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Numerical Solution of the Quaternion Quadratic Matrix Equation $X^2 + BX + C = 0$

Baiquan Hu

Faculty of Science, Hunan University of Technology, Zhuzhou, China
Email: 793874884@qq.com

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Abstract

This paper addresses a class of numerical solution problems for quaternion quadratic matrix equations arising from practical engineering applications. By transforming it into a special Riccati equation, a sufficient condition for the existence of solutions to the corresponding *M*-matrix equation is derived using the theory of comparison matrices. This condition is then applied to investigate the existence of solutions for the quaternion quadratic matrix equation. Finally, numerical examples demonstrate an effective solution method for the quaternion quadratic matrix equation using fixed-point iteration.

Keywords

Quadratic Matrix Equations, Quaternion Matrices, *M*-Matrices, Fixed-Point Iteration Method, Comparison Matrix

1. Introduction

Solving secondary matrix equations holds a central position in computational mathematics. Their extension to noncommutative algebras has become a critical bottleneck in cutting-edge applications such as quantum mechanics (e.g., the Pauli matrix system) [1] and robotic control (quaternion rotation dynamics) [2]. These equations possess significant theoretical importance and broad application contexts across computational mathematics, control theory, signal processing, and applied physics.

This paper considers the following unilateral quadratic matrix equation:

$$X^2 + BX + C = 0, \tag{1}$$

where $B = \operatorname{diag}(b_1, b_2, \cdots, b_n)$ is a diagonal matrix, each b_i being a quaternion, $C \in \mathbb{H}^{n\times n}$ is a quaternion matrix, and

$[C]_{ij} = \mathcal{R}(c_{ij}) + \mathcal{J}(c_{ij}) = \mathcal{R}(c_{ij}) + \mathcal{J}_i(c_{ij})\boldsymbol{i} + \mathcal{J}_j(c_{ij})\boldsymbol{j} + \mathcal{J}_{\boldsymbol{k}}(c_{ij})\boldsymbol{k}$, $[C]_{ij} \in \mathbb{H}^{n\times n}$,

where $\mathcal{J}_i, \mathcal{J}_j, \mathcal{J}_k$ denote the imaginary parts of the quaternion, and I is the identity matrix. In this paper, $\mathbb{R}^{n\times n}$ denotes real $n\times n$ matrices, $\mathbb{H}^{n\times n}$ denotes quaternion $n\times n$ matrices, $\mathbb{C}$ represents the field of complex numbers, $\mathbb{C}_+$ represents the positive half-axis of the complex plane, and $\mathbb{C}_-$ represents the negative half-axis of the complex plane. Equation (1) generalizes the quadratic matrix equation in the real number field to the quaternion space. This generalization extends traditional applications such as system stability analysis, optimal control, the Markov chain noise Wiener-Hopf problem [3], and numerical solutions for differential equations. Moreover, it offers unique advantages and irreplaceable value in recently developed fields such as three-dimensional rotation representations [4], spatial pose computation [5], quantum mechanics, robot kinematics [6], and computer graphics [7]. Within the quaternion matrix domain, the commutative property of multiplication from traditional number field theory often no longer applies. New analytical methods must be developed to address non-commutativity and the unique algebraic structure of quaternions. Consequently, the existence of solutions and numerical methods for solving such equations presents greater challenges.

For various types of quadratic matrix equations, numerous experts and scholars have achieved substantial research outcomes in both theoretical and numerical solution approaches. In the real and complex domains, Higham *et al.* [8] employed generalized Schur decomposition to provide a comprehensive characterization and proposed a novel numerical method. Subsequently, Higham *et al.* [9] combined exact linear search with the Newton method for solving quadratic matrix equations, significantly enhancing the global convergence of the Newton method in both theoretical research and practical applications. Guo [10] transformed the quadratic matrix equations into a special class of asymmetric algebraic Riccati equations, discussing the existence and uniqueness of *M*-matrix solutions and proposing numerical methods for solving them. Yu *et al.* [11] introduced the *M*-matrix to establish a sufficient condition for solution existence and analyzed the convergence properties of both the Newton and Bernoulli methods. Lu *et al.* [12] transformed the quadratic matrix equation into a special asymmetric Riccati equation and solved it using a fixed-point iteration method. In the quaternion domain, Shao *et al.* [13] recently introduced comparison matrices for quaternion matrices to study asymmetric algebraic Riccati equations for quaternion matrices, providing conditions for the existence and uniqueness of extremal solutions.

This paper investigates the existence of solutions and numerical methods for solving quadratic matrix Equations (1) in the quaternion field. By transforming it into a special class of Riccati matrix equations, a sufficient condition for the existence of solutions to the transformed matrix equation is proposed. Under this sufficient condition, utilizing the relevant properties of comparison matrices, the existence and uniqueness of solutions to the quaternion quadratic matrix equation are proven. Finally, a numerical example verifies the conditions for solution existence and employs fixed-point iteration for efficient solution. When B is a diagonal matrix, the quadratic matrix equation can be transformed into a special form of Riccati equation. This allows us to utilize the theories of comparison matrices and *M*-matrices to establish strong existence and uniqueness conditions. However, for the more general case where B is a non-diagonal matrix, the treatment becomes considerably more complex, which will be an important direction for future research.

The remainder of this paper is organized as follows: In Section 2, we review relevant concepts of quaternions, the definition and properties of right eigenvalues for quaternion matrices, as well as some related results on *Z*-matrices, *M*-matrices, and comparison matrices. In Section 3, we transform Equation (1) into a Riccati comparison equation and provide a sufficient condition for the existence of a corresponding *M*-matrix solution. In Section 4, we establish a sufficient condition for the existence of a solution to the quaternion quadratic matrix Equation (1). Finally, numerical experiments are conducted to verify the existence of the solution, and a fixed-point iteration method is employed to solve the equation.

2. Preliminaries

This section begins by reviewing some fundamental concepts of quaternions. The set of quaternions is denoted as $\mathbb{H} = \{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}\}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Here, a_0 represents the real part, while a_1, a_2 and a_3 constitute the imaginary parts. The imaginary units of quaternions satisfy the following relations:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1.$$

For a quaternion $q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, its conjugate is defined as $q^* = a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} - a_3\boldsymbol{k}$, and its norm (magnitude) is given by

$$\|q\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

For more properties of quaternion matrix equations, one may refer to [14] [15].

Given a set of basis vectors $v_1, v_2, \cdots, v_n$, the space spanned by all their linear combinations is denoted as $\mathrm{Span}(v_1, v_2, \cdots, v_n)$.

For a quaternion matrix $A \in \mathbb{H}^{m\times n}$, $[A]_{ij}$ denotes the (i, j)th entry of A; $|A|$ denotes the absolute value of every element in A.

Definition 1 ([13]). Let $A \in \mathbb{H}^{n\times n}$, A vector $v \in \mathbb{H}^n \setminus \{0\}$ is called a right eigenvector of A corresponding to a right eigenvalue $\lambda \in \mathbb{H}$ if the equation

$$Av = v\lambda,$$

hold.

The set of all right eigenvalues of A is denoted by $\sigma_r(A)$. Note that $\sigma_r(A)$ is closed under quaternion similarity, if $Av = v\lambda$ hold, then for all $\alpha \in \mathbb{H} \setminus \{0\}$, we have

$$A(v\alpha) = (v\alpha)(\alpha^{-1}\lambda\alpha),$$

so $v\alpha$ is a right eigenvector of A corresponding to the right eigenvalue

$\alpha^{-1}\lambda\alpha$.

Definition 2 ([13]). If all off-diagonal elements of $A \in \mathbb{R}^{n\times n}$ are non-positive, then A is called a *Z*-matrix. Any *Z*-matrix A can be expressed as $A = sI - B$, where $B \in \mathbb{R}^{n\times n}$ is a non-negative matrix. When s is greater than or equal to the spectral radius of B, *i.e.*, $s \ge \rho(B)$, then A is called an *M*-matrix. Specifically, when $s > \rho(B)$, A is called a nonsingular *M*-matrix; when $s = \rho(B)$, A is called a singular *M*-matrix.

Lemma 1 ([11]). For a *Z*-matrix $A \in \mathbb{R}^{n\times n}$, it is an *M*-matrix if and only if there exists a nonzero vector $v \ge 0$ such that $Av \ge 0$.

Lemma 2 ([11]). For a *Z*-matrix $A \in \mathbb{R}^{n\times n}$, the following statements are equivalent:

1) A is a non-singular *M*-matrix;
2) A is non-singular and satisfies $A^{-1} > 0$;
3) $A\boldsymbol{v} > 0$ for vectors $\boldsymbol{v} > 0$;
4) All eigenvalues of A are positive real numbers.

Lemma 3 ([13]). If $A \in \mathbb{R}^{n\times n}$ is a nonsingular *M*-matrix, then for any *Z*-matrix $B \ge A$, B is also a nonsingular *M*-matrix and $B^{-1} \le A^{-1}$.

Lemma 4 ([13]). For a matrix $A \in \mathbb{H}^{n\times n}$, the comparison matrix is defined as

$$\hat{A}(i,j) = \begin{cases} \mathcal{R}[A]_{ij}, & i = j \\ -\left|[A]_{ij}\right|, & i \ne j \end{cases}$$

where $\mathcal{R}[A]_{ij}$ is the real part of $[A]_{ij}$.

Lemma 5 ([13]). If $A \in \mathbb{H}^{n\times n}$ and $\hat{A}$ is a nonsingular *M*-matrix, then A is nonsingular satisfying $\left|A^{-1}\right| \le \hat{A}^{-1}$, and the right eigenvalues of A have positive real parts.

To obtain the existence of solutions for Equation (1), we need the following lemma:

Lemma 6 ([16]). For the nonsymmetric algebraic Riccati equation

$$XCX - XD - AX + B = 0, \tag{2}$$

where A, B, C, D are real matrices of sizes $m\times m$, $m\times n$, $n\times m$, $n\times n$ respectively.

Define an $(m+n)\times(m+n)$ matrix

$$M = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}.$$

If M is a singular *M*-matrix, then Equation (2) has a unique minimal nonnegative solution S, and both $D - CS$ and $A - BS$ are nonsingular *M*-matrices. When M is an irreducible matrix, both $D - CS$ and $A - BS$ are irreducible.

Lemma 7 ([10]). For a class of quadratic matrix equations

$$X^2 - EX - F = 0, \tag{3}$$

where $E, F, X \in \mathbb{R}^{n\times n}$, E is a diagonal matrix, and F is an *M*-matrix.

If F is a nonsingular M-matrix, then there is exactly one M-matrix solution in Equation (3), and this M-matrix is nonsingular. If F is an irreducible singular M-matrix, then Equation (3) has an M-matrix solution, and all elements of each M-matrix solution are nonzero.

Lemma 8 ([16]). Consider Equation (2). If M is a nonsingular M-matrix or an irreducible singular M-matrix, then there exist nonnegative matrices S_1 and S_2 such that

$$\begin{bmatrix} D & -C \\ B & -A \end{bmatrix}\begin{bmatrix} I & S_2 \\ S_1 & I \end{bmatrix}=\begin{bmatrix} I & S_2 \\ S_1 & I \end{bmatrix}\begin{bmatrix} G_1 & 0 \\ 0 & -G_2 \end{bmatrix}, \tag{4}$$

where both G_1 and G_2 are M-matrices.

Lemma 9 ([16]). Consider Equation (2) and its dual equation

$$XBX - XA - DX + C = 0, \tag{5}$$

If M is a nonsingular M-matrix, then the nonnegative matrices S_1 and S_2 satisfying (4), with G_1 and G_2 being M-matrices, are the unique minimal nonnegative solutions of Equations (2) and (5), respectively. In this case, both G_1 and G_2 are nonsingular, and the matrix

$$\begin{bmatrix} I & S_2 \\ S_1 & I \end{bmatrix}$$

is also nonsingular.

3. Sufficient Conditions for the Existence of the *M*-Matrix Solution to the Comparison Equation Corresponding to Equation (1)

In this section, we consider rewriting Equation (1) to obtain the corresponding comparison equation. We prove that under certain conditions on the coefficient matrix of the comparison equation, its solution exists.

Assume that $X = \alpha I - Y$ ($\alpha > 0$), this assumption helps construct M-matrices, allowing us to utilize their excellent properties. This lays a crucial foundation for our subsequent use of the fixed-point iteration method and proving the existence of solutions. Therefore, Equation (1) can be rewritten as

$$Y^2 - Y(\alpha I) - (\alpha I + B)Y + (\alpha^2 I + \alpha B + C) = 0, \tag{6}$$

and its coefficient matrix is

$$K = \begin{bmatrix} \alpha I & -I \\ -(\alpha^2 I + \alpha B + C) & \alpha I + B \end{bmatrix}.$$

Consider the comparison Equation (6) as follows:

$$Y^2 - Y(\alpha I) - (\alpha I + \mathcal{R}(B))Y + \left|\alpha^2 I + \alpha B + C\right| = 0. \tag{7}$$

Therefore, $\hat{K}$ is the comparison matrix of K, and

$$\hat{K} = \begin{bmatrix} \alpha I & -I \\ -\left|\alpha^2 I + \alpha B + C\right| & \alpha I + \mathcal{R}(B) \end{bmatrix}.$$

Theorem 1. For the comparison matrix

$$\hat{K} = \begin{bmatrix} \alpha I & -I \\ -\left|\alpha^2 I + \alpha B + C\right| & \alpha I + \mathcal{R}(B) \end{bmatrix},$$

assume that

$$\mu_i = \max_i \left\{ \frac{-\mathcal{R}(b_i) + \sqrt{\mathcal{R}^2(b_i) - 4\mathcal{R}(c_{ii})}}{2}, \left|\frac{\mathcal{J}_i(c_{ii})}{\mathcal{J}_i(b_i)}\right|, \left|\frac{\mathcal{J}_j(c_{ii})}{\mathcal{J}_j(b_i)}\right|, \left|\frac{\mathcal{J}_k(c_{ii})}{\mathcal{J}_k(b_i)}\right|, 1 \right\},$$

where $i = 1, 2, \cdots, n$, and $\mathcal{J}_i$, $\mathcal{J}_j$, $\mathcal{J}_k$ denote its three imaginary parts respectively, when $\alpha > \max_i \left\{ \mu_i + \sum_{\substack{j=1 \\ j \neq i}}^n \left|c_{ij}\right| - \mathcal{R}(b_i), 1 \right\}$, $\hat{K}$ is an *M*-matrix.

Proof. Suppose the comparison matrix

$$\hat{K} = \begin{bmatrix} \alpha I & -I \\ -\left|\alpha^2 I + \alpha B + C\right| & \alpha I + \mathcal{R}(B) \end{bmatrix},$$

where $\alpha I = \operatorname{diag}(\alpha, \alpha, \cdots, \alpha)$. According to *Definition* 2, it is obvious that $\hat{K}$ is a *Z*-matrix.

From the definition of μ_i, we have $\mu_i \geq 1$. Therefore, when $\alpha > \max_i \left\{ \mu_i + \sum_{\substack{j=1 \\ j \neq i}}^n \left|c_{ij}\right| - \mathcal{R}(b_i), 1 \right\}$, $\alpha > 1$ always holds.

When $v = (1, 1, \cdots, 1)^{\mathrm{T}}$, we have

$$\hat{K}v = \begin{bmatrix} \alpha I & -I \\ -\left|\alpha^2 I + \alpha B + C\right| & \alpha I + \mathcal{R}(B) \end{bmatrix} v$$

$$= \begin{bmatrix} \alpha - 1 \\ \alpha - 1 \\ \vdots \\ \alpha - 1 \\ -\left|\alpha^2 + \alpha b_1 + c_{11}\right| - \left|c_{12}\right| - \cdots - \left|c_{1n}\right| + \alpha + \mathcal{R}(b_1) \\ -\left|\alpha^2 + \alpha b_2 + c_{22}\right| - \left|c_{21}\right| - \cdots - \left|c_{2n}\right| + \alpha + \mathcal{R}(b_2) \\ \vdots \\ -\left|\alpha^2 + \alpha b_n + c_{nn}\right| - \left|c_{n1}\right| - \cdots - \left|c_{n(n-1)}\right| + \alpha + \mathcal{R}(b_n) \end{bmatrix}.$$

Since $\alpha > 1$, and

$$-\left|\alpha^2 + \alpha b_1 + c_{11}\right| - \left|c_{12}\right| - \cdots - \left|c_{1n}\right| + \alpha + \mathcal{R}(b_1) > 0,$$

$$-\left|\alpha^2 + \alpha b_2 + c_{22}\right| - \left|c_{21}\right| - \cdots - \left|c_{2n}\right| + \alpha + \mathcal{R}(b_2) > 0,$$

$$\vdots$$

$$-\left|\alpha^2 + \alpha b_n + c_{nn}\right| - \left|c_{n1}\right| - \cdots - \left|c_{n(n-1)}\right| + \alpha + \mathcal{R}(b_n) > 0.$$

Therefore, $\hat{K}v > 0$. According to *Lemma* 1, $\hat{K}$ is a nonsingular *M*-matrix.

□

Corollary 1. According to Lemma 6, Equation (7) has a nonsingular *M*-matrix as its solution.

4. Existence of Solutions to Quaternion Quadratic Matrix Equations

This section primarily establishes the existence of solutions for the quaternion quadratic matrix Equation (1) by comparing it with the existence of solutions for Equation (6) in the previous section.

Consider Equation (6) and its dual equation

$$Z\left(\alpha^2 I+\alpha B+C\right)Z-Z\left(\alpha I+B\right)-\left(\alpha I\right)Z+I=0, \tag{8}$$

we have the following theorem.

Theorem 2. Under the assumption of Theorem 1,

$$\hat{K}=\begin{bmatrix} \alpha I & -I \\ -\left|\alpha^2 I+\alpha B+C\right| & \alpha I+\mathcal{R}\left(B\right) \end{bmatrix}$$

is an *M*-matrix. Let $\tilde{\Phi}$ be the minimal nonnegative solution of the comparison Equation (7), then the quaternion Equation (6) has a solution Φ satisfying $\left|\Phi\right|\le\tilde{\Phi}$. Moreover, the right eigenvalues of the nonsingular matrix $\alpha I-\Phi$ have positive real parts, and Φ is the unique numerical solution of Equation (6). Similarly, the dual matrix equation corresponding to Equation (7) is

$$Z\left|\alpha^2 I+\alpha B+C\right|Z-Z\left(\alpha I+\mathcal{R}\left(B\right)\right)-\left(\alpha I\right)Z+I=0, \tag{9}$$

let $\tilde{\Psi}$ be the minimal nonnegative solution of Equation (9), then Equation (8) has a solution Ψ satisfying $\left|\Psi\right|\le\tilde{\Psi}$. Moreover, the right eigenvalues of the nonsingular matrix $\left(\alpha I+B\right)-\left(\alpha^2 I+\alpha B+C\right)\Psi$ have positive real parts, and Ψ is the unique numerical solution of Equation (8).

Proof. Define the linear operators $\mathcal{L}:\mathbb{H}^{n\times n}\to\mathbb{H}^{n\times n}$ and $\tilde{\mathcal{L}}:\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}$ as follows:

$$\mathcal{L}\left(Y\right)=Y\left(\alpha I\right)+\left(\alpha I+B\right)Y,\ \tilde{\mathcal{L}}\left(Y\right)=Y\left(\alpha I\right)+\left(\alpha I+\mathcal{R}\left(B\right)\right)Y.$$

Since

$$\left[\mathcal{L}\left(Y\right)\right]_{ij}=\left[Y\right]_{ij}\alpha+\left(\alpha+b_{ij}\right)\left[Y\right]_{ij},\ \left[\tilde{\mathcal{L}}\left(Y\right)\right]_{ij}=\left(2\alpha+\mathcal{R}\left(b_{ij}\right)\right)\left[Y\right]_{ij}, \tag{10}$$

both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are invertible, where $\mathcal{R}\left(b_{ij}\right)\ge 0$.

Below we prove that $f\left(Y\right)=\mathcal{L}^{-1}\left(Y^2+\left(\alpha^2 I+\alpha B+C\right)\right)$ has a fixed point in the compact convex set $\mathcal{W}=\left\{Y:\left|Y\right|\le\tilde{\Phi}\right\}$. Assume $E=\mathcal{L}\left(Y\right)$, then we have:

$$\begin{aligned}\left|\left[E\right]_{ij}\right|&=\left|\left[Y\right]_{ij}\alpha+\left(\alpha+b_{ij}\right)\left[Y\right]_{ij}\right|\\&=\left|\left(2\alpha+\mathcal{R}\left(b_{ij}\right)\right)\left[Y\right]_{ij}+\left(\mathcal{J}\left(b_{ij}\right)\right)\left[Y\right]_{ij}\right|\\&\ge\left|2\alpha+\mathcal{R}\left(b_{ij}\right)\right|\left|\left[Y\right]_{ij}\right|\\&=\left(2\alpha+\mathcal{R}\left(b_{ij}\right)\right)\left|\left[\mathcal{L}^{-1}\left(E\right)\right]_{ij}\right|.\end{aligned}$$

The above inequality holds because

$$[Y]_{ij} \perp \mathrm{Span}\left\{\boldsymbol{i}[Y]_{ij}, \boldsymbol{j}[Y]_{ij}, \boldsymbol{k}[Y]_{ij}, [Y]_{ij}\boldsymbol{i}, [Y]_{ij}\boldsymbol{j}, [Y]_{ij}\boldsymbol{k}\right\},$$

Taking the inverse linear operator on both sides of $\left|[E]_{ij}\right| \geq \left(2\alpha + \mathcal{R}\left(b_{ij}\right)\right)\left|\left[\mathcal{L}^{-1}(E)\right]_{ij}\right|$ and using (10), we have

$$\left|\mathcal{L}^{-1}(E)\right| \leq \tilde{\mathcal{L}}^{-1}\left(|E|\right).$$

Therefore,

$$\begin{aligned}|Y| &= \left|f(Y)\right| \\ &= \left|\mathcal{L}^{-1}\left(Y^2 + \left(\alpha^2 I + \alpha B + C\right)\right)\right| \\ &\leq \tilde{\mathcal{L}}^{-1}\left(\left|Y^2 + \left(\alpha^2 I + \alpha B + C\right)\right|\right) \\ &\leq \tilde{\mathcal{L}}^{-1}\left(|Y|^2 + \left|\alpha^2 I + \alpha B + C\right|\right) \\ &\leq \tilde{\mathcal{L}}^{-1}\left(\tilde{\Phi}^2 + \left|\alpha^2 I + \alpha B + C\right|\right) \\ &= \tilde{\Phi}.\end{aligned}$$

Therefore

$$f(\mathcal{W}) \subset \mathcal{W}.$$

By Brouwer's fixed point theorem [17], f has a fixed point in $\mathcal{W}$, so Equation (6) has a solution satisfying $|\Phi| \leq \tilde{\Phi}$. Let

$$H = \begin{bmatrix} \alpha I & -I \\ \alpha^2 I + \alpha B + C & -(\alpha I + B) \end{bmatrix}, \ \tilde{H} = \begin{bmatrix} \alpha I & -I \\ \left|\alpha^2 I + \alpha B + C\right| & -\left(\alpha I + \mathcal{R}(B)\right) \end{bmatrix},$$

$$T = \begin{bmatrix} I & \Psi \\ \Phi & I \end{bmatrix}, \ \tilde{T} = \begin{bmatrix} I & \tilde{\Psi} \\ \tilde{\Phi} & I \end{bmatrix},$$

$$R = \alpha I - \Phi, \ \tilde{R} = \alpha I - \tilde{\Phi},$$

$$S = (\alpha I + B) - \left(\alpha^2 I + \alpha B + C\right)\Psi, \ \tilde{S} = \left(\alpha I + \mathcal{R}(B)\right) - \left|\alpha^2 I + \alpha B + C\right|\tilde{\Psi}.$$

From Equations (6) and (8), we obtain

$$HT = T\begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix}, \tag{11}$$

$$\tilde{H}\tilde{T} = \tilde{T}\begin{bmatrix} \tilde{R} & 0 \\ 0 & -\tilde{S} \end{bmatrix}. \tag{12}$$

Since $\hat{K}$ is a nonsingular *M*-matrix, according to *Lemma* 9, $\tilde{T}$ is nonsingular. Define the comparison matrix of $\tilde{T}$ as

$$\hat{\tilde{T}} = \begin{bmatrix} 1 & -\tilde{\Psi} \\ -\tilde{\Phi} & 1 \end{bmatrix},$$

which is clearly a nonsingular M-matrix. Therefore, according to *Lemma* 5, T is nonsingular.

According to *Lemma* 8, $\tilde{R}$ is a nonsingular *M*-matrix. Define the comparison matrix of R as $\hat{R}$ according to *Lemma* 4. Since

$$\mathcal{R}\left([R]_{ii}\right)=\alpha-\mathcal{R}\left([\Phi]_{ii}\right)\geq\alpha-\left|[\Phi]_{ii}\right|\geq\alpha-\left[\tilde{\Phi}\right]_{ii},$$

$$[R]_{ik}=\left|[\Phi]_{ik}\right|\leq\left[\tilde{\Phi}\right]_{ik},$$

therefore $\hat{R}\geq\tilde{R}$. According to *Lemma* 3, it is obvious that $\hat{R}$ is a nonsingular *M*-matrix. Then, according to *Lemma* 5, we can know that the right eigenvalues of R have positive real parts. Similarly, the right eigenvalues of S also have positive real parts.

Here, we only focus on the right eigenvalues of the complex matrix representation of quaternion matrices. Since (11) is a similarity transformation, and it is pointed out in Section 2.1 of [13] that the real parts of the right eigenvalues of quaternions remain unchanged under similarity transformation. Therefore, H has exactly $2n$ complex right eigenvalues in $\mathbb{C}_+$ and exactly $2n$ complex right eigenvalues in $\mathbb{C}_-$. Thus, the right vector space spanned by the columns of $\begin{bmatrix} I \\ \Phi \end{bmatrix}$ is exactly the invariant subspace corresponding to the right eigenvalues with positive real parts.

Finally, we prove the uniqueness of the solution. Assume that $\begin{bmatrix} I \\ Y \end{bmatrix}$ corresponds to the same invariant subspace. Therefore, there exists a nonsingular matrix V such that

$$\begin{bmatrix} I \\ \Phi \end{bmatrix}V=\begin{bmatrix} I \\ Y \end{bmatrix}.$$

From this, we obtain $V=I$ and $Y=\Phi$. The proof of uniqueness for Ψ is similar to the above.

□

Corollary 2. According to Theorem 1, we know that the solution to Equation (6) exists and is unique. Since Equation (6) is obtained from Equation (1) through the transformation $X=\alpha I-Y$, the solution to Equation (1) also exists and is unique.

5. Numerical Example

In this section, we provide a numerical example to illustrate that the numerical solution of the equation considered in this paper exists.

Example 1. For Equation (6), take

$$\begin{cases} B=\begin{bmatrix} 6+1.5i+0.8j+1.2k & 0 \\ 0 & 5+0.9i+1.5j+0.7k \end{bmatrix} \\ C=\begin{bmatrix} 0.4+0.08i+0.06j+0.05k & -0.15+0.03i+0.02j+0.04k \\ -0.12+0.02i+0.05j+0.03k & 0.5+0.07i+0.09j+0.06k \end{bmatrix} \end{cases}$$

We can obtain $\mu_i=1$, $\alpha>1$ satisfying *Theorem* 1. Here, assuming $\alpha=2$, consider the fixed-point iteration scheme

$$Y_{k+1}=\left(2\alpha I+B\right)^{-1}\left(Y_k^2+\alpha^2 I+\alpha B+C\right),$$

with $Y_0 = 0$. After 34 iterations, we obtain an approximate solution to Equation (6)

$$Y = \begin{bmatrix} 0.794 - 0.132i + 0.846j + 1.724k & -0.010 - 0.004i + 0.003j - 0.014k \\ -0.008 - 0.001i + 0.006j - 0.013k & 0.791 - 0.408i + 0.754j + 1.622k \end{bmatrix}.$$

The resulting relative error norm is

$$\frac{\| f(Y) - Y \|}{\| 2\alpha I + B \|} \approx 0.06884106.$$

Figure 1 shows the decreasing trend of relative error and absolute error with the number of iterations, clearly indicating that the fixed-point iteration method exhibits linear convergence.

Through calculation, the right eigenvalues of $\alpha I - Y$ can be obtained as follows

$$\lambda_1 = 1.2041 + 1.9256i,\ \lambda_2 = 1.2041 - 1.9256i,$$

$$\lambda_3 = 1.2113 + 1.8341i,\ \lambda_4 = 1.2113 - 1.8341i.$$

Therefore, Y is an approximate numerical solution. Since $X = \alpha I - Y$, then we can get

$$X = \begin{bmatrix} 1.206 + 0.132i - 0.846j - 1.724k & 0.010 + 0.004i - 0.003j + 0.014k \\ 0.008 + 0.001i - 0.006j + 0.013k & 1.209 + 0.408i - 0.754j - 1.622k \end{bmatrix},$$

and X is an approximate numerical solution of Equation (1).

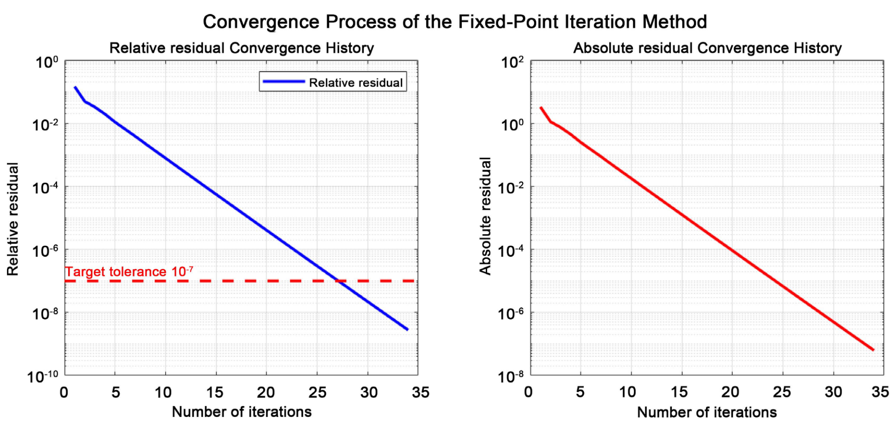

Figure 1. Relative residual and absolute residual vs iteration count.

6. Conclusions

This paper proposes an effective numerical method for the special quaternion quadratic matrix equation $X^2 + BX + C = 0$, where the coefficient matrix B is a diagonal matrix. By introducing the key transformation $X = \alpha I - Y$, we transform the original Equation (1) into a special form of Riccati equation, and successfully utilize the theories of comparison matrices and *M*-matrices to establish sufficient conditions for the existence of solutions.

Numerical examples demonstrate that the fixed-point iteration method constructed based on this theory can effectively converge to the numerical solution of the equation, verifying the correctness of the proposed theory and the feasibility of the method. The results of this research have potential application value in fields such as quantum mechanics and robotics mentioned in the introduction. For example, in quantum mechanics, when the system Hamiltonian possesses specific symmetries, its mathematical description can be simplified to the equation form addressed in this paper; our method provides a solid computational foundation for the quantitative analysis of such problems. In robotics learning and control theory, this solution offers new ideas and tools for handling a class of quadratic optimal control or filtering problems with diagonalized system matrices.

However, the method proposed in this paper also has certain limitations. First, the existence condition for solutions given in *Theorem* 1 is a sufficient condition rather than a necessary one. This means that even if this condition is not satisfied, the equation may still have solutions, but our theoretical framework cannot guarantee their uniqueness, and the convergence of the fixed-point iteration method becomes uncertain. Second, the core limitation of this paper lies in the assumption that matrix B must be diagonal. Although this assumption simplifies the analysis and enables the application of M-matrix theory, it also restricts the scope of application, when B is a non-diagonal matrix, constructing effective transformations and establishing corresponding theories will face fundamental challenges.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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