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Finite Time Domain Dynamics of Massive Vector Fields

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Abstract

We study the finite time domain dynamics of massive vector fields. We consider their standard Lagrangian and Hamiltonian densities and expand them into creation and annihilation operators. We integrate them via coherent states path integral methods and extract the corresponding massive vector fields' Green functions in various dimensions. Then we consider the possible generation of massive vector fields from currents.

Keywords

Massive Vector Fields, Finite Time, Functional Methods, Currents

1. Introduction

Quantum field theory makes up an important area of physics with a variety of applications [1]-[5]. It studies various particles and fields and their possible structure. Here we develop finite time analytic methods for their possible evolution opposed for instance to the numerical simulations of the obeyed dynamic equations [6] [7] or the employment of lattice quantum field theory [5]. Under suitable experimental preparations these methods may give corresponding predictions on finite time results and effects as opposed for instance to the results of asymptotic in/out ($T \to \infty$) scattering theories. In fact, finite time expressions apply after the possible generation of relevant particles or after the removal of possible trapping fields or configurations.

In previous papers we have considered Dirac and scalar fields [8] [9]. Here we consider the dynamics of finite time domain massive vector fields. *I.e.* we study Proca fields [10]-[13]. They are applicable in the case of massive vector Bosons such as the W and Z particles of the electroweak theory. We study them via path integral methods. We begin from their Lagrangian and Hamiltonian densities, expand the electric and the vector fields in term of annihilation and creation operators and integrate the corresponding Hamiltonian according to standard holomorphic representation techniques developed partially by the author as well, to extract their Green functions in various dimensions. These Green functions give the whole dynamical information of possible evolution. They constitute the functional inverse of the dynamic equation that the system satisfies. In fact, here we use them in the study of the generation of fields by currents in spacetime dimensions three and four.

The present paper proceeds as follows. In section 2 we describe the present massive vector field system and the equations it satisfies. We expand in creation and annihilation operators and integrate according to holomorphic representation methods in a finite time interval to extract the transition amplitudes between coherent states and in particular between vacuum states. Then proceeding in section 3.1. we use that result to derive the Green functions for certain spacetime dimensions and in section 3.2. we give certain series representations. In section 4 we use those Green functions to obtain expressions for the potentials of those massive vector fields generated by currents and further we consider their energy momentum tensors. Moreover, in section 5 we give our conclusions. Finally in appendix A, we evaluate the integrals that appear in section 3.2. and in appendix B we give a notational summary.

Throughout the paper we set $c=\hbar=1$. In our expressions we maintain the mass symbol m for clarity. Then the particle's rest energy is mc^2.

If d is the spacetime dimension then we assume the Greek indices to range from 0 to d-1 while the Latin ones to range from 1 to d-1.

2. System and Path Integration

Here we intend to quantize a massive vector field. We work in real time with the metric $g_{\mu\nu}=diag\left(1,\underbrace{-1,\cdots,-1}_{d-1}\right)$ where the first component corresponds to time. We denote the spacetime with $\{x_0\equiv t,\boldsymbol{x}\}$, $\boldsymbol{x}\in R^{d-1}$, and the vector field with $A_\mu=\{A_0,\boldsymbol{A}\}$. Let $F_{\mu\nu}(t,\boldsymbol{x})$ be the curvature tensor. Then the Lagrangian density of the massive vector field couple with a real current J_μ has the form

$$L=-\frac{1}{4}F_{\mu\nu}(t,\boldsymbol{x})F^{\mu\nu}(t,\boldsymbol{x})+\frac{1}{2}m^2A_\mu(t,\boldsymbol{x})A^\mu(t,\boldsymbol{x})-J_\mu(t,\boldsymbol{x})A^\mu(t,\boldsymbol{x}) \quad (1)$$

where m is the mass and

$$F^{\mu\nu}(t,\boldsymbol{x})=\partial^\mu A^\nu(t,\boldsymbol{x})-\partial^\nu A^\mu(t,\boldsymbol{x}) \quad (2)$$

According to variational considerations the field obeys the following equation

$$\partial_\mu F^{\mu\nu}+m^2A^\nu=J^\nu \quad (3)$$

In term of the field A^μ we obtain the Proca equation

$$\Box A^\nu-\partial^\nu\left(\partial_\mu A^\mu\right)+m^2A^\nu=J^\nu \quad (4)$$

where $\square$ is the d'Alembertian. We observe that this equation is not gauge invariant due to the presence of the mass term (see the discussion in the conclusions as well). On taking the d-divergence of the above equation, we get

$$\partial_\mu A^\mu = \partial_\mu J^\mu / m^2 \tag{5}$$

So, for conserved currents

$$\square A^\mu + m^2 A^\mu = J^\mu \tag{6}$$

$$\partial_\mu A^\mu = 0 \tag{7}$$

In contrast to the electromagnetic field the present Proca field has both transverse and longitudinal components. Moreover, from the d components of A^μ only d-1 are independent. In fact, if we set $\nu = 0$ in Equation (3) we get

$$A^0 = \frac{-\nabla \cdot \boldsymbol{E} + J^0}{m^2} \tag{8}$$

$E^i(t, \boldsymbol{x}) = F^{i0}(t, \boldsymbol{x})$ is the electric field. It corresponds to the conjugate momentum. So, we can get the following expression for the Hamiltonian density

$$H(E, A) =: \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{4}F^{ij}F_{ij} + \frac{m^2}{2}\boldsymbol{A}^2 + \frac{1}{2}\frac{(\nabla \cdot \boldsymbol{E})^2}{m^2} + \frac{(J^0)^2}{2m^2} - J^0\frac{\nabla \cdot \boldsymbol{E}}{m^2} - \boldsymbol{J} \cdot \boldsymbol{A}: \tag{9}$$

where the symbol : : means normal ordering. Moreover E, A must satisfy the following equal time canonical commutation relations

$$\left[E^i(t, \boldsymbol{x}), A^j(t, \boldsymbol{y})\right] = i\delta^{ij}\delta^{(d-1)}(\boldsymbol{x} - \boldsymbol{y}) \tag{10}$$

$$\left[E^i(t, \boldsymbol{x}), E^j(t, \boldsymbol{y})\right] = \left[A^i(t, \boldsymbol{x}), A^j(t, \boldsymbol{y})\right] = 0 \tag{11}$$

If we take into account the constraint Equation (8) we obtain further the commutators

$$\left[A^0(t, \boldsymbol{x}), E^j(t, \boldsymbol{y})\right] = \left[-\nabla \cdot \boldsymbol{E}(t, \boldsymbol{x}), E^j(t, \boldsymbol{y})\right] / m^2 = 0 \tag{12}$$

$$\left[A^0(t, \boldsymbol{x}), A^j(t, \boldsymbol{y})\right] = \left[-\nabla \cdot \boldsymbol{E}(t, \boldsymbol{x}), A^j(t, \boldsymbol{y})\right] / m^2 = -\frac{1}{m^2}\partial_j^x \delta^{(d-1)}(\boldsymbol{x} - \boldsymbol{y}) \tag{13}$$

Now we expand the field in terms of creation and annihilation operators. So

$$A^\mu(t, \boldsymbol{x}) = \int \mathrm{d}\tilde{k} \sum_{\lambda=1}^{d-1}\left[a^{(\lambda)+}(\boldsymbol{k})\varepsilon^{\mu *}(\boldsymbol{k}, \lambda)\mathrm{e}^{-ik\cdot x} + a^{(\lambda)}(\boldsymbol{k})\varepsilon^\mu(\boldsymbol{k}, \lambda)\mathrm{e}^{ik\cdot x}\right] \tag{14}$$

where

$$\mathrm{d}\tilde{k} = \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1} 2\omega_{\boldsymbol{k}}} \tag{15}$$

and

$$\omega_{\boldsymbol{k}} = \sqrt{m^2 + \boldsymbol{k}^2} \tag{16}$$

We have $d-1$ vectors $\varepsilon(\boldsymbol{k}, \lambda)$ corresponding to the $d-1$ polarization directions. We choose $d-2$ of them around the direction of motion and another spacelike one with its momentum in the direction of motion, chosen such that $k \cdot \varepsilon(\boldsymbol{k}, \lambda) = 0$. For a massive vector boson moving along the $d-1$ direction we have $k^{\mu}=\left(\omega_{\boldsymbol{k}},\cdots,|\boldsymbol{k}|\right)$. Then on setting $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)=\left(\varepsilon^{1}(\boldsymbol{k},\lambda),\cdots,\varepsilon^{d-1}(\boldsymbol{k},\lambda)\right)$ we get $\varepsilon^{\mu}(\boldsymbol{k},\lambda)=\left(0,\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)\right)$ for $\lambda=1,\cdots,d-2$ and $\varepsilon^{\mu}(\boldsymbol{k},d-1)=\left(\frac{|\boldsymbol{k}|}{m},\cdots,\frac{\omega_{\boldsymbol{k}}}{m}\right)$.

Further the electric field has the form

$$E^{i}(t,\boldsymbol{x})=i\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\left[\omega_{\boldsymbol{k}}a^{(\lambda)+}(\boldsymbol{k})\tilde{\varepsilon}^{i*}(\boldsymbol{k},\lambda)\mathrm{e}^{-ik\cdot x}-\omega_{\boldsymbol{k}}a^{(\lambda)}(\boldsymbol{k})\tilde{\varepsilon}^{i}(\boldsymbol{k},\lambda)\mathrm{e}^{ik\cdot x}\right] \tag{17}$$

From the definition of the electric field upon setting $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{k},\lambda)=\left(\tilde{\varepsilon}^{1}(\boldsymbol{k},\lambda),\cdots,\tilde{\varepsilon}^{d-1}(\boldsymbol{k},\lambda)\right)$ we obtain the relation [4]

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{k},\lambda)=\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)-\frac{\boldsymbol{k}}{\omega_{\boldsymbol{k}}}\varepsilon^{0}(\boldsymbol{k},\lambda)=\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)-\boldsymbol{k}\frac{\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)}{\omega_{\boldsymbol{k}}^{2}} \tag{18}$$

where according to the relations $k\cdot\varepsilon(\boldsymbol{k},\lambda)=\omega_{\boldsymbol{k}}\varepsilon^{0}(\boldsymbol{k},\lambda)-\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)=0$ we have gotten

$$\varepsilon^{0}(\boldsymbol{k},\lambda)=\frac{\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)}{\omega_{\boldsymbol{k}}} \tag{19}$$

In order the above equations to be consistent the following commutation rules must be valid

$$\left[a^{(\lambda)}(\boldsymbol{k}),a^{(\lambda')\dagger}(\boldsymbol{k}')\right]=\delta_{\lambda\lambda'}2\omega_{\boldsymbol{k}}(2\pi)^{d-1}\delta^{(d-1)}(\boldsymbol{k}-\boldsymbol{k}') \tag{20}$$

$$\left[a^{(\lambda)}(\boldsymbol{k}),a^{(\lambda')}(\boldsymbol{k}')\right]=\left[a^{(\lambda)+}(\boldsymbol{k}),a^{(\lambda')+}(\boldsymbol{k}')\right]=0 \tag{21}$$

Then from the above equations we can derive the free field constraint (8) in the form

$$A^{0}=-\frac{\nabla\cdot\boldsymbol{E}}{m^{2}} \tag{22}$$

So, after straightforward calculations we get the following reduced diagonal Hamiltonian in normal order form

$$\begin{aligned}I\left(a^{(\lambda)}(\boldsymbol{k}),a^{(\lambda)+}(\boldsymbol{k}),t\right)&=\int \mathrm{d}^{d-1}xH(E,A)\\&=\int \mathrm{d}^{d-1}x\frac{\left(J^{0}\right)^{2}}{2m^{2}}+\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\left[\omega_{\boldsymbol{k}}a^{(\lambda)+}(\boldsymbol{k})a^{(\lambda)}(\boldsymbol{k})\right.\\&\quad\left.+j^{(\lambda)*}(t,\boldsymbol{k})a^{(\lambda)}(\boldsymbol{k})+j^{(\lambda)}(t,\boldsymbol{k})a^{(\lambda)+}(\boldsymbol{k})\right]\end{aligned} \tag{23}$$

where

$$j^{(\lambda)}(t,\boldsymbol{k})=\varepsilon^{\mu*}(\boldsymbol{k},\lambda)J_{\mu}(t,\boldsymbol{k}) \tag{24}$$

$$J_{\mu}(t,\boldsymbol{k})=\int \mathrm{d}^{d-1}xJ_{\mu}(t,\boldsymbol{x})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{25}$$

The space-like orthonormalized vectors $\varepsilon^{\mu}(\boldsymbol{k},\lambda)$ are simultaneously orthogonal to the time-like vector k^{μ} and therefore if we assume them real

$$\varepsilon(\boldsymbol{k},\lambda)\cdot\varepsilon(\boldsymbol{k},\lambda')=\delta_{\lambda\lambda'} \tag{26}$$

and

$$\sum_{\lambda}\varepsilon^{\mu}(\boldsymbol{k},\lambda)\varepsilon^{\nu}(\boldsymbol{k},\lambda)=-\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{m^{2}}\right) \tag{27}$$

Now we consider the $\boldsymbol{k}$ th mode along the λ polarization direction of the diagonal Hamiltonian (23). It is

$$H_0 = \omega_{\boldsymbol{k}} a^{(\lambda)+}(\boldsymbol{k}) a^{(\lambda)}(\boldsymbol{k}) + j^{(\lambda)*}(t,\boldsymbol{k}) a^{(\lambda)}(\boldsymbol{k}) + j^{(\lambda)}(t,\boldsymbol{k}) a^{(\lambda)+}(\boldsymbol{k}) \tag{28}$$

We intend to construct a path integral representation of the evolution operator

$$U_0(T) = \mathrm{e}^{-i\int_0^T \mathrm{d}\tau H_0(\tau)} \tag{29}$$

We work within the holomorphic representation. So, we introduce complex variables $\alpha^{(\lambda)*}(\boldsymbol{k})$ and proceed via path integral techniques. We represent $a^{(\lambda)+}(\boldsymbol{k})$, $a^{(\lambda)}(\boldsymbol{k})$ by the operators $\alpha^{(\lambda)*}(\boldsymbol{k})$ and $\frac{\partial}{\partial \alpha^{(\lambda)*}(\boldsymbol{k})}$ respectively acting on functions of $\alpha^{(\lambda)*}(\boldsymbol{k})$, which obey the same commutation relations. The Hamiltonian H_0 has then the representation

$$H_0 = \omega_{\boldsymbol{k}} \alpha^{(\lambda)*}(\boldsymbol{k}) \frac{\partial}{\partial \alpha^{(\lambda)*}(\boldsymbol{k})} + j^{(\lambda)*}(t,\boldsymbol{k}) \frac{\partial}{\partial \alpha^{(\lambda)*}(\boldsymbol{k})} + j^{(\lambda)}(t,\boldsymbol{k}) \alpha^{(\lambda)*}(\boldsymbol{k}) \tag{30}$$

At first, we assume a small time t and obtain

$$\begin{aligned}
&\left\langle \alpha^{(\lambda)*}(\boldsymbol{k}) \middle| U_0(t) \middle| \alpha'^{(\lambda)}(\boldsymbol{k}) \right\rangle \\
&= \Bigg[1 - it \Bigg(\omega_{\boldsymbol{k}} \alpha^{(\lambda)*}(\boldsymbol{k}) \frac{\partial}{\partial \alpha^{(\lambda)*}(\boldsymbol{k})} + j^{(\lambda)*}(t,\boldsymbol{k}) \frac{\partial}{\partial \alpha^{(\lambda)*}(\boldsymbol{k})} \\
&\quad + j^{(\lambda)}(t,\boldsymbol{k}) \alpha^{(\lambda)*}(\boldsymbol{k}) + O(t^2) \Bigg) \Bigg] \mathrm{e}^{\alpha^{(\lambda)*}(\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k})} \\
&= \Big[1 - it \Big(\omega_{\boldsymbol{k}} \alpha^{(\lambda)*}(\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k}) + j^{(\lambda)*}(t,\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k}) + j^{(\lambda)}(t,\boldsymbol{k}) \alpha^{(\lambda)*}(\boldsymbol{k}) \Big) \\
&\quad + O(t^2) \Big] \mathrm{e}^{\alpha^{(\lambda)*}(\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k})} \\
&= \mathrm{e}^{\alpha^{(\lambda)*}(\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k})(1 - it\omega_{\boldsymbol{k}}) - it\left(\alpha^{(\lambda)*}(\boldsymbol{k}) j^{(\lambda)}(t,\boldsymbol{k}) + j^{(\lambda)*}(t,\boldsymbol{k}) \alpha'^{(\lambda)}(\boldsymbol{k}) \right)} + O(t^2)
\end{aligned} \tag{31}$$

Then according to the group property

$$\begin{aligned}
&\left\langle \alpha^{(\lambda)*}(\boldsymbol{k}) \middle| U_2 U_1 \middle| \alpha'^{(\lambda)}(\boldsymbol{k}) \right\rangle \\
&= \int \frac{\mathrm{d}\alpha''^{(\lambda)}(\boldsymbol{k}) \mathrm{d}\alpha''^{(\lambda)*}(\boldsymbol{k})}{2\pi i} \left\langle \alpha^{(\lambda)*}(\boldsymbol{k}) \middle| U_2 \middle| \alpha''^{(\lambda)}(\boldsymbol{k}) \right\rangle \\
&\quad \times \mathrm{e}^{-\alpha''^{(\lambda)*}(\boldsymbol{k}) \alpha''^{(\lambda)}(\boldsymbol{k})} \left\langle \alpha''^{(\lambda)*}(\boldsymbol{k}) \middle| U_1 \middle| \alpha'^{(\lambda)}(\boldsymbol{k}) \right\rangle
\end{aligned} \tag{32}$$

After multiple application of the Equations (31), (32) we get the evolution operator at finite time in the following path integral form

$$\begin{aligned}
&\left\langle \alpha_f^{(\lambda)*}(\boldsymbol{k}) \middle| U_0(T) \middle| \alpha_i^{(\lambda)}(\boldsymbol{k}) \right\rangle \\
&= \lim_{N\to\infty} \int \prod_{p=1}^{N-1} \frac{\mathrm{d}\alpha_p^{(\lambda)*}(\boldsymbol{k}) \mathrm{d}\alpha_p^{(\lambda)}(\boldsymbol{k})}{2\pi i} \exp\left[iS_0\left(\alpha_p^{(\lambda)*}(\boldsymbol{k}), \alpha_p^{(\lambda)}(\boldsymbol{k}) \right) \right]
\end{aligned} \tag{33}$$

where

$$\begin{aligned} & S_0\left(\alpha_p^{(\lambda)*}(\boldsymbol{k}),\alpha_p^{(\lambda)}(\boldsymbol{k})\right) \\ & = i\sum_{p=1}^{N-1}\alpha_p^{(\lambda)*}(\boldsymbol{k})\left(\alpha_p^{(\lambda)}(\boldsymbol{k})-\alpha_{p-1}^{(\lambda)}(\boldsymbol{k})\right)-i\alpha_N^{(\lambda)*}(\boldsymbol{k})\alpha_{N-1}^{(\lambda)}(\boldsymbol{k}) \\ & \quad -\omega_{\boldsymbol{k}}\varepsilon\sum_{p=1}^{N}\alpha_p^{(\lambda)*}(\boldsymbol{k})\alpha_{p-1}^{(\lambda)}(\boldsymbol{k})-\varepsilon\sum_{p=1}^{N}\left(\alpha_p^{(\lambda)*}(\boldsymbol{k})j_p^{(\lambda)}(\boldsymbol{k})+j_{p-1}^{(\lambda)*}(\boldsymbol{k})\alpha_{p-1}^{(\lambda)}(\boldsymbol{k})\right) \end{aligned} \tag{34}$$

$$\varepsilon=\frac{T}{N} \tag{35}$$

and

$$\alpha_0^{(\lambda)}(\boldsymbol{k})=\alpha_i^{(\lambda)}(\boldsymbol{k}),\quad \alpha_N^{(\lambda)*}(\boldsymbol{k})=\alpha_f^{(\lambda)*}(\boldsymbol{k}) \tag{36}$$

If we let $N\to\infty$ we obtain the standard path integral representation for the matrix elements of $U_0(T)$ in the form

$$\begin{aligned} & \left\langle\alpha_f^{(\lambda)*}(\boldsymbol{k})\middle|U_0(T)\middle|\alpha_i^{(\lambda)}(\boldsymbol{k})\right\rangle \\ & = \int_{\alpha^{(\lambda)}(0,\boldsymbol{k})=\alpha_i^{(\lambda)}(\boldsymbol{k})}^{\alpha^{(\lambda)*}(T,\boldsymbol{k})=\alpha_f^{(\lambda)*}(\boldsymbol{k})}\frac{D\alpha^{(\lambda)*}(\tau,\boldsymbol{k})D\alpha^{(\lambda)}(\tau,\boldsymbol{k})}{2\pi i}\exp\left[iS_0\left(\alpha^{(\lambda)*}(\tau,\boldsymbol{k}),\alpha^{(\lambda)}(\tau,\boldsymbol{k})\right)\right] \end{aligned} \tag{37}$$

with the action

$$\begin{aligned} & S_0\left(\alpha^{(\lambda)*}(\tau,\boldsymbol{k}),\alpha^{(\lambda)}(\tau,\boldsymbol{k})\right) \\ & = -i\alpha_f^{(\lambda)*}(\boldsymbol{k})\alpha^{(\lambda)}(T,\boldsymbol{k})+\int_0^T\mathrm{d}\tau\left\{\alpha^{(\lambda)*}(\tau,\boldsymbol{k})\left[i\dot{\alpha}^{(\lambda)}(\tau,\boldsymbol{k})-\omega_{\boldsymbol{k}}\alpha^{(\lambda)}(\tau,\boldsymbol{k})\right]\right. \\ & \quad \left.-\alpha^{(\lambda)*}(\tau,\boldsymbol{k})j^{(\lambda)}(\tau,\boldsymbol{k})-j^{(\lambda)*}(\tau,\boldsymbol{k})\alpha^{(\lambda)}(\tau,\boldsymbol{k})\right\} \end{aligned} \tag{38}$$

The path integral (37) with the action (38) is Gaussian and therefore it can be evaluated exactly. By varying $\alpha^{(\lambda)*}(\tau,\boldsymbol{k})$ the saddle point equation yields

$$i\dot{\alpha}^{(\lambda)}(\tau,\boldsymbol{k})-\omega_{\boldsymbol{k}}\alpha^{(\lambda)}(\tau,\boldsymbol{k})-j^{(\lambda)}(\tau,\boldsymbol{k})=0 \tag{39}$$

with solution

$$\alpha^{(\lambda)}(t,\boldsymbol{k})=\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}\alpha_i^{(\lambda)}(\boldsymbol{k})-i\int_0^t\mathrm{d}\tau\,\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-\tau)}j^{(\lambda)}(\tau,\boldsymbol{k}) \tag{40}$$

For completeness we give relations for $\alpha^{(\lambda)*}(\tau,\boldsymbol{k})$ as well. So, on varying $\alpha^{(\lambda)}(\tau,\boldsymbol{k})$ we get

$$-i\dot{\alpha}^{(\lambda)*}(\tau,\boldsymbol{k})-\omega_{\boldsymbol{k}}\alpha^{(\lambda)*}(\tau,\boldsymbol{k})-j^{(\lambda)*}(\tau,\boldsymbol{k})=0 \tag{41}$$

with solution

$$\alpha^{(\lambda)*}(t,\boldsymbol{k})=\mathrm{e}^{-i\omega_{\boldsymbol{k}}(T-t)}\alpha_f^{(\lambda)*}(\boldsymbol{k})-i\int_t^T\mathrm{d}\tau\,\mathrm{e}^{-i\omega_{\boldsymbol{k}}(\tau-t)}j^{(\lambda)*}(\tau,\boldsymbol{k}) \tag{42}$$

In the above equations we have taken into account the boundary conditions given in Equation (36).

Now we can use the differential relation (39) to write Equation (38) in the form

$$S_0\left(\alpha^{(\lambda)*}(\tau,\boldsymbol{k}),\alpha^{(\lambda)}(\tau,\boldsymbol{k})\right)=-i\alpha_f^{(\lambda)*}(\boldsymbol{k})\alpha^{(\lambda)}(T,\boldsymbol{k})-\int_0^T\mathrm{d}\tau\,j^{(\lambda)*}(\tau,\boldsymbol{k})\alpha^{(\lambda)}(\tau,\boldsymbol{k}) \tag{43}$$

Finally, since the integrals are Gaussian, we get

$$\left\langle \alpha_f^{(\lambda)*}(\boldsymbol{k}) \middle| U_0(T) \middle| \alpha_i^{(\lambda)}(\boldsymbol{k}) \right\rangle = F(T)\exp\left[iS_0\left(\alpha^{(\lambda)*}(\tau,\boldsymbol{k}), \alpha^{(\lambda)}(\tau,\boldsymbol{k}) \right) \right] \tag{44}$$

So, on using Equations (40), (43), (44) we get

$$\begin{aligned} &\left\langle \alpha_f^{(\lambda)*}(\boldsymbol{k}) \middle| U_0(T) \middle| \alpha_i^{(\lambda)}(\boldsymbol{k}) \right\rangle \\ &= \exp\left\{ \alpha_f^{(\lambda)*}(\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}T} \alpha_i^{(\lambda)}(\boldsymbol{k}) - i\int_0^T \mathrm{d}t \left(\alpha_f^{(\lambda)*}(\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}(T-t)} j^{(\lambda)}(t,\boldsymbol{k}) \right.\right. \\ &\quad \left.\left. + j^{(\lambda)*}(t,\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}t} \alpha_i^{(\lambda)}(\boldsymbol{k}) \right) - \int_0^T\int_0^t j^{(\lambda)*}(t,\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t')} j^{(\lambda)}(t',\boldsymbol{k}) \mathrm{d}t'\mathrm{d}t \right\} \end{aligned} \tag{45}$$

where the semigroup property of the path integral implies $F(T)=1$.

The Hamiltonian (23) is a superposition of Hamiltonians of the form (28) and so we can obtain the coherent states propagator corresponding to the Hamiltonian (23) in the form

$$\begin{aligned} &U_0\left(\alpha_f^{(\lambda)*}(\boldsymbol{k}), \alpha_i^{(\lambda)}(\boldsymbol{k}), T; J \right) \\ &= \exp\left\{ -i\int_0^T \mathrm{d}t \int \mathrm{d}^{d-1}x \frac{\left(J^0\right)^2}{2m^2} + \int \mathrm{d}\tilde{k} \sum_{\lambda=1}^{d-1} \left[\alpha_f^{(\lambda)*}(\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}T} \alpha_i^{(\lambda)}(\boldsymbol{k}) \right.\right. \\ &\quad - i\int_0^T \mathrm{d}t \left(\alpha_f^{(\lambda)*}(\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}(T-t)} j^{(\lambda)}(t,\boldsymbol{k}) + j^{(\lambda)*}(t,\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}t} \alpha_i^{(\lambda)}(\boldsymbol{k}) \right) \\ &\quad \left.\left. - \int_0^T\int_0^t j^{(\lambda)*}(t,\boldsymbol{k}) \mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t')} j^{(\lambda)}(t',\boldsymbol{k}) \mathrm{d}t'\mathrm{d}t \right] \right\} \end{aligned} \tag{46}$$

In order to evaluate the amplitude in the case of the coherent states evolution operator we have to integrate diagonally [14] [15]. Then for initial and final coherent states we get

$$\begin{aligned} &U\left(\zeta^{(\lambda)*}(\boldsymbol{k}), \eta^{(\lambda)}(\boldsymbol{k}), T; J \right) \\ &= \prod_{\lambda,\boldsymbol{k}} \left[\int \frac{\mathrm{d}^2\alpha^{(\lambda)}(\boldsymbol{k})}{\pi} \right] \prod_{\lambda,\boldsymbol{k}} \left[\left\langle \zeta^{(\lambda)}(\boldsymbol{k}) \middle| \alpha^{(\lambda)}(\boldsymbol{k}) \right\rangle \right] \\ &\quad \times U_0\left(\alpha^{(\lambda)*}(\boldsymbol{k}), \alpha^{(\lambda)}(\boldsymbol{k}), T; J \right) \prod_{\lambda,\boldsymbol{k}} \left[\left\langle \alpha^{(\lambda)}(\boldsymbol{k}) \middle| \eta^{(\lambda)}(\boldsymbol{k}) \right\rangle \right] \end{aligned} \tag{47}$$

So since

$$\begin{aligned} &\left\langle \zeta^{(\lambda)}(\boldsymbol{k}) \middle| \alpha^{(\lambda)}(\boldsymbol{k}) \right\rangle \\ &= \exp\left[-\frac{1}{2}\left|\zeta^{(\lambda)}(\boldsymbol{k})\right|^2 - \frac{1}{2}\left|\alpha^{(\lambda)}(\boldsymbol{k})\right|^2 + \zeta^{(\lambda)*}(\boldsymbol{k})\alpha^{(\lambda)}(\boldsymbol{k}) \right] \end{aligned} \tag{48}$$

and

$$\begin{aligned} &\left\langle \alpha^{(\lambda)}(\boldsymbol{k}) \middle| \eta^{(\lambda)}(\boldsymbol{k}) \right\rangle \\ &= \exp\left[-\frac{1}{2}\left|\alpha^{(\lambda)}(\boldsymbol{k})\right|^2 - \frac{1}{2}\left|\eta^{(\lambda)}(\boldsymbol{k})\right|^2 + \alpha^{(\lambda)*}(\boldsymbol{k})\eta^{(\lambda)}(\boldsymbol{k}) \right] \end{aligned} \tag{49}$$

after the integrations we get

$$
\begin{aligned}
& U\left(\varsigma^{(\lambda)*}(\boldsymbol{k}),\eta^{(\lambda)}(\boldsymbol{k}),T;J\right) \\
& = \exp\left[-i\int_0^T \mathrm{d}t\int \mathrm{d}^{d-1}x \frac{\left(J^0\right)^2}{2m^2}\right] \\
& \quad\times \exp\left\{\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\left(-\frac{1}{2}\left|\varsigma^{(\lambda)}(\boldsymbol{k})\right|^2-\frac{1}{2}\left|\eta^{(\lambda)}(\boldsymbol{k})\right|^2+\frac{\varsigma^{(\lambda)*}(\boldsymbol{k})\eta^{(\lambda)}(\boldsymbol{k})}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}}-\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)\right)\right. \\
& \quad + i\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\int_0^T \mathrm{d}t\left(\varsigma^{(\lambda)*}(\boldsymbol{k})\frac{\mathrm{e}^{i\omega_{\boldsymbol{k}}t}}{1-\mathrm{e}^{i\omega_{\boldsymbol{k}}T}}j^{(\lambda)}(t,\boldsymbol{k})-j^{(\lambda)*}(t,\boldsymbol{k})\frac{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}}\eta^{(\lambda)}(\boldsymbol{k})\right) \\
& \quad \left. -\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\int_0^T\int_0^t j^{(\lambda)*}(t,\boldsymbol{k})\xi(\boldsymbol{k},t,t',T)j^{(\lambda)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right\}
\end{aligned}
\tag{50}
$$

We have set

$$
\xi(\boldsymbol{k},t,t',T)=\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t')}+\frac{2\cos\left(\omega_{\boldsymbol{k}}(t-t')\right)}{1-\mathrm{e}^{i\omega_{\boldsymbol{k}}T}}=i\frac{\cos\left(\omega_{\boldsymbol{k}}(t-t')-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}
\tag{51}
$$

So finally

$$
\begin{aligned}
& U\left(\varsigma^{(\lambda)*}(\boldsymbol{k}),\eta^{(\lambda)}(\boldsymbol{k}),T;J\right) \\
& = \exp\left[-i\int_0^T \mathrm{d}t\int \mathrm{d}^{d-1}x \frac{\left(J^0\right)^2}{2m^2}\right] \\
& \quad\times \exp\left\{\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\left(-\frac{1}{2}\left|\varsigma^{(\lambda)}(\boldsymbol{k})\right|^2-\frac{1}{2}\left|\eta^{(\lambda)}(\boldsymbol{k})\right|^2+\frac{\varsigma^{(\lambda)*}(\boldsymbol{k})\eta^{(\lambda)}(\boldsymbol{k})}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}}-\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)\right)\right. \\
& \quad + i\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\int_0^T \mathrm{d}t\left(\varsigma^{(\lambda)*}(\boldsymbol{k})\frac{\mathrm{e}^{i\omega_{\boldsymbol{k}}t}}{1-\mathrm{e}^{i\omega_{\boldsymbol{k}}T}}j^{(\lambda)}(t,\boldsymbol{k})-j^{(\lambda)*}(t,\boldsymbol{k})\frac{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}}\eta^{(\lambda)}(\boldsymbol{k})\right) \\
& \quad \left. -i\frac{1}{2}\int \mathrm{d}\tilde{k}\sum_{\lambda=1}^{d-1}\int_0^T\int_0^T j^{(\lambda)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(\lambda)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right\}
\end{aligned}
\tag{52}
$$

We can obtain the finite time interval Green function of massive vector fields from the present considerations. We do that in the next section.

3. Green Function

We procced to the extraction of the Green function and the presentation of a series of possible representations.

3.1. Derivation

According to the discussion of the previous section the correlation functions' generating functional has the form

$$Z(J) = U(0,0,T;J) \tag{53}$$

Then the d dimensional Green function satisfies the expressions

$$G^{(d)\mu\nu}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) = -\frac{1}{Z(0)}\frac{\delta^2}{\delta J_\mu \delta J_\nu} Z(J)\bigg|_{J=0} \tag{54}$$

So, on performing the functional derivations according to Equations (52), (53), (54) we get

$$\begin{aligned} &G^{(d)\mu\nu}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) \\ &= i\frac{g^{\mu 0}g^{\nu 0}}{m^2}\delta(t-t')\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{x}') \\ &\quad -i\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{2\omega_{\boldsymbol{k}}\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right) \end{aligned} \tag{55}$$

where $0 \le t,t' \le T$. To derive Equation (55) we have applied Equation (27).

Further we would like to remove $\frac{k^\mu k^\nu}{m^2}$ by replacing the various k^μ, k^ν with appropriate derivatives. However, if $\mu = 0$ or $\nu = 0$ then we have time derivatives which act on the time step functions. They appear due to the absolute value of the time in Equation (55). As we can check their contribution cancels the non-covariant delta function term appearing in Equation (55). So, we have the form

$$\begin{aligned} &G^{(d)0i}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) = G^{(d)i0}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) \\ &= i\frac{1}{2m^2}\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} k^i \frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)} \end{aligned} \tag{56a}$$

and

$$\begin{aligned} &G^{(d)\mu\nu}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) \\ &= -i\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right)\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{2\omega_{\boldsymbol{k}}\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)} \end{aligned} \tag{56b}$$

for the rest of the matrix elements.

Equations (56a), (56b) give the Green function that describes the dynamics of massive vector fields. We can use it in the study of their generation and propagation in a spacetime of a finite time interval.

Proceeding to the study of that Green function we write the sine function in terms of exponentials. Then

$$\begin{aligned} &G^{(d)0i}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) = G^{(d)i0}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) \\ &= -\frac{1}{m^2}\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} k^i \frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}} \mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \end{aligned} \tag{57a}$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(x-x')}\frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\omega_{\boldsymbol{k}}\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)}\mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \tag{57b}$$

for the rest of the matrix elements.

In numerical calculations we can circumvent the poles of the above equations if we introduce a parameter o with $0<o<1$ and replace the above expressions with the formulas

$$G^{(d)0i}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)=G^{(d)i0}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=-\frac{1}{m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(x-x')}k^i\frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{1-o\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}}\mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \tag{58a}$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(x-x')}\frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\omega_{\boldsymbol{k}}\left(1-o\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)}\mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \tag{58b}$$

for the rest of the matrix elements.

3.2. Limiting Cases

Alternatively, we can expand the denominators of Equations (57a), (57b) in geometric series to get

$$G^{(d)0i}\left(\boldsymbol{x},\tau;T\right)=G^{(d)i0}\left(\boldsymbol{x},\tau;T\right)$$
$$=-\frac{1}{2m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}k^i\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{59a}$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x},\tau;T\right)$$
$$=\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{59b}$$

for the rest of the matrix elements.

Now we give expressions for the various values of the indices. So

$$G^{(d)0i}\left(\boldsymbol{x},\tau;T\right)=G^{(d)i0}\left(\boldsymbol{x},\tau;T\right)$$
$$=-\frac{1}{2m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]k^i \tag{60}$$
$$=A^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)x^i$$

where

$$A^{(d)}\left(\boldsymbol{x}^2,\tau;T\right) = -\frac{1}{2m^2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}(\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{61}$$

If $\mu=\nu=0$ we obtain

$$G^{(d)00}\left(\boldsymbol{x},\tau;T\right) = \left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{62}$$

Moreover

$$\begin{aligned} G^{(d)ij}\left(\boldsymbol{x},\tau;T\right) &= -\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\left(\delta^{ij}+\frac{k^ik^j}{m^2}\right) \\ &= \left[B^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)+C^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)\right]\delta^{ij}+D^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)x^ix^j \end{aligned} \tag{63}$$

where

$$\begin{aligned} B^{(d)}\left(\boldsymbol{x}^2,\tau;T\right) &= -\frac{1}{m^2}\frac{1}{d-2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left[-(\boldsymbol{k}\cdot\boldsymbol{x})^2+\boldsymbol{k}^2\boldsymbol{x}^2\right] \\ &\quad\times\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \end{aligned} \tag{64}$$

$$\begin{aligned} D^{(d)}\left(\boldsymbol{x}^2,\tau;T\right) &= -\frac{1}{m^2}\frac{1}{d-2}\frac{1}{\left(\boldsymbol{x}^2\right)^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left[(d-1)(\boldsymbol{k}\cdot\boldsymbol{x})^2-\boldsymbol{k}^2\boldsymbol{x}^2\right] \\ &\quad\times\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \end{aligned} \tag{65}$$

and

$$C^{(d)}\left(\boldsymbol{x}^2,\tau;T\right) = -\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{66}$$

In the present subsection's forms, we observe that if we let $T\to\infty$ the expressions in the parentheses which include T on exponentials collapse to the factor $\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}$. Then we get the standard Proca propagator [4].

In the Appendix A we give expressions for the above integrals for various dimensions.

4. Application

Now we apply the above expressions to the study of the generation of massive vector fields from currents. Let the current be $J_\nu(\boldsymbol{x},t)$ where the vector $\boldsymbol{x}$ has dimension $d-1$. Then

$$A^\mu(t,\boldsymbol{x}) = -i\int_0^T\mathrm{d}t'\int\mathrm{d}^{d-1}\boldsymbol{x}'G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)J_\nu\left(\boldsymbol{x}',t'\right) \tag{67}$$

We consider a charged particle on a trajectory $\boldsymbol{y}(t)$. Then the current d-vector has the standard form

$$J_\nu(\boldsymbol{x},t) = Q\mathrm{v}_\nu(t)\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}(t)) \tag{68}$$

where $\mathrm{v}_\nu(t)$ $\nu = 0,1,\cdots,d-1$ is its velocity and Q its charge. Moreover, we set

$$\mathrm{v} = (1,\mathrm{v}_1,\cdots,\mathrm{v}_{d-1}) = (1,\mathbf{v}) \tag{69}$$

So

$$A^0(t,\boldsymbol{x}) = -iQ\int_0^T \Big\{ G^{(d)00}(\boldsymbol{x}-\boldsymbol{y}(t'),t-t';T) - A^{(d)}\Big((\boldsymbol{x}-\boldsymbol{y}(t'))^2,t-t';T\Big)(\boldsymbol{x}-\boldsymbol{y}(t'))^j \mathrm{v}_j(t')\Big\}\mathrm{d}t' \tag{70}$$

and

$$\begin{aligned} A^i(t,\boldsymbol{x}) = -iQ\int_0^T \Big\{ & A^{(d)}\Big((\boldsymbol{x}-\boldsymbol{y}(t'))^2,t-t';T\Big)(\boldsymbol{x}-\boldsymbol{y}(t'))^i \\ & -\Big[B^{(d)}\Big((\boldsymbol{x}-\boldsymbol{y}(t'))^2,t-t';T\Big)+C^{(d)}\Big((\boldsymbol{x}-\boldsymbol{y}(t'))^2,t-t';T\Big)\Big]\mathrm{v}^i(t') \\ & -D^{(d)}\Big((\boldsymbol{x}-\boldsymbol{y}(t'))^2,t-t';T\Big)(\boldsymbol{x}-\boldsymbol{y}(t'))^i(\boldsymbol{x}-\boldsymbol{y}(t'))^j \mathrm{v}_j(t')\Big\}\mathrm{d}t' \end{aligned} \tag{71}$$

Now we apply the above theory to an explicit trajectory. We consider a uniformly moving charge with velocity v_0 along the x-direction. Then Equation (69) becomes

$$\mathrm{v} = (1,\mathrm{v}_0,0,\cdots,0) \tag{72}$$

and therefore

$$\boldsymbol{y}(t) = (\mathrm{v}_0 t,0,\cdots,0) \tag{73}$$

If $\mathrm{v}_0 = 0$ then the particle is stationary, $A^0(t,\boldsymbol{x})$ is spherically symmetric and under rotations $A^i(t,\boldsymbol{x})$ takes values on spheres. Otherwise, $A^0(t,\boldsymbol{x})$ takes values on cylinders with axis of symmetry along the x direction and $A^i(t,\boldsymbol{x})$ has a mixed tensorial structure with cylindrical symmetry.

Further, the massive vector field symmetric energy-momentum tensor $\mathrm{T}^{\mu\nu}$ has the form

$$\mathrm{T}^{\mu\nu} = F^{\mu\alpha}F_\alpha{}^\nu + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + m^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A^\alpha A_\alpha\right) \tag{74}$$

At first, we set $d = 3$. *I.e.* we consider a 2 + 1 spacetime. Then the energy density is

$$T^{00} = \frac{1}{2}\left(\boldsymbol{E}^2 + B^2\right) + \frac{1}{2}m^2\sum_{\rho=0}^{2}\left(A^\rho\right)^2 \tag{75}$$

while the Poynting vector is

$$T^{i0} = S^i = \varepsilon^{ij}E^jB + m^2A^iA^0 \tag{76}$$

ε^{ij} is the two-dimensional antisymmetric tensor. We notice that $F^{\mu\nu}(t,\boldsymbol{x}) = \partial^\mu A^\nu(t,\boldsymbol{x}) - \partial^\nu A^\mu(t,\boldsymbol{x})$. Then $F^{00} = 0$, $E^i(t,\boldsymbol{x}) = F^{i0}(t,\boldsymbol{x}) = -F^{0i}(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x}) = F^{12}(t,\boldsymbol{x})$. The latter is a pseudoscalar. We can observe that the Poynting vector equals also the momentum density T^{0i}. Further the space-space components T^{ij} form instead a two-dimensional symmetric tensor representing the momentum flux. They have the form

$$T^{ij} = \frac{1}{2}\left(\boldsymbol{E}^2 + B^2 + A^{\alpha}A_{\alpha}\right)\delta^{ij} - E^i E^j - B^2 + m^2 A^i A^j \tag{77}$$

Further we consider the case $d = 4$. *I.e.* we study the dynamics in a 3 + 1 spacetime. Then the energy density is

$$T^{00} = \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{2}m^2\sum_{\rho=0}^{3}\left(A^{\rho}\right)^2 \tag{78}$$

and the Poynting vector has the form

$$T^{i0} = S^i = \left(\boldsymbol{E}\times\boldsymbol{B}\right)^i + m^2 A^i A^0 \tag{79}$$

Similarly to the previous case after setting $F^{\mu\nu}\left(t,\boldsymbol{x}\right) = \partial^{\mu}A^{\nu}\left(t,\boldsymbol{x}\right) - \partial^{\nu}A^{\mu}\left(t,\boldsymbol{x}\right)$ we get the standard expressions $F^{00} = 0$, $E^i\left(t,\boldsymbol{x}\right) = F^{i0}\left(t,\boldsymbol{x}\right) = -F^{0i}\left(t,\boldsymbol{x}\right)$ and further $F^{ij}\left(t,\boldsymbol{x}\right) = -\varepsilon^{ijk}B^k\left(t,\boldsymbol{x}\right)$. Also, the Poynting vector equals the momentum density T^{0i} while the space-space components T^{ij} form a three-dimensional symmetric tensor representing the momentum flux. They are given as

$$T^{ij} = \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2 + A^{\alpha}A_{\alpha}\right)\delta^{ij} - E^i E^j - B^i B^j + m^2 A^i A^j \tag{80}$$

5. Conclusions

In the present paper we considered the case of a massive vector field, within a finite time interval, described by the Proca equation. We integrated it and extracted its Green function via path integral methods. Further we gave integral as well as series expressions of that Green function and we applied it to the study of the generation of such massive vector fields from currents of various types in the case of spacetime dimension equal to three and four. Those results can be used in the extraction of finite time predictions.

We notice that the present techniques constitute a possible alternative compared for instance to the numerical integration of the dynamic equations or the employment of lattice field theoretic methods.

As far as gauge invariance is concerned, we can observe that the Proca Lagrangian and the Proca equation are not invariant under local gauge transformations due to the present of the mass term. This is related with the high singularity of the present propagator as the mass tends to zero and the non-renormalizability of the theory. The whole point can be circumvented via the idea of spontaneous breaking of gauge symmetry. This is the natural generalization of the spontaneous global symmetry breaking.

In subsequent publications we intend to consider, within finite time intervals, the case of other interacting or free quantum fields and study their dynamics.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: Green Function Numerical Forms

We intend to give expressions for the integrals (61), (62), (64), (65), (66) for spacetime dimension d equal to three and four. In order to perform the angular integrations in all of the expressions below in the case of $d = 3$ we direct the $\boldsymbol{x}$ vector along the x-axis and expand $\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ in term of exponentials and Bessel functions [16] while if $d = 4$ we direct the $\boldsymbol{x}$ vector along the z-axis and expand $\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ in term of Legendre polynomials and spherical Bessel functions [16]. Then we obtain the relations below concerning $G^{(d)00}\left(\boldsymbol{x},\tau;T\right)$, $A^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)$, $B^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)$, $C^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)$ and $D^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)$.

Proceeding if $d = 3$ $G^{(3)00}\left(\boldsymbol{x},\tau;T\right)$ takes the form

$$
\begin{aligned}
&G^{(3)00}\left(\boldsymbol{x},\tau;T\right)\\
&=\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int\frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=\frac{1}{4\pi}\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int_0^{\infty}\mathrm{d}k\,kJ_0\left(k|\boldsymbol{x}|\right)\frac{1}{\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=-i\frac{1}{4\pi}\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}}\exp\left[-im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right]\\
&=-i\frac{1}{4\pi m^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2\right]^{5/2}}\Bigg[2\left(nT+|\tau|\right)^2+|\boldsymbol{x}|^2\\
&\quad+im\left[2\left(nT+|\tau|\right)^2+|\boldsymbol{x}|^2\right]\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}-m^2\left(nT+|\tau|\right)^4\\
&\quad+m^2\left(nT+|\tau|\right)^2|\boldsymbol{x}|^2+m^2\left[\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2\right]^2\Bigg]\exp\left[-im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right]
\end{aligned}
\tag{A1}
$$

To handle the first sum in the third equality we have made the replacement $n+1\rightarrow n$ and then we have set $n\rightarrow -n$. We procced in the same way with the rest of the integrals and perform the same actions in the relevant expressions.

Therefore, if $d = 4$ we get for the corresponding $G^{(4)00}\left(\boldsymbol{x},\tau;T\right)$ the expression

$$
\begin{aligned}
&G^{(4)00}\left(\boldsymbol{x},\tau;T\right)\\
&=\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=\frac{1}{2\pi^2}\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int_0^{\infty}\mathrm{d}k\,k^2j_0\left(k|\boldsymbol{x}|\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=-i\frac{m}{4\pi^2}\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}}K_1\left(im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right)\\
&=-i\frac{1}{4\pi^2m}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2\right]^{5/2}}\Bigg\{im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\\
&\quad\times\left(3\left(nT+|\tau|\right)^2+|\boldsymbol{x}|^2\right)K_0\left(im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right)\\
&\quad+\Bigg[m^2\left[\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2\right]^2+2\left(3\left(nT+|\tau|\right)^2+|\boldsymbol{x}|^2\right)\\
&\quad-m^2\left(nT+|\tau|\right)^2\left(\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2\right)\Bigg]K_1\left(im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right)\Bigg\}
\end{aligned}
\tag{A2}
$$

If we let $d=3$ $A^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ is

$$\begin{aligned}
&A^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)\\
&=-\frac{1}{2m^2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}(\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}e^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=-i\frac{1}{4\pi m^2}\frac{1}{|\boldsymbol{x}|}\int_0^{\infty}\mathrm{d}k\,k^2J_1\left(k|\boldsymbol{x}|\right)\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=\sqrt{\frac{2}{\pi m}}\frac{1}{4\pi}\mathrm{e}^{\frac{i\pi}{4}}\left[\sum_{n=0}^{\infty}\frac{\left((n+1)T-|\tau|\right)K_{\frac{5}{2}}\left(im\sqrt{\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2\right)^{7/4}}\right.\\
&\left.+\sum_{n=0}^{\infty}\frac{\left(|\tau|+nT\right)K_{\frac{5}{2}}\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{7/4}}\right]
\end{aligned}\tag{A3}$$

while $A^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ obeys the following relations if $d=4$

$$\begin{aligned}
&A^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)\\
&=-\frac{1}{2m^2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}(\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=-i\frac{1}{4\pi^2m^2}\frac{1}{|\boldsymbol{x}|}\int_0^{\infty}\mathrm{d}k\,k^3j_1\left(k|\boldsymbol{x}|\right)\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&=-\frac{1}{4\pi^2\sqrt{m}}\mathrm{e}^{-\frac{i\pi}{4}}\left[\sum_{n=0}^{\infty}\frac{\left((n+1)T-|\tau|\right)K_3\left(im\sqrt{\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2\right)^{9/4}}\right.\\
&\left.+\sum_{n=0}^{\infty}\frac{\left(|\tau|+nT\right)K_3\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{9/4}}\right]
\end{aligned}\tag{A4}$$

Moreover $B^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ where $d=3$ becomes

$$\begin{aligned}
&B^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)\\
&=-\frac{1}{m^2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[-(\boldsymbol{k}\cdot\boldsymbol{x})^2+\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&-\frac{1}{4\pi m^2}\int_0^{\infty}\mathrm{d}k\,k^3\left[J_0\left(k|\boldsymbol{x}|\right)+J_2\left(k|\boldsymbol{x}|\right)\right]\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\
&-i\frac{1}{4\pi m^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{3/2}}\left(1+im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)\exp\left(-im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)
\end{aligned}\tag{A5}$$

where we have used the identity

$$J_0\left(k|\boldsymbol{x}|\right)+J_2\left(k|\boldsymbol{x}|\right)=\frac{2}{k|\boldsymbol{x}|}J_1\left(k|\boldsymbol{x}|\right)\tag{A6}$$

Proceeding further if $d=4$, $B^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ is

$$\begin{aligned} &B^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)\\ &=-\frac{1}{2m^2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[-(\boldsymbol{k}\cdot\boldsymbol{x})^2+\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=-\frac{1}{6\pi^2m^2}\int_0^{\infty}\mathrm{d}k\,k^4\left(j_0\left(k|\boldsymbol{x}|\right)+j_2\left(k|\boldsymbol{x}|\right)\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}K_2\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{A7}$$

where we have applied the expression

$$j_0\left(k|\boldsymbol{x}|\right)+j_2\left(k|\boldsymbol{x}|\right)=\frac{3}{k|\boldsymbol{x}|}j_1\left(k|\boldsymbol{x}|\right)=3\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{1}{k|\boldsymbol{x}|}\right)^{3/2}J_{3/2}\left(k|\boldsymbol{x}|\right) \tag{A8}$$

$C^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ corresponding to $d=3$ has the form

$$\begin{aligned} &C^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)\\ &=-\int\frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=-\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}k\,kJ_0\left(k|\boldsymbol{x}|\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=i\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}}\exp\left[-im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right] \end{aligned} \tag{A9}$$

while for $C^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ corresponding to $d=4$ we get

$$\begin{aligned} &C^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)\\ &=-\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=-\frac{1}{2\pi^2}\int_0^{\infty}\mathrm{d}k\,k^2j_0\left(k|\boldsymbol{x}|\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=i\frac{m}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{1}{2}}}K_1\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{A10}$$

For $d=3$ $D^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ obeys

$$\begin{aligned} &D^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)\\ &=-\frac{1}{m^2}\frac{1}{\left(\boldsymbol{x}^2\right)^2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[2(\boldsymbol{k}\cdot\boldsymbol{x})^2-\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=\frac{1}{2\pi m^2}\frac{1}{|\boldsymbol{x}|^2}\int_0^{\infty}\mathrm{d}k\,k^3J_2\left(k|\boldsymbol{x}|\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right]\\ &=-\sqrt{\frac{2m}{\pi}}\frac{1}{4\pi}\mathrm{e}^{-\frac{i\pi}{4}}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{5}{4}}}K_{\frac{5}{2}}\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{A11}$$

and finally, $D^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ corresponding to $d=4$ takes the form

$$\begin{aligned} &D^{(4)}\left(\boldsymbol{x}^2,\tau;T\right) \\ &=-\frac{1}{m^2}\frac{1}{2}\frac{1}{\left(\boldsymbol{x}^2\right)^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[3\left(\boldsymbol{k}\cdot\boldsymbol{x}\right)^2-\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \\ &=\frac{1}{2\pi^2m^2}\frac{1}{|\boldsymbol{x}|^2}\int_0^\infty\mathrm{d}k\,k^4 j_2\left(k|\boldsymbol{x}|\right)\frac{1}{2\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \\ &=i\frac{m}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{3}{2}}}K_3\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{A12}$$

We notice that in the above equations we can apply the identity

$$K_\nu\left(iz\right)=\frac{-\pi i}{2}\mathrm{e}^{-\frac{\pi}{2}\nu i}H_{-\nu}^{(2)}\left(z\right) \tag{A13}$$

In the equations (A1) to (A13) J_l are Bessel functions of the first kind, j_l are spherical Bessel functions, K_l are modified Bessel functions of the third kind and $H_l^{(2)}$ are Hankel functions of the second kind.

Appendix B: Notation

d = Spacetime dimension;
$x_0, \boldsymbol{x}$ = Spacetime coordinates;
m = Particle's mass;
$A_0, \vec{A}$ = Vector field;
$F_{\mu\nu}$ = Curvature tensor;
J_μ = Current;
$g_{\mu\nu}$ = Metric;
$\square$ = D'Alambertian;
$E^i\left(t,\boldsymbol{x}\right)$ = Electric field;
$B\left(t,\boldsymbol{x}\right)$ = Magnetic field;
L = Lagrangian density;
H = Hamiltonian density;
$\boldsymbol{k}$ = Wavevector;
$\omega_{\boldsymbol{k}}$ = Particle's energy;
$a^{(\lambda)+}\left(\boldsymbol{k}\right)$, $a^{(\lambda)}\left(\boldsymbol{k}\right)$ = Creation and annihilation operators;
$\alpha^{(\lambda)*}\left(\boldsymbol{k}\right)$, $\alpha^{(\lambda)}\left(\boldsymbol{k}\right)$ = Coherent states variables;
T = Time;
S_0 = Action;
$G^{(d)\mu\nu}$ = Green function;
v = Velocity;
$T^{\mu\nu}$ = Energy momentum tensor.