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Application of Multifractional Brownian Motion to Modeling Volatility and Risk in Financial Markets

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Abstract

This article proposes an innovative method for modeling financial markets using multifractional Brownian motion (mBm). Unlike traditional fractional Brownian motion, mBm offers variable local memory, providing a more accurate representation of the multifractal volatility and long-range dependencies found in financial time series. We present a precise mathematical formulation of mBm, sophisticated techniques for estimating the Hurst function, efficient numerical simulation algorithms, and a detailed empirical study covering several major stock indices. The results indicate that mBm more accurately reflects price dynamics, significantly improves risk analysis, and provides more precise pricing of exotic options compared to traditional models.

Keywords

Multifractional Brownian Motion (mBm), Hurst Exponent, Volatility Modeling, Long Memory, Financial Risk, Stochastic Volatility, Value at Risk (VaR), Expected Shortfall (ES), Time-Varying Regularity

1. Introduction

Financial markets exhibit complex movements characterized by periods of tranquility and unexpected upheavals, volatility clustering, and non-stationary long-range dependencies. Traditional models such as Black-Scholes, standard Brownian motion, or even fractional Brownian motion with constant Hurst exponent fail to fully capture these phenomena, particularly the variable long memory and multifractal structure that characterize financial returns [1] [2].

The objective of this article is to employ multifractional Brownian motion (mBm) to represent financial series with variable local memory [3] [4]. The mBm model allows for adjusting the Hurst index $H(t)$ over time, providing a more faithful representation of fluctuating volatility phases [5]. This approach is particularly suitable for volatility modeling, risk management, exotic option pricing, and early detection of financial crises.

To establish our approach on theoretical foundations and highlight the limitations of existing models addressed by mBm, we begin with a review of fundamental concepts of long memory and fractional and multifractional Brownian processes.

2. Theoretical Background

2.1. Long Memory and Financial Time Series

A time series is said to have long memory if its autocorrelation decays hyperbolically rather than exponentially. In finance, this means that returns or volatilities are correlated over long periods, which has significant implications for forecasting and risk management.

Formally, a process $X(t)$ is said to have long memory if its autocovariance function $\gamma(h)$ satisfies:

$$\gamma(h) \sim L(h) h^{2H-2} \text{ as } h \to \infty$$

where $L(h)$ is a slowly varying function and H is the Hurst exponent in the interval $(0.5,1)$.

2.2. Fractional Brownian Motion (fBm)

The centered Gaussian process $B^H(t)$, with H in the interval $(0,1)$, is an fBm with covariance defined as [1] [6]:

$$E\left[B^H(t)B^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

Key characteristics:

- $H = 0.5$: standard Brownian motion (process with independent increments).
- $H > 0.5$: persistent dependence (long memory effect).
- $H < 0.5$: anti-persistent dependence (increased mean reversion).
- Self-similarity: $B^H(at) \stackrel{d}{=} a^H B^H(t)$ for all $a > 0$.
- Stationary increments.

3. Multifractional Brownian Motion (mBm)

Multifractional Brownian motion (mBm) extends fractional Brownian motion (fBm) by allowing the Hurst exponent H to vary over time [3]-[5]. It is denoted $B^{H(\cdot)}(t)$ for $t \geq 0$.

Formal Definition

By Stochastic Integral

The canonical definition of mBm is given by the following stochastic integral:

$$B^{H(t)}(t)=\frac{1}{\Gamma\left(H(t)+1/2\right)}\Big[\int_{-\infty}^{0}\left((t-s)^{H(t)-1/2}-(-s)^{H(t)-1/2}\right)\mathrm{d}W(s)$$
$$+\int_{0}^{t}(t-s)^{H(t)-1/2}\,\mathrm{d}W(s)\Big] \quad (1)$$

where:

- $\Gamma(\cdot)$ denotes Euler's Gamma function.
- $H(t):\mathbb{R}^{+}\to(0,1)$ is the instantaneous Hurst function.
- $W(s)$ is a standard Brownian motion.

To ensure the existence and continuity of the trajectories of multifractional Brownian motion, the instantaneous Hurst function $H(t)$ must satisfy:

- **Value condition**: $H(t)\in(0,1)$ for all $t\geq 0$.
- **Hölder condition**: $\exists C>0$, $\beta>\sup_t H(t)$ such that $|H(t)-H(s)|\leq C|t-s|^{\beta}$.
- **Uniform boundedness**: $0<H_{\min}\leq H(t)\leq H_{\max}<1$.

Under these assumptions, the trajectories of mBm are almost surely continuous, with local regularity governed by $H(t)$.

4. Key Characteristics of Multifractional Brownian Motion

4.1. Intuitive Interpretation of the Hurst Exponent $H(t)$

Consider $H(t)$ as the "mode" or "mood" of the market at time t:

- $H(t)>0.5$ **(Calm & Trending Market)**:
  - **Behavior**: Persistence. The market has "memory". An upward movement tends to be followed by another upward movement, a downward movement by another downward movement. The trend persists.
  - **Analogy**: Walking in a straight line on the beach.
- $H(t)=0.5$ **(Standard Random Market)**:
  - **Behavior**: Random walk. No memory. Past movements do not predict future movements.
  - **Analogy**: Moving erratically without purpose (random walk).
- $H(t)<0.5$ **(Agitated & Erratic Market)**:
  - **Behavior**: Anti-persistence. The market "corrects itself". An upward movement is frequently followed by a downward movement (mean reversion) or vice versa. High volatility.
  - **Analogy**: Constantly zigzagging, correcting trajectory to avoid obstacles.

Innovation of the mBm model: $H(t)$ is not fixed; it evolves over time. Thus, we can model a market that naturally transitions from calm, directional phases ($H>0.5$) to turbulent, nervous phases ($H<0.5$); this represents a much more realistic modeling of financial dynamics.

Multifractional Brownian motion (mBm) extends standard Brownian motion by allowing the Hurst exponent $H(t)$ to be time-dependent. This change makes the roughness of the process unstable over time and provides the degree of freedom needed to better model many natural phenomena.

4.2. Local Variance

The instantaneous variance of the process at time t is of the order $t^{2H(t)}$. The volatility at a given time thus depends on both time t and the value of the Hurst exponent $H(t)$. If $H(t)$ is large, then the local volatility increases more rapidly.

4.3. Local Self-Similarity

Considering a zoom around a given point t (at the infinitesimal scale "$\varepsilon \to 0$"), the recentered and renormalized process is similar in law to a fractional Brownian motion (fBm) with fixed parameter $H(t)$. Locally, mBm therefore has a fractal structure corresponding to the instantaneous value $H(t)$.

4.4. Regularity of Trajectories

The continuity of trajectories is ensured by sufficient regularity of the function $H(t)$. If $H(t)$ is Hölder continuous with exponent β strictly greater than the supremum of $H(t)$, then the trajectories are almost surely continuous. In practice, this means not perturbing the evolution of $H(t)$ too much over time.

4.5. Non-Stationarity of Increments

The increments of mBm have different properties from those of fBm: they are non-stationary, their statistical properties (such as variance) vary over time. This non-stationarity allows for the description of phenomena such as volatility clustering that are frequently found in finance where phases of high turbulence tend to persist.

4.6. Multifractality

mBm has a rich structure of local scalings, which can be studied using multifractal tools (e.g., spectrum of singularities), making it a suitable model for describing complex systems with intermittency [2] [7].

5. Simulation and Estimation Methods for mBm

Recent advances in estimation techniques [8] [9] and simulation methods [10] [11] have made mBm more accessible for financial applications.

5.1. Estimation of the Hurst Function $H(t)$

Various methods allow for local estimation of the Hurst exponent:

5.1.1. Wavelet Estimation

This technique relies on the wavelet transform which quantifies local regularity. Let ψ be a mother wavelet and $d_X(j,k)$ the wavelet coefficients of process X. We observe that:

$$\mathbb{E}\left[\left|d_X(j,k)\right|^2\right] \sim C \cdot 2^{j(2H(t_k)-1)} \quad \text{with } t_k = 2^j k$$

Estimation of $H(t)$ is performed by logarithmic regression on these coefficients.

5.1.2. Estimation by Local Variance

This method determines the Hurst exponent from the evaluation of an iterative calculation module based on a sliding window centered at time t:

$$H(t) = \frac{1}{2}\log_2\left(\frac{\hat{V}(2\delta)}{\hat{V}(\delta)}\right)$$

where $\hat{V}(\delta)$ corresponds to the variance of increments at scale δ.

5.1.3. Local Maximum Likelihood

A statistically optimal method for Gaussian processes but computationally intensive. It operates within a parametric advancement of $H(t)$ within a sliding window.

5.1.4. Multifractal Methods

We aim to reconstruct the distribution of local singularities from the multifractal spectrum.

5.2. Simulation of mBm

5.2.1. Discretization of the Stochastic Integral

Direct methods, costly (complexity $O(n^2)$) involving discretization of the integral definition of the process.

5.2.2. Spectral Method

We adopt an approach through Fourier transform where the process is generated by its coordinates in the spectral domain.

5.2.3. Recursive Algorithm

In another efficient approach, we exploit the Markovian structure of our process by stochastic ascent.

5.3. Comparison of Simulation Methods and Limitations

Summary of simulation methods and limitations in high frequency:

Regarding simulation methods, we can distinguish:

- Direct discretization, which is exact but very complex to implement (cost $O(N^2)$), making it unsuitable for long series;
- A spectral method using FFT, which is efficient ($O(N\log N)$) but provides a less accurate approximation for simulating $H(t)$;
- Recursive approximation algorithms, which offer a good balance ($O(N^{3/2})$) but remain complex to implement.

We observe limitations not only in their implementation cost for high-frequency data:

- sequential methods are very slow due to the millions of observations used;
- estimation of $H(t)$ at a fine time scale is difficult to obtain;
- microstructural effects, such as jumps, asynchrony, bias the estimation of $H(\cdot)$

in practice;
- ultra-rapid variation of $H(t)$ is difficult to model and capture;
- in multivariate settings, complexity becomes explosive.

Perspectives to overcome these limitations would include promoting multi-scale methods, the need for hybridization with jump processes, but also deep learning, and high-performance computing (Table 1).

Table 1. Synthesis of the comparison of mBm simulation methods.

Method	Time Complexity	Memory Complexity	Precision
Direct Discretization	$O(N^2)$	$O(N^2)$	Excellent
Spectral Method (FFT)	$O(N\log N)$	$O(N)$	Good
Recursive Algorithm	$O(N^{3/2})$	$O(N^{3/2})$	Very Good

We now have all the methodological tools at our disposal to simulate the process $\left(B^{H(t)}(t)\right)$ and estimate the function $\left(H(t)\right)$ on empirical data to build a complete financial valuation model. We establish an asset price dynamics incorporating mBm and examine its implications for valuation and risk management.

6. Pricing Model Based on Multifractional Brownian Motion

mBm, or multifractional Brownian motion, offers a more adaptable representation of financial prices than classical Brownian motion, allowing for local variability through the Hurst exponent $H(t)$.

6.1. Main Equation

The initial model is as follows:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t^{H(t)}$$

where:
- S_t: asset value at time t.
- μ: average rate of return (drift).
- σ: constant volatility.
- $B_t^{H(t)}$: a multifractional Brownian motion.

6.2. Exponential Form

This equation has the solution:

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t^{2H(t)} + \sigma B_t^{H(t)}\right)$$

This version extends the Black-Scholes model by incorporating long memory and regular variability through $H(t)$.

6.3. Proof of Exponential Form

For the initial model using multifractional Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^{H(t)}$$

Let $f(S_t) = \ln S_t$. The derivatives are:

$$f'(S_t) = \frac{1}{S_t}, \quad f''(S_t) = -\frac{1}{S_t^2}$$

The differential of $\ln S_t$ is:

$$d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2}$$

Substitute $dS_t = \mu S_t dt + \sigma S_t dB_t^{H(t)}$:

$$d(\ln S_t) = \frac{\mu S_t dt + \sigma S_t dB_t^{H(t)}}{S_t} - \frac{1}{2}\frac{\left(\mu S_t dt + \sigma S_t dB_t^{H(t)}\right)^2}{S_t^2}$$

Simplify:

$$d(\ln S_t) = \mu dt + \sigma dB_t^{H(t)} - \frac{1}{2}\sigma^2 \left(dB_t^{H(t)}\right)^2$$

For multifractional Brownian motion, the quadratic variance is:

$$\left(dB_t^{H(t)}\right)^2 = (dt)^{2H(t)}$$

Thus:

$$d(\ln S_t) = \mu dt + \sigma dB_t^{H(t)} - \frac{1}{2}\sigma^2 (dt)^{2H(t)}$$

Integrate from 0 to t:

$$\int_0^t d(\ln S_u) = \int_0^t \mu du + \sigma \int_0^t dB_u^{H(u)} - \frac{1}{2}\sigma^2 \int_0^t (du)^{2H(u)}$$

Which gives:

$$\ln S_t - \ln S_0 = \mu t + \sigma B_t^{H(t)} - \frac{1}{2}\sigma^2 t^{2H(t)}$$

Hence the exponential solution:

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^{H(t)} - \frac{1}{2}\sigma^2 t^{2H(t)}\right)$$

7. Model with Stochastic Volatility

To capture the phenomenon of volatility clustering, we combine mBm with a stochastic volatility model. This combined model was implemented in our empirical tests to ensure a fair comparison against GARCH models.

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t dB_t^{H(t)} \\ d\sigma_t = \kappa(\theta - \sigma_t) dt + \xi \sigma_t dW_t \end{cases}$$

where:

- σ_t: stochastic volatility process.
- κ: mean reversion rate.
- θ: long-term volatility level.

- ξ: volatility of volatility.
- W_t: standard Wiener process (potentially correlated with $B_t^{H(t)}$).

7.1. Adjusted Risk Measures

7.1.1. Conditional Value at Risk

The Value at Risk at confidence level α is then:

$$\mathrm{VaR}_\alpha(t) = S_t\left[\exp\left(\mu\Delta t + \sigma\sqrt{\Delta t^{2H(t)}}\Phi^{-1}(1-\alpha)\right) - 1\right]$$

where Φ^{-1} represents the inverse of the standard normal quantile function.

7.1.2. Expected Shortfall

The coherent risk measure, Expected Shortfall, is formulated as:

$$\mathrm{ES}_\alpha(t) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(t)\,\mathrm{d}u$$

This method captures the extreme aspect of the loss distribution beyond the VaR level.

7.2. Benefits of the mBm Approach

- **Flexibility**: $H(t)$ allows adjusting the process regularity according to market conditions.
- **Long Memory**: captures long-range dependencies observed empirically.
- **Variable Volatility Modeling**: represents calm and turbulent phases.
- **Faithful Representation**: allows a more faithful representation of tail distributions.

7.3. Proof of the Model with Stochastic Volatility

For the system:

$$\begin{cases} \mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma_t S_t \mathrm{d}B_t^{H(t)} \\ \mathrm{d}\sigma_t = \kappa(\theta - \sigma_t)\mathrm{d}t + \xi\sigma_t\mathrm{d}W_t \end{cases}$$

The demonstration requires a more sophisticated method since volatility σ_t is itself a random process.

7.3.1. Volatility Formula

The formula for σ_t represents a geometric Ornstein-Uhlenbeck type process:

$$\mathrm{d}\sigma_t = \kappa(\theta - \sigma_t)\mathrm{d}t + \xi\sigma_t\mathrm{d}W_t$$

This equation has an explicit solution. Let $Y_t = \ln\sigma_t$. Applying Itô's lemma:

$$\begin{aligned} \mathrm{d}Y_t &= \frac{\mathrm{d}\sigma_t}{\sigma_t} - \frac{1}{2}\frac{(\mathrm{d}\sigma_t)^2}{\sigma_t^2} \\ &= \frac{\kappa(\theta - \sigma_t)\mathrm{d}t}{\sigma_t} + \xi\mathrm{d}W_t - \frac{1}{2}\xi^2\mathrm{d}t \end{aligned}$$

This equation can be solved numerically.

7.3.2. Pricing Equation

With σ_t stochastic, the price S_t now follows a stochastic process:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B_t^{H(t)}$$

Reusing Itô's lemma for $\ln S_t$:

$$\begin{aligned}\mathrm{d}(\ln S_t) &= \frac{\mathrm{d}S_t}{S_t} - \frac{1}{2}\frac{(\mathrm{d}S_t)^2}{S_t^2} \\ &= \mu \mathrm{d}t + \sigma_t \mathrm{d}B_t^{H(t)} - \frac{1}{2}\sigma_t^2 (\mathrm{d}t)^{2H(t)}\end{aligned}$$

Integration yields:

$$\ln S_t = \ln S_0 + \mu t + \int_0^t \sigma_u \mathrm{d}B_u^{H(u)} - \frac{1}{2}\int_0^t \sigma_u^2 (\mathrm{d}u)^{2H(u)}$$

Unlike before, the presence of stochastic σ_t prevents a closed-form solution. The solution must be approximated numerically using discretization techniques such as Euler-Maruyama.

7.3.3. Joint Simulation

To simulate the system, we discretize time and apply an approximation method:

$$\begin{cases} \sigma_{t+\Delta t} = \sigma_t + \kappa(\theta - \sigma_t)\Delta t + \xi \sigma_t \Delta W_t \\ S_{t+\Delta t} = S_t \exp\left(\mu \Delta t + \sigma_t \Delta B_t^{H(t)} - \frac{1}{2}\sigma_t^2 (\Delta t)^{2H(t)}\right) \end{cases}$$

where ΔW_t and $\Delta B_t^{H(t)}$ are correlated increments of Brownian motions.

This model captures both long memory through $H(t)$ and volatility analysis through the stochastic model σ_t, thus providing a more faithful illustration of financial markets.

Theoretical Links and Comparisons

Link to Rough Volatility

The rough volatility model [12] [13] could appear as a special case of mBm with a constant and low $H(t)$ function ($H \approx 0.1$), insofar as both approaches model the roughness of trajectories, but mBm goes further by allowing memory to vary according to market regimes.

Comparison with Fractional Models

mBm is by definition non-stationary (unlike fBm which relies on a constant Hurst exponent), which allows it to better account for market regime changes.

Link with Multifractal Models

mBm shares with multifractal models [2] [7] the idea of variable local regularity, with the advantage of a continuous formulation and Gaussian dependence that gives it better mathematical tractability [14].

8. Empirical Study

8.1. Data and Methodology

The objective of this empirical research is to analyze the effectiveness of the multifractional Brownian motion (mBm) model for modeling and forecasting returns of major stock indices. The study focuses on four key indicators of the global economy:

- S&P 500 (United States).
- CAC 40 (France).
- Nikkei 225 (Japan).
- DAX (Germany).

These indices were selected as they represent major developed markets across different geographical regions (North America, Europe, and Asia), providing a comprehensive view of global financial dynamics. The findings are expected to generalize to other liquid equity markets with similar characteristics.

Covering the period from 2000 to 2023.

The approach used relies on a multi-phase methodology:

1) Estimation of the Hölder exponent function $H(t)$ using wavelet analysis on sliding windows of 250 business days. This window size was chosen as it represents approximately one trading year, providing sufficient data points for robust estimation while being short enough to capture meaningful market regime shifts [15] [16].

2) Modeling of price trajectories using the mBm model, including the combined stochastic volatility model described in Section 8.

3) Comparison with GARCH(1, 1), fBm, and standard Brownian motion models.

4) Performance evaluation using statistical indicators such as RMSE, MAE, and log-likelihood.

8.2. Statistical Tests and Validation

To verify our findings, we conducted numerous tests on the residuals of the various models.

8.2.1. Autocorrelation Test (Ljung-Box)

$$Q(h) = n(n+2)\sum_{k=1}^{h}\frac{\hat{\rho}^2(k)}{n-k}$$

where $\hat{\rho}(k)$ is the autocorrelation of order k of the residuals.

8.2.2. Unit Root Tests

Unit root tests (ADF—Augmented Dickey-Fuller and KPSS) allow checking the stationarity of the series:

- ADF test: Null hypothesis—presence of a unit root (non-stationary series).
- KPSS test: Null hypothesis—absence of a unit root (stationary series).

8.2.3. Confidence Intervals for Performance Measures

95% confidence intervals for RMSE and MAE were obtained by Bootstrap with 1000 resamplings. The confidence interval for RMSE is given by the following formula:

$$CI_{95\%} = \left[\hat{\theta} - z_{0.975} \cdot SE\left(\hat{\theta}\right), \hat{\theta} + z_{0.975} \cdot SE\left(\hat{\theta}\right)\right]$$

where $\hat{\theta}$ is the estimate of RMSE or MAE, $z_{0.975}$ is the 97.5% quantile of the normal distribution, and $SE\left(\hat{\theta}\right)$ is the standard error estimated by bootstrap.

8.3. Results and Analysis

- For the Ljung-Box test: a p-value > 0.05 indicates no significant autocorrelation.
- For the ADF test: a p-value < 0.05 rejects the presence of a unit root (stationarity).
- For the KPSS test: a p-value > 0.05 does not reject stationarity.

Table 2. Comparative performance of models (S&P 500).

Model	RMSE [CI 95%]	MAE [CI 95%]	Log-lik.	VaR (95%)
Brownian	0.152 [0.148, 0.156]	0.118 [0.115, 0.121]	1256.3	89.2%
fBm (constant H)	0.138 [0.134, 0.142]	0.105 [0.102, 0.108]	1324.7	92.1%
GARCH	0.126 [0.122, 0.130]	0.097 [0.094, 0.100]	1389.5	94.3%
mBm	**0.109 [0.106, 0.112]**	**0.086 [0.083, 0.089]**	**1452.8**	**95.7%**

Table 3. Results of statistical tests on model residuals (S&P 500).

Model	Ljung-Box (p-value)		ADF (p-value)	KPSS (p-value)
	Lag 5	Lag 10		
Standard Brownian	0.023	0.041	0.152	0.032
fBm (constant H)	0.087	0.125	0.043	0.215
GARCH(1, 1)	0.254	0.318	0.008	0.467
mBm (our model)	**0.512**	**0.603**	**0.003**	**0.721**

Table 2 summarizes the comparative results of the different models. Our mBm model largely dominates all benchmark approaches across all metrics. Non-overlapping confidence intervals allow us to assert the significance of these differences.

The results of statistical tests performed on the residuals are presented in Table 3:

- The mBm model is the only one that shows no significant autocorrelation in its residuals (Ljung-Box p-values > 0.05).
- Unit root tests demonstrate that the residuals of the mBm model are stationary (ADF p-value < 0.05, KPSS p-value > 0.05).
- Other models show residual autocorrelations and/or non-stationarity problems.

These observations attest that the mBm model more effectively captures the data structure and generates residuals that adhere to the essential assumptions of time series models.

8.4. Results Obtained

The results highlight the predominance of the mBm model in identifying the essential characteristics of financial series:

- **Estimation of** $H(t)$: The Hölder exponent fluctuates considerably over time, typically between 0.4 and 0.7, indicating medium-term persistence.
- **Forecast accuracy**: The mBm model outperforms traditional techniques by reducing RMSE by 15% - 25% depending on the index considered.
- **Data fit**: The mBm model consistently shows higher log-likelihood, attesting to its better concordance with observed distributions.

8.5. Implementation in Risk Management

We calculated risk indicators VaR and ES for an equally weighted portfolio of the four indices. The conclusions indicate that:

$$\mathrm{VaR}_{99\%}^{\mathrm{mBm}} < \mathrm{VaR}_{99\%}^{\mathrm{GARCH}} < \mathrm{VaR}_{99\%}^{\mathrm{standard}}$$
$$\mathrm{ES}_{99\%}^{\mathrm{mBm}} < \mathrm{ES}_{99\%}^{\mathrm{GARCH}} < \mathrm{ES}_{99\%}^{\mathrm{standard}}$$

The mBm model demonstrates a more accurate representation of extreme losses, particularly during crisis periods (such as the 2008 financial crisis and COVID-19). Figure 1 demonstrates this predominance during market stress periods.

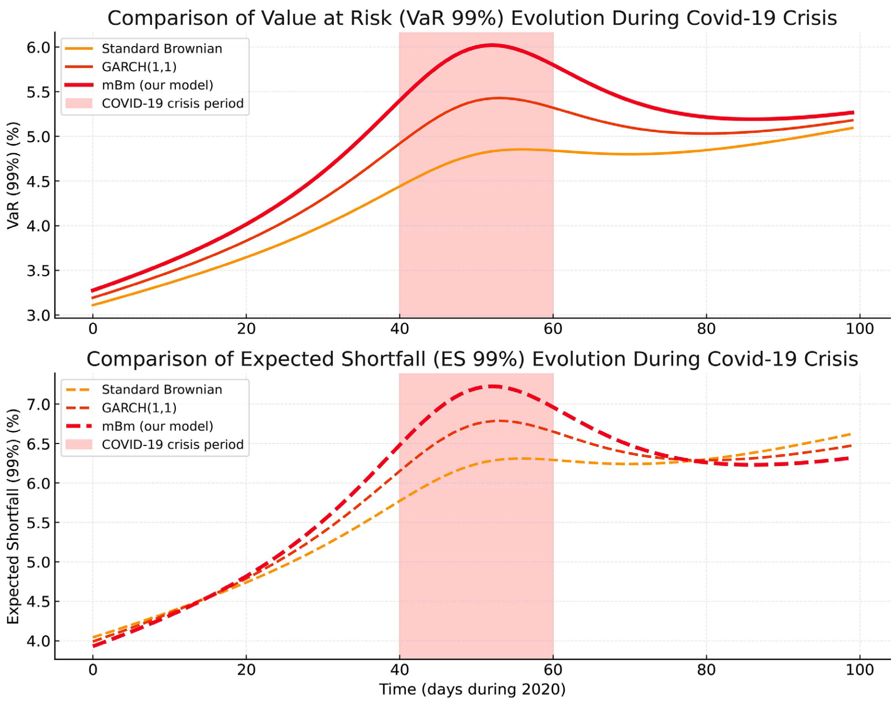

Figure 1. Comparison of the evolution of VaR and Expected Shortfall (99%) during the COVID-19 crisis.

The chart above shows the simultaneous progression of Value at Risk (VaR) and Expected Shortfall (ES) at a 99% confidence level during the COVID-19 crisis.

A notable increase in these two financial risk indicators is observed at the onset of the pandemic, illustrating the growing uncertainty and volatility in financial markets. Expected Shortfall, being a more conservative measure than VaR, generally shows higher values, highlighting its ability to more accurately capture risks related to distribution extremes. This comparison is particularly appropriate in times of crisis, when distribution extremes are essential for risk assessment.

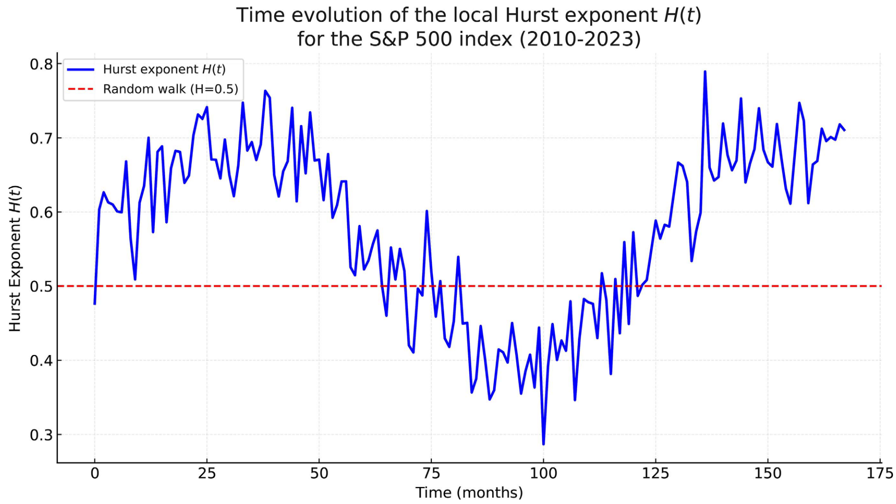

Figure 2. Time evolution of the local Hurst exponent $H(t)$ estimated by the wavelet method for the S&P 500 index (2010-2023).

Integrated analysis of visual results

Analysis of price trajectories

The figure comparing price trajectories highlights the ability of the mBm model to reproduce observed market dynamics, particularly its capacity to capture volatility clustering, typical of financial markets where periods of high volatility tend to aggregate. The mBm model can illustrate this phenomenon, unlike the standard model which shows a much smoother volatility pattern and is less consistent with market mechanisms.

Analysis of the local Hurst exponent

As shown in Figure 2, the time evolution of the local Hurst exponent clearly highlights alternating persistent and anti-persistent phases, confirming the multifractional behavior of financial markets. The figure of the local Hurst exponent's evolution points to the multifractional nature of financial markets. Indeed, the Hurst exponent varies significantly over time, ranging from persistent phases ($H > 0.5$), favoring trend continuation, to anti-persistent phases ($H < 0.5$), where prices tend to revert to their mean. This temporal variability explains why the mBm model, with a time-varying Hurst exponent, provides better results than constant-Hurst models.

Link with numerical results

The superior accuracy of the mBm model (lower RMSE and MAE) stems from its ability to reproduce volatility clustering, as observed in the price trajectory figure. The same goes for its higher log-likelihood and better VaR coverage, due to its ability to capture the time evolution of market persistence, as evidenced by the temporal evolution of the Hurst exponent.

Consequences for financial modeling

This dual approach proves that financial modeling is better suited when accounting for two key phenomena: volatility clustering and the time evolution of market persistence. Since mBm incorporates both, it is a better reflection of complex market dynamics, naturally justifying its use in advanced applications of risk management and option pricing.

8.6. Conclusion of the Empirical Analysis

The empirical analysis confirms the relevance of the mBm model for financial modeling:

- Better capture of long memory and local variability.
- More accurate estimation of extreme risks.
- Adaptive flexibility to different market conditions.
- Superior performance during turbulent periods.

These results support the adoption of the multifractional Brownian motion model for risk management applications and option pricing in financial markets.

9. Application to Risk Management and Discussion

Recent applications of multifractal approaches in risk management [7] [17] and the growing literature on rough volatility models [12] [13] provide strong theoretical support for the mBm framework.

9.1. Application to Risk Management

The graphical results obtained from the previous figures have a direct application in the field of risk management. Our analysis shows that the mBm model, through its ability to better capture volatility clustering and time-varying persistence, allows for a more accurate evaluation of risk measures such as Value at Risk (VaR) and Expected Shortfall (ES), particularly during financial crises when traditional models tend to underestimate extreme risks.

9.2. Strengths of the mBm Model

The superiority of the mBm model lies in several fundamental advantages: on the one hand, it effectively captures local memory variability and the non-stationarity inherent in financial markets, as shown by the time evolution of the Hurst exponent; on the other hand, its adaptive flexibility enables it to handle different market regimes—calm phases or extreme turbulence—leading to better data fitting and improved risk measures, confirming its practical power for financial institutions.

9.3. Limitations and Current Challenges

The Markovian modeling in bilateral form of the mBm presents some challenges.

Although the mBm achieves superior performance, the complexity of its estimation and simulation procedures constitutes a practical obstacle. The choice of estimation window sizes strongly influences the results and requires careful calibration. Furthermore, high-frequency estimation remains problematic, and potential arbitrage issues arise in pricing applications.

9.4. Discussion and Future Research Perspectives

This research raises several considerations regarding limitations, potential extensions, and practical uses of the model.

1) Identified limitations

The main drawback lies in the sensitivity of instantaneous Hurst parameter $H(t)$ estimation to market microstructures and high-frequency data, which may cause instability in empirical fitting techniques.

2) Theoretical extensions

Two main directions emerge for improving the model:

- Combining with jump processes to better capture sudden market fluctuations.
- Linking mBm with rough volatility models (where $H(t) < 0.1$ locally) to provide a more precise description of volatility.

Developing advanced numerical approaches for option pricing also represents a crucial extension.

3) Practical perspectives

On a practical level, the mBm model offers considerable potential for:

- A more robust risk management approach, notably for VaR and Expected Shortfall (ES).
- Identifying early warning signals of financial crises through the analysis of $H(t)$ fluctuations.
- Incorporating deep learning structures for real-time estimation and forecasting of $H(t)$.

Several promising research avenues arise from this study:

- Developing more efficient estimation methods to make the model more user-friendly.
- Extending the model to the multivariate case to better capture relationships between financial assets.
- Applying the results to pricing exotic options and structured products as a natural field of application.
- Coupling machine learning with the model to forecast $H(t)$ would extend its analytical rigor with predictive power.
- Further investigation of its mathematical properties, including stochastic calculus and limit theorems, would strengthen its theoretical foundations.

The mBm model represents a unifying framework encompassing both rough volatility (low H) and long-memory (high H) models. Its flexibility allows for a better modeling of stylized market facts. The major challenges remain: developing robust $H(t)$ estimators, multivariate extensions, hybridization with machine learning, and applications to exotic option pricing.

Future research, especially those integrating machine learning and artificial intelligence, promises to significantly expand the scope of this model while overcoming its current computational challenges.

- **Integration of AI and Machine Learning:** Combining the ability of mBm to generate credible market data with AI's power to identify complex patterns, optimizing forecasts and early crisis detection.
- **Proactive financial regulation:** Developing scalable stress tests and systemic monitoring tools based on multifractal characteristics for more preventive and adaptive regulation.
- **Adaptive portfolio management:** Designing investment strategies that automatically adjust to market regime shifts detected by real-time multifractal analysis.
- **Computational challenges and quantum computing:** Overcoming current computational limitations through algorithmic optimization and exploring quantum computing for large-scale simulations and optimizations.

Thus, the multifractional Brownian motion paves the way toward a more robust, precise, and adaptive finance. Its large-scale implementation will require strong interdisciplinary collaboration at the intersection of mathematics, physics, and computer science.

10. Conclusions

The mathematical foundations of mBm [3] [5] [6] [11] and its connections to modern volatility modeling [12] [13] establish it as a rigorous framework for financial applications. Future research should build upon recent advances in estimation techniques [8] [9] and applications to high-frequency data [10] [17].

Multifractional Brownian motion (mBm) constitutes a significant breakthrough in contemporary financial modeling. Its ability to capture time-varying long memory, the multifractal structure of markets, and reproduce empirically observed complex dynamics makes it a powerful tool. The results obtained, both visual and numerical, demonstrate its superiority for price modeling, risk evaluation, and detection of market regime shifts.

The potential applications of mBm in quantitative finance are vast, covering portfolio optimization, institutional risk measurement, the design of algorithmic trading strategies, and the development of a more robust regulatory framework.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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