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Heuristic Discussion on the Riemann Hypothesis

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Abstract

This study tries to establish a heuristic basis for discussing the Riemann hypothesis. The backbone of this description lies in the use of graphical description and numerical non-linear least squares fitting of the Riemann function non-trivial zeros versus prime numbers.

Keywords

Riemann Function, Riemann Function Non-Trivial Zeros, Riemann Function Auxiliary Zeros, Riemann Hypothesis, Heuristic Assessment of Riemann Hypothesis, Empirical Bijective Map Between Primes and Auxiliary Zeros of the Riemann Function, Diagonal Cartesian Product of Ordered Sets, Non-Linear Least Squares

1. Introduction

The Riemann function [1]:

$$\forall z \in \mathbb{C} : \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} n^{-z} = 1 + 2^{-z} + 3^{-z} + 4^{-z} + \cdots + n^{-z} + \cdots \quad (1)$$

It has been the source of the so-called Riemann hypothesis via its non-trivial zeros.

The problem of an infinite number of such zeros has generated a substantial amount of literature, which, to mention a few, relates the non-trivial zeros of the Riemann zeta function to many other subjects. The connections are mathematical, associated with varied physical content, or both.

1.1. Literature

The starting point for the extensive collection of studies related to the Riemann zeta function is the original reference [1]. Besides, one can quote the volumes of Edwards [2], Mazur and Stein [3], and the nice, apparently naïve, but deeply mathematically rooted book of Derbyshire [4]. Afterward, the large studies by Bombieri [5] and various authors [6]-[8] can be considered a source of comprehensive information. A recent paper claims to have found a heuristic relationship based on Brownian motion [9]. Furthermore, there are almost monthly new publications on this subject, claiming more or less openly the resolution of the Riemann hypothesis problem.

1.2. Graphical Information of Interest

Except for Figure 7, which appears in Conrey's study [10], the present author is aware of no attempt in the literature to obtain a graphical representation of the non-trivial zeros, compared with other numerical sources, such as the prime numbers. Conrey includes a comparison in the mentioned figure, using together three circular representations, consisting of 1) the first 40 Riemann function non-trivial zeros, 2) the eigenvalues of a (40×40) unitary random matrix, and 3) a chosen set of 40 random points.

One can comment that such a last display appears a mystery to the reader because an infinite number of 40 eigenvalues can be obtained from infinite randomly constructed square matrices of the appropriate dimension. Besides, one can indefinitely generate 40 random points that can alternatively be employed in this circular display. Thus, one can ask why these points were chosen to be shown and not others. Sorrowfully, in the present author's opinion, such a representation shows nothing of interest but no more than three different sets of points drawn on three circles of the same radius.

Besides, some authors do not refer to Figure 7 in reference [10] or discuss it within the text of the quoted reference [10], making the display even more puzzling to possible readers.

2. The Existence of a Heuristic Bijection between Non-Trivial Zeros of the Riemann Function and the Prime Numbers

The notes that readers can find below attempt to follow an empirical path to gather a collection of clues about Riemann's hypothesis. The main direction of this route is to try to achieve *heuristic*[1] proof of the existence of a bijection between Riemann non-trivial zeros and prime numbers.

For this purpose, one can write the following facts:

2.1. The Structure of Non-Trivial Zeros

The Riemann hypothesis might be described such that all the non-trivial zeros of

[1]The term *heuristic* is taken here in one of the possible meanings offered by the *Oxford Dictionary*: 1798—"Of, relating to, or enabling discovery or problem-solving, esp. through relatively unstructured methods such as experimentation, evaluation, trial…".

the Riemann zeta function are to be found in the critical line L, defined by:

$$\forall a \in \mathbb{R}^{+} : z = \frac{1}{2} + ia \in \mathrm{L}\,; \tag{2}$$

that is, according to this hypothesis, one can write the structure of the set Z_R of non-trivial zeros of the Riemann zeta function using:

$$\exists z_I \in \mathrm{Z}_R \subset \mathbb{C} : \zeta(z_I) = 0 \Rightarrow z_I = \frac{1}{2} \pm ia_I \wedge a_I \in \mathrm{A} \subset \mathbb{R}^{+}. \tag{3}$$

1) If the set Z_R, identifies the set of complex non-trivial zeros of the Riemann function, then if the Riemann Hypothesis holds, one can also write: $\mathrm{Z}_R \subset \mathrm{L}$.

2) The positive definite real set A, defined in the Equation (3), could be called the set of the Riemann function's auxiliary (non-trivial) zeros.

3) The non-trivial zeros as written in the Equation (3) are observed in pairs, as the complex conjugate of a non-trivial zero is also a zero. That is, the following expression holds:

$$\forall z_I \in \mathrm{Z}_R : \zeta(z_I) = 0 \Rightarrow \exists z_I^{*} \in \mathrm{Z}_R \rightarrow \zeta(z_I^{*}) = 0\,, \tag{4}$$

4) Here, one could consider the role of the set of auxiliary non-trivial zeros A, which, as a real set, is invariant upon conjugation of the Z_R elements. To contemplate this, and taking into account the nature of the complex zeros, the set M of positive modules of the zeros will be used instead:

$$\forall z_I \in \mathrm{Z}_R : m_I = |z_I| = +\left|\frac{1}{4} + a_I^2\right|^{\frac{1}{2}} \rightarrow m_I \in \mathrm{M} \subset \mathbb{R}^{+}. \tag{5}$$

2.2. Heuristics

Heuristic computational evidence [11]-[13] that many non-trivial zeros exist, all within the critical line L.

Thus, one can assume there is already a piece of computational heuristic evidence in the literature pointing towards an infinite number of Riemann function non-trivial zeros in L.

2.3. Prime Numbers

It has been proven in various ways, as seen in [14], starting with Euclid [15] [16], that the number of prime natural numbers is infinite. Such affirmation constitutes the so-called prime number theorem. Several proofs of the theorem can be invoked and are available from various sources, as explained in the well-structured book by Derbyshire [4].

Let us name P the set of primes. The set P can be assumed as an ordered set, that is:

$$\mathrm{P} = \{p_1; p_2; p_3; \cdots; p_N; \cdots\} \subset \mathbb{N} \rightarrow p_1 < p_2 < p_3 < \cdots < p_N < \cdots, \tag{6}$$

being $\mathbb{N}$ the set of natural numbers.

2.4. Bijection

If evidence exists, even empirical[2], showing that a bijective map between the set of prime numbers P and the Riemann zeta function non-trivial zeros is present, then one can write:

$$\begin{aligned}&\forall z_I : \zeta(z_I) = 0 \to z_I \in \mathrm{Z}_R \wedge \forall p_I \in \mathrm{P}\\&\Rightarrow \varphi_Z : \mathrm{Z}_R \to \mathrm{P} \wedge \varphi_Z^{-1} : \mathrm{P} \to \mathrm{Z}_R\end{aligned} \tag{7}$$

However, one can also consider that the positive real set M forms the critical line L. Then alternatively, one can also write, if the previous bijective map in the Equation (7) exists, that there also might exist a bijective map between the sets M and P, like:

$$\varphi : \mathrm{M} \to \mathrm{P} \wedge \varphi^{-1} : \mathrm{P} \to \mathrm{M} \tag{8}$$

2.5. Hipothesis under Diagonal Cartesian Product

One must note that one can consider the set of Riemann function auxiliary zeros A as an ordered set, an order transmitted to the set of the modules of the zeros M. That is:

$$\mathrm{A} = \{a_1; a_2; a_3; \cdots; a_N; \cdots\} \subset \mathbb{R}^+ \to a_1 < a_2 < a_3 < \cdots < a_N < \cdots$$

and therefore

$$\mathrm{M} = \{m_1; m_2; m_3; \cdots; m_N; \cdots\} \subset \mathbb{R}^+ \to m_1 < m_2 < m_3 < \cdots < m_N < \cdots \tag{9}$$

1) One can also hypothesize that a trivial correspondence between the ordered sets M and P might exist.

Then, expressing such a correspondence using the *diagonal Cartesian product*, which is defined now as:

$$\begin{aligned}&\mathrm{M} = \{m_I \,|\, I = 1,2,3,\cdots\} \wedge \mathrm{P} = \{p_I \,|\, I = 1,2,3,\cdots\} \Rightarrow\\&\mathrm{F}_2 = \mathrm{M} * \mathrm{P} = \{f_I = (a_I; p_I) \,|\, I = 1,2,3,\cdots\}\end{aligned} \tag{10}$$

2) The described mathematical construction might be applied to *any pair* of ordered sets. According to the definition (10), the diagonal Cartesian product differs from the usual definition of the Cartesian product. Diagonal product tuples are accepted as elements if they hold a pair of elements of the implied sets containing the same ordering numeral. It is also not difficult to see that the diagonal Cartesian product contains the underlying structure of the so-called inward product of two vectors, for example, [17], equivalent to the so-called Hadamard or diagonal matrix product.

3) As defined in the Equation (10), the resultant diagonal Cartesian product set $\mathrm{F}_2 = \mathrm{M} * \mathrm{P}$, contains ordered pairs of elements, which can supposedly constitute an initial set of two-dimensional points. These points could later be associated

[2]The term *empirical* is taken here in one of the possible meanings offered by the *Oxford Dictionary*: 1588—"That pursues knowledge by means of direct observation, investigation, or experiment (as distinct from deductive reasoning, abstract theorizing, or…

with an implicit form of the map φ in the equation (8). Meanwhile, the reversed diagonal Cartesian product set $G_2 = P * M$, can be associated with the inverse map φ^{-1}.

2.6. Graphical Evidence

The empirical evidence of the existence of some diagonal Cartesian product connected with a bijective map φ between the sets M and P can be easily observed when plotting the points of the diagonal Cartesian product between the sets M and P. Plotting the points of $F_2 = M * P$ presents $\varphi : M \to P$ and doing the same with $G_2 = P * M$ a representation of $\varphi^{-1} : P \to M$ is obtained.

1) Owing to the ordered nature of both M and P sets, one can draw the previous Cartesian products in a three-dimensional fashion using, as shown in **Figure 1**, the modules of the zeros in the Z-axis, the order of natural numbers in the Y-axis, and the ordered prime numbers in the X-axis. Also, **Figure 2** shows a similar arrangement but interchanges the X-axis's role with the Z-axis. Both Figures are constructed using the graphical tool of reference [18]. The increasing size of the points in both figures indicates a greater closeness of the points to the observer in the plot.

2) Such three-dimensional visualizations can be associated with an extension of the two-dimensional diagonal Cartesian products as defined in the equation (10), but now constructed, using the ordered set of natural numbers $\mathbb{N}$, as a new coordinate:

$$F_3 = M * \mathbb{N} * P \wedge G_3 = P * \mathbb{N} * M \quad (11)$$

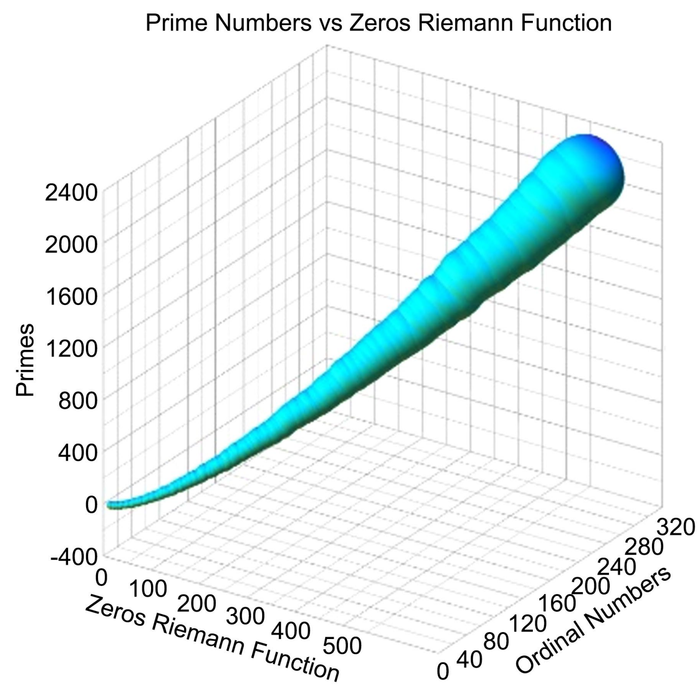

Figure 1. Prime numbers vs auxiliary zeros of the Riemann function for a diagonal Cartesian product of dimension 309.

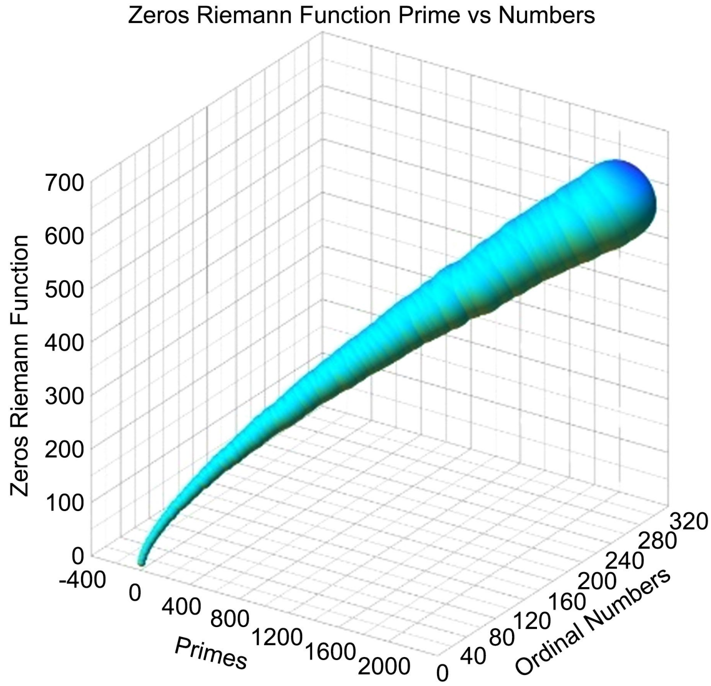

Figure 2. Auxiliary zeros of the Riemann function vs Prime numbers for a diagonal Cartesian product of dimension 309.

2.7. Discussion about the Present Figures

Thus, as shown in Figure 1 and Figure 2 below, an empirical form representing the diagonal Cartesian product exists, which can be associated and visualized like a bijective map between the sets M and P.

This indicates that one can safely consider the number of non-trivial Riemann zeta function zeros on the line L, which might be connected in a one-to-one correspondence with the prime numbers P.

Figure 1 shows a shape resembling the positive branch of the hyperbolic sine, while Figure 2 acquires the form of the positive branch of the hyperbolic tangent.

Consequently, considering the bijection between M and P, the number of non-trivial zeros of the Riemann function could be considered heuristically infinite.

2.8. Evidence of Bijection

Therefore, if the Riemann function's non-trivial zeros set Z_R is shown to be heuristically infinite, there is no need to consider that some could be placed outside the critical line L.

It is sufficient to contemplate the existence of the diagonal Cartesian product associated with the bijective map, involving every module of the Riemann function non-trivial zero in Z_R with every prime number in P.

2.9. Non-Linear Correlations Principles

However, one can obtain a non-linear correlation between the elements of the sets Z_R and P in the way recently described [19] and already used [20] to find a relation between the cardinality of prime numbers and Mersenne numbers.

2.10. Non-Linear Correlations Results

The specific functional form of the plot, which has to involve statistical gear if calculated, will be just briefly resumed here when no order variable is included, leaving a deeper study for further insight, if necessary. The function chosen is of the form:

$$y^{\nu} = ax^{\mu} + b,$$

and numerical results are obtained with the first $N = 128$ primes and auxiliary zeros of the Riemann function. It is interesting to note that a better correlation in the statistical sense:

$$\left(r = 0.9998 \middle| a = 1.0135 \middle| \mu = 1.336 \middle| b = -0.01301 \middle| \nu = 0.8668\right) \quad (12)$$

has been found when the roles of prime numbers and auxiliary zeros of the Riemann function are $\{y, x\}$; while a lower correlation index is found when the roles are reversed as $\{x, y\}$:

$$\left(r = 0.9915 \middle| a = 0.8297 \middle| \mu = 0.5871 \middle| b = 0.2051 \middle| \nu = 0.5871\right). \quad (13)$$

Such a result seems coherent with intuition because representing natural numbers with real numbers yields a better fit than representing real numbers with a set of natural numbers.

However, in both cases, the results shown in Equations (12) and (13) are good heuristic evidence of a non-linear relation between prime numbers and Riemann function zeros, at least for these 128 dimensions.

3. Conclusion

The Riemann hypothesis is admissible from a heuristic point of view. There exists a non-linear correspondence between the auxiliary zeros of the Riemann function and prime numbers that can be computed or easily visualized.

Caution

However, it is unclear whether augmenting the dimensions of the fittings and plots beyond higher dimensions will result in a random succession different from the hyperbolic-like branches shown in **Figure 1** and **Figure 2** or the close non-linear relationships in Equations (12) and (13).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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