Journal of Applied Mathematics and Physics, 2025, 13(11), 3681-3694
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

NUVO Space II: Analysis and Variational Structure on NUVO Space

Rickey W. Austin

St Claire Scientific, Albuquerque, NM, USA
Email: rickeywaustin@stclairescientific.com

How to cite this paper: Austin, R.W. (2025) NUVO Space II: Analysis and Variational Structure on NUVO Space. *Journal of Applied Mathematics and Physics*, **13**, 3681-3694.
https://doi.org/10.4236/jamp.2025.1311205

Received: October 5, 2025
Accepted: October 31, 2025
Published: November 3, 2025

Abstract

We develop the analytic, geometric, and variational framework on NUVO space, the conformally flat manifold (M,g) with $g=\lambda^2\eta$ introduced in Part I. Weighted divergence and Stokes theorems, curvature identities, and the Laplace-Beltrami operator are derived in full detail. We construct the variational principles governing geodesic motion and scalar currents and prove the existence and regularity of solutions to representative nonlinear scalar field equations. Together with Part I, this paper provides the mathematical foundation required for subsequent applications to gravitation and field dynamics.

Keywords

NUVO Manifold, Conformal Laplacian, Bochner Identity, Weighted Sobolev Spaces, Scalar Field Dynamics, Conformal Variational Geometry

1. Introduction

Part I of this series established the geometric foundation of *NUVO space* as a conformally flat manifold (M,g) determined by a flat background metric η and a smooth, positive scalar field $\lambda: M\to(0,\infty)$. The induced metric $g=\lambda^2\eta$ defines a unit-constrained frame structure that fixes local scaling while preserving the global topology of M; compare standard conformal geometry texts [1]-[3]. Unlike general relativity or Brans-Dicke theory, the NUVO framework treats the conformal factor λ as a geometric field intrinsic to the background rather than as an external scalar coupled to curvature, thereby preserving flat topological structure while allowing curvature to emerge from scalar modulation.

Purpose of the present paper. The objective of Part II is to develop the complete analytical and variational machinery required for applications of NUVO geometry to dynamical and gravitational problems. We extend the purely geometric structure of Part I to include:

1) the weighted differential operators—gradient, divergence, and Laplace-Beltrami—and the associated Stokes and Gauss theorems in the λ-weighted measure;

2) explicit curvature formulas for the conformal metric $g=\lambda^2\eta$ together with energy identities and Bochner-type relations [1]-[3];

3) the variational and geodesic principles governing motion on NUVO space, including conservation currents defined purely by scalar geometry;

4) the existence, regularity, and stability of weak solutions to representative nonlinear scalar field equations [4]-[6].

These developments complete the mathematical backbone of NUVO space, allowing the scalar field λ to be treated as a geometric quantity obeying well-posed equations rather than an auxiliary rescaling factor.

Structure of the paper. Section 2 introduces the λ-weighted differential operators and establishes the divergence and Stokes theorems. Section 3 derives the curvature tensors and energy identities, culminating in the scalar curvature functional. Section 4 formulates the variational principle for geodesic motion and the conservation laws for the scalar-weighted (sinertia) current. Section 5 presents the analysis of nonlinear scalar field equations, proving existence and regularity of solutions under general structural conditions. Section 6 collects analytical examples and limiting cases, while Section 7 summarizes the results and outlines the transition to the physical applications pursued in later papers, including the gravitational field equations and PPN analysis.

Notation and conventions. Indices are raised and lowered using the background metric η unless otherwise stated. Differential operators ∇_η and Δ_η denote the flat background gradient and Laplacian, while ∇_g and Δ_g denote their counterparts associated with $g=\lambda^2\eta$. Volume and surface measures satisfy $\mathrm{d}V_g=\lambda^n\mathrm{d}V_\eta$ and $\mathrm{d}S_g=\lambda^{n-1}\mathrm{d}S_\eta$. All functions and fields are assumed sufficiently smooth for the stated operations to be well defined.

Relation to subsequent work. The formulas and identities established here will be used directly in the derivation of the NUVO gravitational field equation and its weak- and strong-field limits. They also supply the analytical tools for defining conserved quantities, variational energies, and perturbation theory in the scalar framework. For the geometric base of this paper, see Part I [7].

2. Weighted Differential Operators and Divergence Theorems

The conformal metric $g=\lambda^2\eta$ introduces a natural λ-weighted calculus on M. All differential operators associated with g can be written explicitly in terms of the background operators defined by η and the scalar field λ. Basic conformal operator relations appear in standard references [1]-[3]. This section establishes the gradient, divergence, and Laplace-Beltrami operators, together with their integral identities and Stokes-type theorems. The analysis is purely geometric and independent of any physical interpretation.

2.1. Weighted Gradient and Divergence

Let (M,η) be a flat n-dimensional manifold with coordinates $x=(x^1,\cdots,x^n)$ and background metric η_{ij}. For any smooth scalar field f and vector field $X = X^i\partial_i$, the gradient and divergence with respect to g are

$$\nabla^g f = g^{ij}\partial_j f\partial_i,\ \operatorname{div}_g X = \frac{1}{\sqrt{|\det g|}}\partial_i\left(\sqrt{|\det g|}X^i\right). \tag{1}$$

Since $\det g = \lambda^{2n}\det\eta$, one obtains the compact formula

$$\boxed{\operatorname{div}_g X = \lambda^{-n}\operatorname{div}_\eta\left(\lambda^n X\right).} \tag{2}$$

Equation (2) defines the λ-weighted divergence on NUVO space.

Remark 1. The expression (2) shows that all integral identities involving divergence on g can be expressed as weighted identities on the flat background η. This observation underlies the λ-weighted versions of the divergence and Stokes theorems proved below.

2.2. Integral Identities

Let $\mathrm{d}V_g = \lambda^n \mathrm{d}V_\eta$ denote the volume measure of g. For any compact domain $\Omega\subset M$ with smooth boundary $\partial\Omega$ and outward g-unit normal n, integration of (2) gives

$$\begin{aligned}\int_\Omega \operatorname{div}_g X\mathrm{d}V_g &= \int_\Omega \lambda^{-n}\operatorname{div}_\eta\left(\lambda^n X\right)\lambda^n\mathrm{d}V_\eta\\ &= \int_\Omega \operatorname{div}_\eta\left(\lambda^n X\right)\mathrm{d}V_\eta\\ &= \int_{\partial\Omega}\eta\left(X,n_\eta\right)\lambda^n\mathrm{d}S_\eta.\end{aligned} \tag{3}$$

Because $n=\lambda^{-1}n_\eta$ and $\mathrm{d}S_g = \lambda^{n-1}\mathrm{d}S_\eta$, the surface term becomes

$$\eta\left(X,n_\eta\right)\lambda^n\mathrm{d}S_\eta = \lambda\eta\left(X,n_\eta\right)\mathrm{d}S_g = g\left(X,n\right)\mathrm{d}S_g.$$

Hence the fundamental identity:

Theorem 2 (Divergence theorem on NUVO space) For every smooth vector field X and domain $\Omega\subset M$ with smooth boundary,

$$\boxed{\int_\Omega \operatorname{div}_g X\mathrm{d}V_g = \int_{\partial\Omega} g\left(X,n\right)\mathrm{d}S_g.} \tag{4}$$

Proof. The result follows directly from (2), the change of measure $\mathrm{d}V_g = \lambda^n\mathrm{d}V_\eta$, and the relation $n_\eta = \lambda n$ between the unit normals. □

Corollary 1 (Gauss identity). For any scalar field f and vector field X,

$$\int_\Omega f\operatorname{div}_g X\mathrm{d}V_g = -\int_\Omega g\left(\nabla^g f, X\right)\mathrm{d}V_g + \int_{\partial\Omega} fg\left(X,n\right)\mathrm{d}S_g.$$

Remark 3. Setting $\lambda\equiv 1$ reduces (4) to the classical Stokes theorem on the flat background (M,η), confirming internal consistency of the formalism.

2.3. Weighted Laplace-Beltrami Operator

The Laplace-Beltrami operator Δ_g acting on a scalar function f is defined by

$\Delta_g f = \mathrm{div}_g\left(\nabla^g f\right)$. Using $g^{ij} = \lambda^{-2}\eta^{ij}$ and formula (2), we compute

$$\begin{aligned}\Delta_g f &= \lambda^{-n}\mathrm{div}_\eta\left(\lambda^n g^{ij}\partial_j f\partial_i\right)\\ &= \lambda^{-n}\partial_i\left(\lambda^n\lambda^{-2}\eta^{ij}\partial_j f\right)\\ &= \lambda^{-2}\left(\Delta_\eta f + (n-2)\nabla_\eta\varphi\cdot\nabla_\eta f\right),\ \varphi = \log\lambda.\end{aligned} \tag{5}$$

Proposition 4 (Explicit Laplace-Beltrami operator). On NUVO space $\left(M, g = \lambda^2\eta\right)$ the scalar Laplace-Beltrami operator is

$$\boxed{\Delta_g f = \lambda^{-2}\left(\Delta_\eta f + (n-2)\nabla_\eta\varphi\cdot\nabla_\eta f\right),\ \varphi = \log\lambda.} \tag{6}$$

Remark 5. The operator Δ_g is self-adjoint in $L^2\left(M, \lambda^n \mathrm{d}V_\eta\right)$ and satisfies the usual maximum and mean-value principles; see, e.g., [4] [5].

Remark 6. Equation (6) immediately implies self-adjointness of Δ_g in the weighted Hilbert space $L^2\left(M, \lambda^n \mathrm{d}V_\eta\right)$:

$$\int_M f\Delta_g h\mathrm{d}V_g = \int_M h\Delta_g f\mathrm{d}V_g,\quad f, h \in C_c^\infty(M).$$

2.4. Weighted Sobolev Spaces

To analyze integral and variational properties, we introduce the appropriate functional framework.

Definition 7 (Weighted Sobolev space). Let λ be a positive function bounded above and below on M. Define

$$W_\lambda^{1,2}(M) = \left\{u \in L^2\left(M, \lambda^n\mathrm{d}V_\eta\right) : \nabla_\eta u \in L^2\left(M, \lambda^n\mathrm{d}V_\eta\right)\right\}.$$

The norm is

$$\|u\|^2_{W_\lambda^{1,2}} = \int_M\left(\left|\nabla_\eta u\right|^2 + |u|^2\right)\lambda^n\mathrm{d}V_\eta.$$

Lemma 8 (Poincaràinequality). If λ satisfies $0 < \lambda_{\min} \le \lambda(x) \le \lambda_{\max} < \infty$, then there exists $C > 0$ such that

$$\int_\Omega|u - \bar{u}|^2\lambda^n\mathrm{d}V_\eta \le C\int_\Omega\left|\nabla_\eta u\right|^2\lambda^n\mathrm{d}V_\eta,$$

for all $u \in W_\lambda^{1,2}(\Omega)$, where $\bar{u}$ is the λ-weighted mean of u on Ω.

Compact embeddings and Poincaré inequalities in the weighted setting follow from standard arguments in elliptic theory [4] [5].

Remark 9. The space $W_\lambda^{1,2}$ forms the natural variational domain for elliptic equations involving Δ_g, as will be used in Section 5.

3. Curvature and Energy Identities

We next compute the curvature tensors associated with $g = \lambda^2\eta$ and derive several integral and variational identities that will later underpin both field equations and conservation principles on NUVO space.

3.1. Ricci and Scalar Curvature of a Conformal Metric

Let ∇_η denote the Levi-Civita connection of the flat background η, and set

$\varphi = \log\lambda$. The connection coefficients of $g = \lambda^2\eta$ were obtained in Part I as

$$\Gamma^k{}_{ij} = \delta^k{}_i \partial_j \varphi + \delta^k{}_j \partial_i \varphi - \eta_{ij} \eta^{k\ell} \partial_\ell \varphi.$$

Theorem 10 (Curvature of a conformal metric). For $g = \lambda^2\eta$ with $\varphi = \log\lambda$, the Ricci and scalar curvatures are

$$\mathrm{Ric}_g = -(n-2)\left(\nabla_\eta^2 \varphi - \nabla_\eta \varphi \otimes \nabla_\eta \varphi\right) - \left(\Delta_\eta \varphi + (n-2)\left|\nabla_\eta \varphi\right|^2\right)\eta, \tag{7}$$

$$R_g = -2(n-1)\lambda^{-1}\Delta_\eta \lambda - (n-1)(n-2)\lambda^{-2}\left|\nabla_\eta \lambda\right|^2. \tag{8}$$

Proof. The result follows from classical conformal transformation formulas for curvature (see, e.g., Chavel and Lee [1] [2]; also Jost [3]). Starting from the connection difference tensor $C^k{}_{ij} = \Gamma^k{}_{ij} - (\Gamma_\eta)^k{}_{ij}$, a direct computation of $R^k{}_{\ell ij} = \partial_i C^k{}_{\ell j} - \partial_j C^k{}_{\ell i} + C^k{}_{mi} C^m{}_{\ell j} - C^k{}_{mj} C^m{}_{\ell i}$ and its traces yields (7) and (8). ☐

Corollary 2 (Flatness condition). If λ is constant, then $\mathrm{Ric}_g = 0$ and $R_g = 0$. Hence constant λ corresponds to a globally flat geometry identical to (M,η) up to overall scale.

Remark 11. Curvature is governed entirely by first and second derivatives of λ. Gradients $\nabla_\eta \lambda$ produce anisotropic corrections, while $\Delta_\eta \lambda$ encodes isotropic dilation or compression of the conformal volume element.

3.2. Bochner and Energy Identities

The curvature expressions above give rise to standard energy identities for scalar fields on NUVO space.

Proposition 12 (Bochner identity on NUVO space). For every $f \in C^\infty(M)$,

$$\frac{1}{2}\Delta_g \left|\nabla^g f\right|^2 = \left|\nabla_g^2 f\right|^2 + \mathrm{Ric}_g\left(\nabla^g f, \nabla^g f\right) + \nabla^g f \cdot \nabla^g\left(\Delta_g f\right). \tag{9}$$

Proof. Identity (9) follows from standard Weitzenböck formulas and remains valid for any Levi-Civita connection. ☐

Integrating (9) over a compact domain and applying the divergence theorem of Section 2 gives

$$\int_\Omega \left|\nabla_g^2 f\right|^2 \mathrm{d}V_g + \int_\Omega \mathrm{Ric}_g\left(\nabla^g f, \nabla^g f\right)\mathrm{d}V_g = \frac{1}{2}\int_{\partial\Omega} \partial_n\left(\left|\nabla^g f\right|^2\right)\mathrm{d}S_g.$$

Such relations will be central to later energy estimates and stability analyses.

Bochner and Weitzenböck identities in this conformal setting are standard [1] [3].

3.3. Scalar Curvature Energy Functional

The scalar curvature R_g admits a natural global integral interpretable as a conformal energy of the field λ.

Definition 13 (Scalar curvature energy functional). Define

$$\mathcal{E}[\lambda] = \int_M R_g \mathrm{d}V_g = \int_M \left[-2(n-1)\lambda^{n-1}\Delta_\eta \lambda - (n-1)(n-2)\lambda^{n-2}\left|\nabla_\eta \lambda\right|^2\right]\mathrm{d}V_\eta. \tag{10}$$

Proposition 14 (First variation). The first variation of $\mathcal{E}[\lambda]$ under $\lambda \mapsto \lambda + \epsilon h$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{E}[\lambda+\epsilon h]\bigg|_{\epsilon=0} = -2(n-1)\int_M \lambda^{n-1}\left(\Delta_\eta h + (n-2)\lambda^{-1}\nabla_\eta\lambda\cdot\nabla_\eta h\right)\mathrm{d}V_\eta.$$

Stationary points of $\mathcal{E}$ therefore satisfy

$$\Delta_\eta\lambda + (n-2)\lambda^{-1}\left|\nabla_\eta\lambda\right|^2 = 0, \tag{11}$$

which is precisely the harmonic condition for the conformal factor in dimension $n > 2$.

Remark 15. Equation (11) defines the flat-space harmonic gauge for NUVO geometry. In subsequent sections this variational structure will extend to geodesic and scalar-field equations governing dynamics on (M, g).

4. Variational Geodesics and Conservation Currents

The conformal structure $(M, g = \lambda^2\eta)$ admits a natural variational principle that generates geodesic motion and corresponding conservation laws. This section establishes the variational derivation of the geodesic equation, identifies the associated conserved current, and outlines stability properties of nearby trajectories.

4.1. Variational Principle and Geodesic Equation

Let $\gamma:[a,b]\to M$ be a smooth curve with velocity $\dot\gamma = \mathrm{d}\gamma/\mathrm{d}t$. The action functional

$$S[\gamma] = \int_a^b \lambda(\gamma(t))\left\|\dot\gamma(t)\right\|_\eta \mathrm{d}t \tag{12}$$

defines the *scalar-weighted arc length* on NUVO space. Its stationary curves coincide with the geodesics of g.

The Euler-Lagrange derivation uses standard variational calculus; see, for instance, Evans [5].

Theorem 16 (Geodesic equation on NUVO space). A smooth curve γ is stationary for $S[\gamma]$ if and only if it satisfies

$$\ddot x^k + \Gamma^k{}_{ij}\dot x^i\dot x^j = 0,\ \Gamma^k{}_{ij} = \delta^k{}_i\partial_j\varphi + \delta^k{}_j\partial_i\varphi - \eta_{ij}\eta^{k\ell}\partial_\ell\varphi, \tag{13}$$

where $\varphi = \log\lambda$ and derivatives are taken with respect to the background coordinates of η.

Proof. Let $L(x,\dot x) = \lambda(x)\left\|\dot x\right\|_\eta$. The Euler-Lagrange equations $\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_{\dot x^k}L\right) - \partial_{x^k}L = 0$ give

$$\lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta_{kj}\dot x^j}{\left\|\dot x\right\|_\eta}\right) + \partial_k\lambda\left\|\dot x\right\|_\eta - \frac{\partial_k\eta_{ij}\dot x^i\dot x^j}{2\left\|\dot x\right\|_\eta}\lambda = 0.$$

Because η is flat, $\partial_k\eta_{ij} = 0$. Expanding the total derivative and simplifying yields the Christoffel expression (13). □

Corollary 3 (Affine parametrization). Reparametrizing γ by the g-arc length $s=\int_a^t \lambda(\gamma(\tau))\|\dot\gamma(\tau)\|_\eta \mathrm{d}\tau$ renders $\|\dot\gamma\|_g$ constant, giving an affine parameterization for which the geodesic equation retains the form (13).

Remark 17. The scalar factor λ rescales local arc length, so that motion in regions of larger λ appears contracted when measured by η. Geodesics thus represent extremal scalar-weighted lengths rather than extremal coordinate distances.

4.2. Existence, Uniqueness, and Energy Conservation

Standard ODE theory provides local well-posedness for (13).

Theorem 18 (Existence and uniqueness). If $\lambda \in C^{1,1}(M)$ and η is smooth, then for any initial position and velocity (x_0, v_0) there exists a unique local geodesic $\gamma(t)$ satisfying (13). The solution depends continuously on initial data.

Proof. The right-hand side of (13) is locally Lipschitz in $(x,\dot x)$ for $\lambda \in C^{1,1}$, hence the Picard-Lindelöf theorem applies. □

Proposition 19 (Energy integral). Along any geodesic of g the quantity

$$E=g(\dot\gamma,\dot\gamma)=\lambda^2\eta_{ij}\dot x^i\dot x^j$$

is constant.

Proof. Taking the covariant derivative of E along γ and using $\nabla_g g=0$ yields $\mathrm{d}E/\mathrm{d}t=0$. □

Remark 20. The constancy of E expresses the reparametrization invariance of the variational principle. Null, timelike, and spacelike geodesics in (M,g) correspond to η-trajectories scaled by λ. Local well-posedness follows from ODE theory with Lipschitz right-hand sides (textbook methods; cf. [5]).

4.3. Sinertia Current and Continuity Law

The scalar weighting that defines $S[\gamma]$ also determines a conserved current for any scalar density ρ.

Definition 21 (Sinertia current). Let ρ be a scalar field and u^μ a g-normalized vector field, $g_{\mu\nu}u^\mu u^\nu=-1$ (or +1 in Euclidean signature). Define

$$J^\mu=\lambda\rho u^\mu. \tag{14}$$

The term sinertia (from "scalar inertia"? denotes the effective inertia carried by the scalar field itself, representing a conserved flow of scalar-weighted momentum through the geometry.

Proposition 22 (Continuity equation). The current J^μ is divergence-free in g,

$$\nabla_\mu^g J^\mu=0 \Leftrightarrow \mathrm{div}_g(\lambda\rho u)=0.$$

Proof. Applying ∇_μ^g and using $\mathrm{div}_g(\lambda\rho u)=\lambda^{-n}\mathrm{div}_\eta(\lambda^{n+1}\rho u)$ from formula (2) shows that the λ-weighted measure renders the flux through $\partial\Omega$ zero for compact domains, establishing conservation. □

Remark 23. The continuity law expresses the conservation of scalar-weighted density along flow lines of u. In the dynamical interpretation of later papers, J^μ

will represent the conserved sinertia flux associated with geodesic motion or scalar field evolution.

4.4. Jacobi Fields and Stability of Nearby Trajectories

Let $\gamma_s(t)$ be a smooth one-parameter family of geodesics and $J(t)=\partial_s\gamma_s(t)\big|_{s=0}$ the corresponding *Jacobi field*. Differentiating (13) with respect to s gives

$$\nabla_t^2 J + R(J,\dot{\gamma})\dot{\gamma} = 0, \tag{15}$$

where R is the Riemann curvature tensor of g.

Theorem 24 (Stability estimate). If λ and its first two derivatives are bounded on a compact region $K \subset M$, then any Jacobi field J along a geodesic $\gamma \subset K$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|J\|_g^2 \le C\|J\|_g^2,$$

for some constant C depending only on $\left\|\nabla_\eta^2\lambda\right\|_\infty$. Hence perturbations of nearby geodesics remain bounded in K.

Proof. The estimate follows from the energy identity obtained by contracting (15) with J and substituting curvature bounds derived from (7). □

Remark 25. Bounded curvature of λ ensures exponential stability of geodesic congruences within finite domains, providing the geometric foundation for later analyses of focusing and defocusing phenomena.

5. Nonlinear Scalar Field Equations on NUVO Space

The scalar field λ that defines the conformal metric $g=\lambda^2\eta$ can itself satisfy nonlinear partial differential equations whose structure is compatible with the λ-weighted geometry. We now formulate and analyze a general class of such equations, prove existence and regularity of weak solutions, and discuss symmetry, uniqueness, and stability.

5.1. Model Equations and Variational Structure

We consider scalar field equations whose structure is compatible with the λ-weighted geometry, framed variationally via standard elliptic PDE methods [4] [5] and monotone operator theory [6]. Such nonlinear forms arise naturally in conformally invariant scalar-tensor and nonlinear-σ models, where the potential $U(\lambda)$ encodes self-interaction or curvature back-reaction. The present choice represents the minimal structure preserving ellipticity and geometric self-consistency, consistent with recent analyses of conformal scalar-tensor analogues and emergent-gravity formulations [8] [9].

$$-\Delta_\eta\lambda = F(\lambda,\nabla_\eta\lambda) \text{ or equivalently } -\Delta_g\lambda = \mathcal{G}(\lambda,\nabla^g\lambda), \tag{16}$$

where F and $\mathcal{G}$ satisfy structural conditions ensuring ellipticity and monotonicity. When F derives from a potential $U(\lambda)$ through $F(\lambda)=U'(\lambda)$, these equations admit a variational formulation.

Definition 26 (Energy functional). For a smooth potential $U:\mathbb{R}^+\to\mathbb{R}$, define

$$\mathcal{I}[\lambda]=\int_\Omega\left(\frac{1}{2}\left|\nabla_\eta\lambda\right|^2-U(\lambda)\right)\lambda^n\mathrm{d}V_\eta, \tag{17}$$

where $\Omega\subset M$ is a bounded domain. Critical points of $\mathcal{I}$ satisfy

$$-\Delta_\eta\lambda-n\lambda^{-1}\left|\nabla_\eta\lambda\right|^2=U'(\lambda), \tag{18}$$

interpreted weakly in $W_\lambda^{1,2}(\Omega)$.

Remark 27. The λ^n factor in (17) arises from the volume element $\mathrm{d}V_g=\lambda^n\mathrm{d}V_\eta$ and ensures that the Euler-Lagrange equations correspond to the Laplace-Beltrami operator Δ_g acting on λ. The Euler-Lagrange correspondence and weak formulation follow the classical framework [4] [5].

5.2. Existence of Weak Solutions

We now prove existence of minimizers for $\mathcal{I}[\lambda]$ under standard coercivity and monotonicity hypotheses.

Theorem 28 (Existence of minimizers). Assume $U\in C^1(\mathbb{R}^+)$ satisfies:

1) coercivity: $\lim_{\lambda\to\infty}\frac{U(\lambda)}{\lambda^2}=+\infty$;

2) lower boundedness: $U(\lambda)\ge -C_0$ for some $C_0>0$;

3) monotonicity: $\lambda U'(\lambda)\ge 0$ for all $\lambda>0$.

Then $\mathcal{I}$ attains a minimizer $\lambda\in W_\lambda^{1,2}(\Omega)$ with $\lambda>0$ that satisfies the weak equation (18). Positivity and regularity are obtained by maximum principle and elliptic estimates [4] [5].

Proof. Conditions (i)-(ii) guarantee coercivity and weak lower semicontinuity of $\mathcal{I}$ on $W_\lambda^{1,2}(\Omega)$. The direct method of the calculus of variations therefore yields a minimizer. Positivity follows by the maximum principle applied to the weak formulation. □

Corollary 4 (Regularity). If $U\in C^k(\mathbb{R}^+)$ and $\partial\Omega$ is $C^{1,\alpha}$, then any weak solution λ of (18) belongs to $C^{k+1,\alpha}(\bar{\Omega})$.

Proof. Apply standard elliptic regularity for uniformly elliptic operators with smooth coefficients (cf. Gilbarg-Trudinger). (cf. classical elliptic regularity [4] [5].) □

5.3. Symmetry and Decay of Ground States

On the full space $\mathbb{R}^n$, finite-energy solutions exhibit strong symmetry properties.

Theorem 29 (Radial symmetry and monotonicity). Let $\lambda>0$ be a finite-energy solution of $-\Delta_\eta\lambda=U'(\lambda)$ on $\mathbb{R}^n$ with $U'(\lambda)\lambda\ge 0$ and $U(0)=0$. Then λ is radially symmetric and strictly decreasing in $r=|x|$. The proof follows the moving planes method of Gidas-Ni-Nirenberg [10].

Proof. The proof follows the method of moving planes of Gidas, Ni, and Nirenberg: one reflects the solution about a plane and uses the maximum principle to enforce equality, obtaining spherical symmetry. □

Corollary 5 (Asymptotic decay). Under the assumptions of Theorem 29, $\lambda(r)$ satisfies $\lambda(r)\to\lambda_\infty$ as $r\to\infty$ with exponential or power-law decay depending on the asymptotic form of $U'(\lambda)$.

Remark 30. Radial symmetry ensures that curvature and energy densities derived from λ remain isotropic, which will simplify subsequent applications to spherically symmetric gravitational solutions.

5.4. Uniqueness and Linearized Stability

To study stability and uniqueness of weak solutions we examine the linearized equation obtained by setting $\lambda=\lambda_0+\epsilon h$ in (16).

Theorem 31 (Uniqueness). Suppose $F(\lambda)$ in (16) satisfies a Lipschitz-monotone condition

$$\left(F(\lambda_1)-F(\lambda_2)\right)(\lambda_1-\lambda_2)>0 \text{ for all } \lambda_1\neq\lambda_2.$$

Then the weak solution of (16) in $W_\lambda^{1,2}(\Omega)$ is unique. This is a standard application of monotonicity methods [6].

Proof. Subtract the equations for two solutions, multiply by $(\lambda_1-\lambda_2)$, and integrate. The monotonicity condition forces the difference to vanish. □

Theorem 32 (Linearized stability). Let λ_0 be a smooth stationary solution of (16) and h a small perturbation satisfying

$$-\Delta_g h-\partial_\lambda\mathcal{G}\left(\lambda_0,\nabla^g\lambda_0\right)h=0.$$

If $\partial_\lambda\mathcal{G}\left(\lambda_0,\nabla^g\lambda_0\right)>0$ in Ω, then the quadratic form

$$Q[h]=\int_\Omega\left(\left|\nabla_g h\right|^2+\partial_\lambda\mathcal{G}\left(\lambda_0,\nabla^g\lambda_0\right)h^2\right)\mathrm{d}V_g$$

is positive definite and the equilibrium λ_0 is linearly stable.

Proof. Multiply the linearized equation by h and integrate by parts using the divergence theorem of Section 2. Positivity of $\partial_\lambda\mathcal{G}$ implies $Q[h]>0$. □

Remark 33. The positivity of the quadratic form $Q[h]$ ensures that small perturbations of λ produce bounded oscillations in the weighted energy norm, establishing stability of scalar configurations in the absence of external forcing.

6. Analytical Consequences and Examples

The preceding sections provide the complete analytic framework for scalar geometry on NUVO space. We now illustrate several limiting and representative cases that demonstrate how the λ-weighted operators, curvature, and variational structures behave in practice.

6.1. Harmonic and Constant Limits

When λ is constant or harmonic with respect to η, the conformal geometry of (M,g) reduces to the flat background.

Proposition 34 (Harmonic limit). If λ satisfies $\Delta_\eta\lambda=0$, then $\mathrm{Ric}_g=0$ and $R_g=0$. Consequently, (M,g) is locally flat and all geodesics coincide with straight lines in η. Substituting $\Delta_\eta \lambda = 0$ into the conformal curvature formulas (7)-(8) [1]-[3] eliminates all curvature terms.

Proof. Substituting $\Delta_\eta \lambda = 0$ into (7) and (8) eliminates all curvature terms. □

Corollary 6 (Constant field). For $\lambda \equiv \lambda_0 > 0$, one has $g = \lambda_0^2 \eta$, $\nabla_g = \nabla_\eta$, and $\Delta_g = \lambda_0^{-2} \Delta_\eta$. The entire scalar calculus reduces to uniform rescaling of η.

Remark 35. This limit verifies that the NUVO calculus is a genuine generalization of flat geometry: the background metric η is recovered when the scalar field ceases to vary.

6.2. Radial Power-Law Fields

Nontrivial curvature arises for spatially varying λ. A simple and analytically tractable case is the radial power-law profile

$$\lambda(r) = 1 + \varepsilon r^{-p}, \quad \varepsilon, p > 0, \tag{19}$$

in *n*-dimensional Euclidean background $\eta_{ij} = \delta_{ij}$. These computations are consistent with the general conformal-curvature identities [1] [2].

Lemma 36 (Gradient and Laplacian). For λ of the form (19),

$$\nabla_\eta \lambda = -p\varepsilon r^{-p-2} x, \quad \Delta_\eta \lambda = p(p-n+2)\varepsilon r^{-p-2}.$$

Proposition 37 (Asymptotic curvature). For λ given by (19), the scalar curvature to first order in ε is

$$R_g = -2(n-1)\, p(p-n+2)\varepsilon r^{-p-2} + \mathcal{O}(\varepsilon^2).$$

Hence curvature decays as $r^{-(p+2)}$ and the geometry is asymptotically flat for $p > 0$.

Proof. Substitute the expressions for $\nabla_\eta \lambda$ and $\Delta_\eta \lambda$ into (8) and retain terms linear in ε. □

Remark 38. Choosing $p = n-2$ yields $\Delta_\eta \lambda = 0$, so the metric becomes conformally harmonic and curvature vanishes. For other exponents, curvature behaves as an inverse power of distance, resembling long-range fields in classical potentials.

6.3. Curvature Consistency Check

The energy and curvature formulas derived in Theorem 10 can be cross-verified by explicit computation for a Gaussian-type scalar field. Let

$$\lambda(x) = 1 + \alpha e^{-\beta r^2}, \quad \alpha, \beta > 0.$$

Then

$$\nabla_\eta \lambda = -2\alpha\beta x e^{-\beta r^2}, \quad \Delta_\eta \lambda = 2\alpha\beta e^{-\beta r^2}\left(2\beta r^2 - n\right).$$

Substituting into (8) gives

$$R_g(r) = -4(n-1)\alpha\beta e^{-\beta r^2}\left[\frac{(n-2)\alpha\beta r^2 e^{-\beta r^2} + \left(2\beta r^2 - n\right)\left(1+\alpha e^{-\beta r^2}\right)}{\left(1+\alpha e^{-\beta r^2}\right)^3}\right]. \tag{20}$$

For small α and large r, $R_g \sim 8(n-1)\alpha\beta^2 r^2 \mathrm{e}^{-\beta r^2}$, confirming smooth decay of curvature and finite total scalar energy.

Remark 39. Equation (20) explicitly verifies the analytic consistency of the scalar curvature formula (8) and demonstrates that $\mathcal{E}[\lambda]$ from (10) is convergent for rapidly decaying scalar profiles.

6.4. Summary of Analytic Behavior

1) The conformal geometry $(M, g = \lambda^2\eta)$ is flat if and only if λ is harmonic with respect to η.

2) For power-law $\lambda(r) = 1 + \varepsilon r^{-p}$, curvature decays as $r^{-(p+2)}$, ensuring asymptotic flatness for $p > 0$.

3) Rapidly decaying fields such as Gaussian profiles yield finite total curvature energy $\mathcal{E}[\lambda]$.

Remark 40. These results demonstrate that the analytic and variational frameworks derived for NUVO space reproduce familiar geometric limits of classical conformal metrics while remaining fully consistent with the weighted calculus developed in previous sections.

7. Discussion and Conclusions

The results developed in this paper complete the analytic and variational construction of NUVO space. Together with the geometric framework established in Part I, they define a self-consistent conformal calculus that is both mathematically rigorous and structurally compact. The scalar field λ now possesses a precise analytic meaning: it is a positive function that determines not only the conformal metric $g = \lambda^2\eta$ but also the weighted differential operators, curvature tensors, and variational energies acting on M.

Summary of principal results.

1) The λ-weighted gradient, divergence, and Laplace-Beltrami operators were derived in closed form, and the corresponding divergence and Stokes theorems were proven for the measure $\mathrm{d}V_g = \lambda^n \mathrm{d}V_\eta$.

2) The curvature tensors of $g = \lambda^2\eta$ were computed explicitly, leading to the Ricci and scalar curvature formulas (7)-(8) [1]-[3]. The scalar-curvature energy functional $\mathcal{E}[\lambda]$ and its first variation were obtained, yielding the harmonic condition (11) for stationary points.

3) The variational principle (12) generated the geodesic Equation (13), whose integral of motion $E = g(\dot{\gamma}, \dot{\gamma})$ is conserved. The associated sinertia current $J^\mu = \lambda\rho u^\mu$ obeys the continuity law $\nabla^g_\mu J^\mu = 0$.

4) Existence, regularity, and symmetry of solutions to the nonlinear scalar equations (16) were established under general monotonicity and coercivity conditions [4]-[6] [10], ensuring that the scalar field λ defines a well-posed elliptic problem.

5) Analytical examples demonstrated that harmonic λ yields exact flatness, while power-law and Gaussian profiles produce asymptotically flat curvature con-

sistent with theoretical predictions.

Conceptual implications. The mathematical structure presented here shows that a single scalar degree of freedom λ suffices to encode both local curvature and global scaling on a flat background. Weighted differential operators and curvature expressions derived from λ are internally consistent, conserve total scalar flux, and reduce to standard Euclidean or Minkowskian forms in the harmonic limit. The theory thus supplies a conformally exact but globally flat alternative to conventional curved-space formalisms.

The coercivity and monotonicity assumptions on $U(\lambda)$ ensure bounded curvature and prevent collapse of the conformal volume element, thereby excluding geometric singularities. In asymptotically constant regimes they guarantee global flatness, providing natural boundary conditions for physical space-times.

Outlook. The analytical foundations developed in Parts I [7] and II provide the necessary tools for constructing the *NUVO gravitational field equation*, in which the curvature and variational principles derived here determine the effective dynamics of matter and light. The next paper in this sequence, "*NUVO Gravity Equations and Parameterized Post-Newtonian Analysis*," will apply these operators to weak- and strong-field regimes, test classical limits against observational data, and explore the transition between scalar-modulated geometry and standard general relativity.

Concluding remark. From the geometric definition of $g = \lambda^2 \eta$ to the variational, differential, and energetic structures detailed here, NUVO space is now fully defined as a mathematical object. All subsequent physical models can be developed directly on this foundation, ensuring analytic coherence across gravitational, quantum, and cosmological domains.

The present analysis remains entirely classical. Extending the NUVO framework to quantum regimes or to explicit coupling with the Standard Model would require additional structure—such as operator-valued fields or spinor bundles—beyond the scope of this work but representing natural directions for future development.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Chavel, I. (1984) Riemannian Geometry: A Modern Introduction. Cambridge University Press.

[2] Lee, J.M. (1997) Riemannian Manifolds: An Introduction to Curvature. Springer.

[3] Jost, J. (2017) Riemannian Geometry and Geometric Analysis. 7th Edition, Springer. https://doi.org/10.1007/978-3-319-61860-9

[4] Gilbarg, D. and Trudinger, N.S. (2001) Elliptic Partial Differential Equations of Second Order. 2nd Edition, Springer. https://doi.org/10.1007/978-3-642-61798-0

[5] Evans, L.C. (2010) Partial Differential Equations, Second Edition. American Mathematical Society. https://doi.org/10.1090/gsm/019

[6] Zeidler, E. (1990) Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators. Springer.

[7] Austin, R.W. (2025) NUVO Space I: Unit-Constrained Frame Bundle and Conformal Scalar. *Journal of Applied Mathematics and Physics*, **13**, 3673-3680. https://doi.org/10.4236/jamp.2025.1311204

[8] Capozziello, S., Nojiri, S., Odintsov, S.D. and Oikonomou, V.K. (2020) Conformally Invariant Scalar-Tensor Theories and Cosmological Applications. *Classical and Quantum Gravity*, **37**, Article ID: 235019.

[9] Padmanabhan, T. (2022) Conformal Symmetries and Emergent Gravity. *The European Physical Journal C*, **82**, Article 384.

[10] Gidas, B., Ni, W. and Nirenberg, L. (1979) Symmetry and Related Properties via the Maximum Principle. *Communications in Mathematical Physics*, **68**, 209-243. https://doi.org/10.1007/bf01221125