002886297768	0.002886296922
$(1.5,4,4)$	0.001106665963	0.001106665963	0.001106665002

Table 4. The absolute errors between actual values and its approximations of the cdf of the normalized $M_n - m_n$ with $(n,\sigma)=(1000,1)$ for $\lambda > 0$.

(v,λ,x)	$\lvert U-U_1\rvert$	$\lvert U-U_2\rvert$	$\lvert U-U_3\rvert$
$(2,3,0.5)$	0.005999364766	0.005999364765	0.005999364763
$(2,3,1)$	0.017944923928	0.017944923929	0.017944923885
$(2,3,2.5)$	0.002824078186	0.002824078186	0.002824078085
$(2,3,4)$	0.002001450902	0.002001450902	0.002001450791

Continued

$(2,4,0.5)$	0.004541252964	0.004539358060	0.004539358059
$(2,4,1)$	0.013292745311	0.013292745298	0.013292745271
$(2,4,2.5)$	0.002126531624	0.002126531624	0.002126531561
$(2,4,4)$	0.000708637712	0.000708637712	0.000708637643
$(1.5,3,0.5)$	0.004614674689	0.004614674687	0.004614674680
$(1.5,3,1)$	0.005183691518	0.005183691518	0.005183691473
$(1.5,3,2.5)$	0.001109438827	0.001109438827	0.001109438732
$(1.5,3,4)$	0.000390670111	0.000390670111	0.000390670003
$(1.5,4,0.5)$	0.003463618270	0.003463618268	0.003463618263
$(1.5,4,1)$	0.003877046225	0.003877046212	0.003877046176
$(1.5,4,2.5)$	0.000832418148	0.000832418148	0.000832418071
$(1.5,4,4)$	0.000295375652	0.000295375652	0.000295375566

$$\begin{cases} L(y) = P\left(M_n - m_n \le v_n(y)\right), \\ L_1(y) = \Phi_v(y), \\ L_2(y) = L_1(y) + b_n^{-2} \int_{-\infty}^{+\infty} \mathrm{e}^{-x} \Lambda(x) \Phi_v(y) s_\lambda(x) \mathrm{d}x, \\ L_3(y) = L_2(y) + b_n^{-4} \int_{-\infty}^{+\infty} \mathrm{e}^{-x} \Lambda(x) \Phi_v(y) r_\lambda(x) \mathrm{d}x, \end{cases} \tag{25}$$

for $\lambda < 0$; and

$$\begin{cases} U(x) = P\left(M_n - m_n \le \tilde{u}_n(x)\right), \\ U_1(x) = \Phi_v(x), \\ U_2(x) = L_1(x) + \bar{b}_n^{-2} \int_{-\infty}^{+\infty} \mathrm{e}^{y} \Lambda(-y) \Phi_v(x) s_{-\lambda}(-y) \mathrm{d}y, \\ U_3(x) = L_2(x) + \bar{b}_n^{-4} \int_{-\infty}^{+\infty} \mathrm{e}^{y} \Lambda(-y) \Phi_v(x) r_{-\lambda}(-y) \mathrm{d}y, \end{cases} \tag{26}$$

for $\lambda > 0$, where $s_\lambda(x)$, $r_\lambda(x)$, $s_{-\lambda}(-y)$ and $r_{-\lambda}(-y)$ are given in Theorem1, respectively.

Next, we use R to calculate the absolute errors with sample sizes $n = 100$ and $n = 1000$ under normalized constants a_n, $\bar{a}_n$, b_n and $\bar{b}_n$. From Tables 1-4, we have the following findings: 1) The accuracy of higher-order asymptotics improves with increasing n; 2) First-order, second-order and third-order asymptotics are all close to the actual values; 3) The accuracy of the three approximations is comparable, as the second-order and third-order corrections do not lead to a significant improvement in the approximation. Therefore, it cannot be concluded that the third-order approximation is markedly superior; it only marginally approaches the true value in numerical terms. To achieve further improvements in accuracy, subsequent studies could employ the same approach as presented in this work to calculate higher-order terms.

4. Auxiliary Lemmas

Before proving the main conclusion, we first give the following lemmas. Lemma 1 and Lemma 2 have been proved by Yang and Hu (2025).

Lemma 1. Let $F_\lambda(x)$ denote the cdf of $STN(\sigma^2,\lambda,v)$. For large x, we have

(i). for $\lambda<0$,

$$\begin{aligned}1-F_\lambda(x) &= -\frac{\sqrt{2}\Gamma\left(\frac{v+1}{2}\right)\sigma^{v+3}v^{\frac{v}{2}}}{\pi\Gamma\left(\frac{v}{2}\right)\lambda^3}\mathrm{e}^{-\frac{\lambda^2}{2\sigma^2}}\exp\left\{-\int_1^x\frac{g(t)}{f(t)}\mathrm{d}t\right\}\left\{1-\left(\frac{v(v+1)\sigma^2}{2}\right.\right.\\&\quad\left.+\frac{(v+4)\sigma^2}{\lambda^2}\right)x^{-2}+\left[\frac{v^2(v+1)(v+3)\sigma^4}{8}\right.\\&\quad\left.\left.+\frac{v(v+1)(v+6)\sigma^4}{2\lambda^2}+\frac{(v^2+9v+23)\sigma^4}{\lambda^4}\right]x^{-4}+O\left(x^{-6}\right)\right\}\end{aligned} \tag{27}$$

where $g(t)=1+\frac{(v+3)\sigma^2}{\lambda^2t^2}$ and $f(t)=\frac{\sigma^2}{\lambda^2}t^{-1}$.

(ii). for $\lambda>0$,

$$\begin{aligned}1-F_\lambda(x) &= \frac{2\Gamma\left(\frac{v+1}{2}\right)\sigma^{v}v^{\frac{v}{2}-1}}{\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}\exp\left\{-\int_1^x\alpha(t)t^{-1}\mathrm{d}t\right\}\left\{1-\frac{(v^3+v^2)\sigma^2}{2(v+2)}x^{-2}\right.\\&\quad\left.+\left[\frac{(v^4-v^2)\sigma^4}{8}+\frac{3\sigma^4v^2}{(v+2)(v+4)}+\frac{(v^3-v^2)\sigma^4}{2(v+2)}\right]x^{-4}+O\left(x^{-6}\right)\right\}\end{aligned} \tag{28}$$

where $\alpha(t)=v$.

Lemma 2. Let $f_\lambda(x)$ be the density function of the skew-t-normal distribution.

(i). for $\lambda<0$, we have

$$\begin{aligned}na_nf_\lambda(u_n(x)) &= \mathrm{e}^{-x}\left\{1+\left(\frac{(v+3)\sigma^2}{\lambda^2}-\frac{\sigma^2}{2\lambda^2}x^2-\frac{(v+2)\sigma^2}{\lambda^2}x\right)b_n^{-2}\right.\\&\quad+\left[\left(\frac{v(v+1)\sigma^4}{\lambda^2}+\frac{2\sigma^4}{\lambda^4}-\frac{(v+3)(v+2)\sigma^4}{\lambda^4}\right)x\right.\\&\quad-\frac{2(v+4)\sigma^4}{\lambda^4}-\frac{v(v+1)\sigma^4}{\lambda^2}+\frac{(v+1)(v+3)\sigma^4}{2\lambda^4}x^2\\&\quad\left.\left.+\frac{(v+2)\sigma^4}{2\lambda^4}x^3+\frac{\sigma^4}{8\lambda^4}x^4\right]b_n^{-4}+o\left(b_n^{-4}\right)\right\}\end{aligned} \tag{29}$$

(ii). for $\lambda>0$, we have

$$
\begin{aligned}
n\bar{a}_n f_\lambda\left(\tilde{v}_n(y)\right) = \mathrm{e}^y \Bigg\{ & 1 + \left(\frac{(v+3)\sigma^2}{\lambda^2} - \frac{\sigma^2}{2\lambda^2} y^2 + \frac{(v+2)\sigma^2}{\lambda^2} y \right) \bar{b}_n^{-2} \\
& + \Bigg[-\left(\frac{v(v+1)\sigma^4}{\lambda^2} + \frac{2\sigma^4}{\lambda^4} - \frac{(v+3)(v+2)\sigma^4}{\lambda^4} \right) y \\
& - \frac{2(v+4)\sigma^4}{\lambda^4} - \frac{v(v+1)\sigma^4}{\lambda^2} + \frac{(v+1)(v+3)\sigma^4}{2\lambda^4} y^2 \\
& - \frac{(v+2)\sigma^4}{2\lambda^4} y^3 + \frac{\sigma^4}{8\lambda^4} y^4 \Bigg] \bar{b}_n^{-4} + o\left(\bar{b}_n^{-4}\right) \Bigg\}
\end{aligned} \tag{30}
$$

Lemma 3. With $u_n(x)$, $\tilde{u}_n(x)$, $v_n(y)$ and $\tilde{v}_n(y)$ given by (5) and (7), we have

(i). for $\lambda < 0$,

$$
\begin{aligned}
& \left[F_\lambda\left(u_n(x)\right) - F_\lambda\left(u_n(x) + v_n(-y)\right) \right]^n \\
& = \Lambda(x)\Phi_v(y)\left\{ 1 + \kappa_\lambda(x) b_n^{-2} + \left[\omega_\lambda(x) + \frac{\kappa_\lambda^2(x)}{2} \right] b_n^{-4} \right\} + o\left(b_n^{-4}\right);
\end{aligned} \tag{31}
$$

(ii). for $\lambda > 0$,

$$
\begin{aligned}
& \left[F_\lambda\left(\tilde{u}_n(x) + \tilde{v}_n(y)\right) - F_\lambda\left(\tilde{v}_n(y)\right) \right]^n \\
& = \Phi_v(x)\Lambda(-y)\left\{ 1 + \kappa_{-\lambda}(-y)\bar{b}_n^{-2} + \left[\omega_{-\lambda}(-y) + \frac{\kappa_{-\lambda}^2(-y)}{2} \right] \bar{b}_n^{-4} \right\} + o\left(\bar{b}_n^{-4}\right).
\end{aligned} \tag{32}
$$

where $\kappa_\lambda(x)$, $\omega_\lambda(x)$, $\kappa_{-\lambda}(-y)$ and $\omega_{-\lambda}(-y)$ are given in Theorem 1.

Proof. 1). For $\lambda < 0$, by lemma 1 and $1 - F_\lambda(b_n) = n^{-1} = 1 - F_{-\lambda}\left(\bar{b}_n\right)$, we have

$$
\begin{aligned}
z_\lambda(x,y) & := n\log\left[F_\lambda\left(u_n(x)\right) - F_\lambda\left(u_n(x) + v_n(-y)\right) \right] + \mathrm{e}^{-x} + y^{-v} \\
& = n\log\left[1 - \left(1 - F_\lambda\left(u_n(x)\right) + F_\lambda\left(u_n(x) + v_n(-y)\right)\right) \right] + \mathrm{e}^{-x} + y^{-v} \\
& = -n\left[\left(1 - F_\lambda\left(u_n(x)\right)\right) + \frac{1}{2}\left(1 - F_\lambda\left(u_n(x)\right)\right)^2 (1 + o(1)) - n^{-1}\mathrm{e}^{-x} \right] \\
& \quad - n\Big[\left(1 - F_{-\lambda}\left(-u_n(x) + v_n(y)\right)\right) + \frac{1}{2}\left(1 - F_{-\lambda}\left(-u_n(x) + v_n(y)\right)\right)^2 (1 + o(1)) \\
& \quad - n^{-1}y^{-v} \Big] - n\left(1 - F_{-\lambda}\left(-u_n(x) + v_n(y)\right)\right)\left(1 - F_\lambda\left(u_n(x)\right)\right)(1 + o(1)) \\
& = \mathrm{e}^{-x}\Bigg\{ \left(\frac{\sigma^2}{2\lambda^2}x^2 + \frac{(v+3)\sigma^2}{\lambda^2}x \right) b_n^{-2} - \Bigg[\frac{2\sigma^2}{\lambda^2}\left(\frac{v(v+1)\sigma^2}{2} + \frac{(v+4)\sigma^2}{\lambda^2} \right) x \\
& \quad + \frac{(v+3)\sigma^4}{2\lambda^4}x^2 + \frac{1}{2}\left(\frac{\sigma^2}{2\lambda^2}x^2 + \frac{(v+3)\sigma^2}{\lambda^2}x \right)^2 \Bigg] b_n^{-4} \Bigg\} + o\left(b_n^{-4}\right)
\end{aligned} \tag{33}
$$

for large n. Hence by (33), we get

$$
\lim_{n\to\infty} b_n^2 z_\lambda(x,y) = \kappa_\lambda(x), \quad \lim_{n\to\infty} b_n^2\left(b_n^2 z_\lambda(x,y) - \kappa_\lambda(x)\right) = \omega_\lambda(x) \tag{34}
$$

with $\kappa_\lambda(x)$ and $\omega_\lambda(x)$ given in Theorem 1. It can be easily derived that as $n \to \infty$

$$\begin{aligned}
&b_n^2\left\{b_n^2\left[\left(F_\lambda\left(u_n(x)\right)-F_\lambda\left(u_n(x)+v_n(-y)\right)\right)^n-\Lambda(x)\Phi_v(y)\right]-\kappa_\lambda(x)\Lambda(x)\Phi_v(y)\right\}\\
&=b_n^2\left\{b_n^2\left[\exp\left(z_\lambda(x,y)\right)-1\right]-\kappa_\lambda(x)\right\}\Lambda(x)\Phi_v(y)\\
&=\left\{b_n^2\left[b_n^2 z_\lambda(x,y)-\kappa_\lambda(x)\right]+b_n^4 z_\lambda^2(x,y)\left(\frac{1}{2}+z_\lambda(x,y)\sum_{i=3}^{\infty}\frac{z_\lambda^{i-3}(x,y)}{i!}\right)\right\}\Lambda(x)\Phi_v(y)\\
&\to\left[\omega_\lambda(x)+\frac{\kappa_\lambda^2(x)}{2}\right]\Lambda(x)\Phi_v(y).
\end{aligned}\tag{35}$$

So, combining with (35), we can obtain

$$\begin{aligned}
&\left[F_\lambda\left(u_n(x)\right)-F_\lambda\left(u_n(x)+v_n(-y)\right)\right]^n\\
&=\Lambda(x)\Phi_v(y)\left\{1+\kappa_\lambda(x)b_n^{-2}+\left[\omega_\lambda(x)+\frac{\kappa_\lambda^2(x)}{2}\right]b_n^{-4}\right\}+o\left(b_n^{-4}\right).
\end{aligned}\tag{36}$$

(ii). The proof is based on a similar approach to that used in the proof of (i). For $\lambda>0$, by (27) and (28), we have

$$\begin{aligned}
&\tilde{z}_\lambda(x,y):=n\log\left[F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right]+x^{-v}+\mathrm{e}^y\\
&=n\log\left[1-\left(1-F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)+F_\lambda\left(\tilde{v}_n(y)\right)\right)\right]+x^{-v}+\mathrm{e}^y\\
&=\mathrm{e}^y\left\{\left(\frac{\sigma^2}{2\lambda^2}y^2-\frac{(v+3)\sigma^2}{\lambda^2}y\right)\bar{b}_n^{-2}-\left[-\frac{2\sigma^2}{\lambda^2}\left(\frac{v(v+1)\sigma^2}{2}+\frac{(v+4)\sigma^2}{\lambda^2}\right)y\right.\right.\\
&\quad\left.\left.+\frac{(v+3)\sigma^4}{2\lambda^4}y^2+\frac{1}{2}\left(\frac{\sigma^2}{2\lambda^2}y^2-\frac{(v+3)\sigma^2}{\lambda^2}y\right)^2\right]\bar{b}_n^{-4}\right\}+O\left(\bar{b}_n^{-6}\right)
\end{aligned}\tag{37}$$

for large n. Thus by (37) we have

$$\lim_{n\to\infty}\bar{b}_n^2\tilde{z}_\lambda(x,y)=\kappa_{-\lambda}(-y),\quad \lim_{n\to\infty}\bar{b}_n^2\left(\bar{b}_n^2\tilde{z}_\lambda(x,y)-\kappa_{-\lambda}(-y)\right)=\omega_{-\lambda}(-y),\tag{38}$$

Similar to (35), we have

$$\begin{aligned}
&\bar{b}_n^2\left\{\bar{b}_n^2\left[\left(F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right)^n-\Lambda(-y)\Phi_v(x)\right]\right.\\
&\quad\left.-\kappa_{-\lambda}(-y)\Lambda(-y)\Phi_v(x)\right\}\\
&\to\left[\omega_{-\lambda}(-y)+\frac{\kappa_{-\lambda}^2(-y)}{2}\right]\Lambda(-y)\Phi_v(x).
\end{aligned}\tag{39}$$

and then combining with (39) we can get

$$\begin{aligned}
&\left[F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right]^n\\
&=\Phi_v(x)\Lambda(-y)\left\{1+\kappa_{-\lambda}(-y)\bar{b}_n^{-2}+\left[\omega_{-\lambda}(-y)+\frac{\kappa_{-\lambda}^2(-y)}{2}\right]\bar{b}_n^{-4}\right\}+o\left(\bar{b}_n^{-4}\right).
\end{aligned}\tag{40}$$

□

5. Proofs

Proof of Theorem 1. It follows from Yang and Hu (2025) that the density function of the normalized (M_n, m_n) is

$$g_{n,\lambda}(x,y)=\begin{cases} n(n-1)a_n\bar{b}_n\left(F_\lambda(u_n(x))-F_\lambda(v_n(y))\right)^{n-2} f_\lambda(u_n(x))\,f_\lambda(v_n(y)), & \lambda<0,\\ n(n-1)\bar{a}_n b_n\left(F_\lambda(\tilde{u}_n(x))-F_\lambda(\tilde{v}_n(y))\right)^{n-2} f_\lambda(\tilde{u}_n(x))\,f_\lambda(\tilde{v}_n(y)), & \lambda>0. \end{cases} \tag{41}$$

(i). For $\lambda<0$, let $q_{n,\lambda}(x,y)$ be the density function of $P(M_n\le u_n(x), M_n-m_n\le v_n(y))$. By the convolution formula, we can get $q_{n,\lambda}(x,y)=g_{n,\lambda}\left(x,\frac{a_nx+b_n}{\bar{b}_n}-y\right)$. Then we have

$$\begin{aligned} &P(M_n-m_n\le v_n(y))\\ &=\int_{-\infty}^{y}\int_{-\infty}^{+\infty} q_{n,\lambda}(x,s)\,\mathrm{d}x\mathrm{d}s\\ &=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{y} g_{n,\lambda}\left(x,\frac{a_nx+b_n}{\bar{b}_n}-s\right)\mathrm{d}s\right]\mathrm{d}x\\ &=na_n\int_{-\infty}^{+\infty}\left[F_\lambda(u_n(x))-F_\lambda(u_n(x)+v_n(-y))\right]^{n-1} f_\lambda(u_n(x))\,\mathrm{d}x. \end{aligned} \tag{42}$$

Note that $1-F_\lambda(u_n(x))=O(n^{-1})$ and $F_\lambda(u_n(x)+v_n(-y))=O(n^{-1})$, so we have

$$\begin{aligned} &\left[F_\lambda(u_n(x))-F_\lambda(u_n(x)+v_n(-y))\right]^{n-1}\\ &=\left[F_\lambda(u_n(x))-F_\lambda(u_n(x)+v_n(-y))\right]^{n}\left(1+O(n^{-1})\right). \end{aligned} \tag{43}$$

It follows from the dominated convergence theorem, (29) and (31), we have

$$\begin{aligned} &P(M_n-m_n\le v_n(y))\\ &=na_n\int_{-\infty}^{+\infty}\left[F_\lambda(u_n(x))-F_\lambda(v_n(-y))\right]^{n} f_\lambda(u_n(x))\left(1+O(n^{-1})\right)\mathrm{d}x\\ &=\int_{-\infty}^{+\infty}\mathrm{e}^{-x}\Lambda(x)\Phi_v(y)\left[1+s_\lambda(x)b_n^{-2}+r_\lambda(x)b_n^{-4}+o(b_n^{-4})\right]\mathrm{d}x\\ &\to\Phi_v(y). \end{aligned} \tag{44}$$

as $n\to\infty$.

(ii). Similarly, for $\lambda>0$, let $\tilde{q}_{n,\lambda}(x,y)$ be the density function of $P(M_n-m_n\le\tilde{u}_n(x), m_n\le\tilde{v}_n(y))$. Then $g_{n,\lambda}(x,y)$ and $\tilde{q}_{n,\lambda}(x,y)$ satisfy $\tilde{q}_{n,\lambda}(x,y)=g_{n,\lambda}\left(x+\frac{\bar{a}_ny-\bar{b}_n}{b_n},y\right)$, so we have

$$\begin{aligned} &P(M_n-m_n\le\tilde{u}_n(x))\\ &=\int_{-\infty}^{x}\int_{-\infty}^{+\infty}\tilde{q}_{n,\lambda}(s,y)\,\mathrm{d}y\mathrm{d}s\\ &=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} g_{n,\lambda}\left(s+\frac{\bar{a}_ny-\bar{b}_n}{b_n},y\right)\mathrm{d}s\right]\mathrm{d}y\\ &=n\bar{a}_n\int_{-\infty}^{+\infty}\left[F_\lambda(\tilde{u}_n(x)+\tilde{v}_n(y))-F_\lambda(\tilde{v}_n(y))\right]^{n-1} f_\lambda(\tilde{v}_n(y))\,\mathrm{d}y. \end{aligned} \tag{45}$$

Since $1-F_\lambda(\tilde{u}_n(x)+\tilde{v}_n(y))=O(n^{-1})$ and $F_\lambda(\tilde{v}_n(y))=O(n^{-1})$, then we have

$$\begin{aligned}&\left[F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right]^{n-1}\\&=\left[F_\lambda\left(\tilde{u}_n(x)+\tilde{v}_n(y)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right]^{n}\left(1+O\left(n^{-1}\right)\right).\end{aligned}\tag{46}$$

Combining with (30) and (32), we have

$$\begin{aligned}&P\left(M_n-m_n\le\tilde{u}_n(x)\right)\\&=n\bar{a}_n\int_{-\infty}^{+\infty}\left[F_\lambda\left(\tilde{u}_n(x)\right)-F_\lambda\left(\tilde{v}_n(y)\right)\right]^n f_\lambda\left(\tilde{v}_n(y)\right)\left(1+O\left(n^{-1}\right)\right)\mathrm{d}y\\&=\int_{-\infty}^{+\infty}\mathrm{e}^y\Lambda(-y)\Phi_v(x)\left[1+s_{-\lambda}(-y)\bar{b}_n^{-2}+r_{-\lambda}(-y)\bar{b}_n^{-4}+o\left(\bar{b}_n^{-4}\right)\right]\mathrm{d}y\\&\to\Phi_v(x)\end{aligned}\tag{47}$$

as $n\to\infty$. The proof is finished. □

6. Conclusions

This paper systematically investigates the limiting distribution properties of the normalized sample range under the skew-t-normal distribution. We first derive higher-order expansions for its extreme value distribution and normalization constants, thereby determining the optimal convergence rate of the distribution of normalized sample range M_n-m_n. In particular, the convergence rate obtained remains of the same order regardless of whether the skewness parameter λ is positive or negative, reflecting an intrinsic symmetry in the distributional morphology with respect to its convergence behavior. More importantly, a comparative analysis in this study reveals that the skew-t-normal, skew-normal (cf. [14]), and general error distributions (cf. [15]) have the same order of optimal convergence rate for the normalized sample range under normalized constants b_n and $\bar{b}_n$. However, the specific forms and convergence behaviors of the higher-order (second- and third-order) expansions of sample range for the skew-t-normal distribution are further dictated by the interplay between the skewness parameter λ and the degrees of freedom v.

In the numerical analysis section, we observe that the second- and third-order asymptotics provide no significant improvement in accuracy, indicating that further incorporation of lower-order terms within the current expansion framework offers limited refinement. To achieve substantial breakthroughs in approximation precision, future work must account for the influence of fourth-order and higher-order terms. Therefore, systematic investigation into the mathematical structure and numerical effects of higher-order expansion terms represents a critical pathway to overcoming current precision bottlenecks. Simultaneously, the limitations inherent in the current expansion order clearly delineate the theoretical boundaries of existing methodologies and highlight principal directions for subsequent research.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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