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On the (Δ + 2)-Total-Colorability of Planar Graphs with 7-Cycles Containing at Most Two Chords

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Abstract

The Total Coloring Conjecture (TCC) proposes that every simple graph G is (Δ + 2)-totally-colorable, where Δ is the maximum degree of G. For planar graph, TCC is open only in case Δ = 6. In this paper, we prove that TCC holds for planar graph with Δ = 6 and every 7-cycle contains at most two chords.

Keywords

Planar Graph, 7-Cycle, 8-Totally-Colorable, Maximum Degree

1. Introduction

In this paper, we consider only finite, simple, undirected graphs, and follow [1] for the undefined terminology and notation here. Given a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$ (or simply V, E and Δ), respectively. A cycle of length k is called a k-cycle, two cycles C_1 and C_2 are said to *intersecting* (resp., *adjacent*) if C_1 and C_2 share at least one vertex (resp., one edge). A 3-cycle is also called a *triangle*. For a cycle C of G, an edge $xy \in E(G)\setminus E(C)$ is called a *chord* of C if $x, y \in V(C)$.

A k-*total-coloring* of graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is k-*totally-colorable* if it admits a k-total-coloring. Obviously, at least Δ + 1 colors are necessitated to color G totally. Behzad [2] and Vizing [3] independently put forward the following conjecture, which is well known as the *Total Coloring Conjecture* (**TCC**).

Conjecture Every graph is (Δ + 2)-totally-colorable.

Clearly, TCC is true for $\Delta \le 2$. Rosenfeld [4] and Vijayaditya [5] solved it for $\Delta = 3$. Kostochka solved the cases of $\Delta = 4$ [6] and $\Delta = 5$ [7] successively. For planar graphs, more results are known, TCC is true for $\Delta = 7$ [8], and $\Delta \ge 8$ [9]. So the only open case for planar graphs is $\Delta = 6$. While TCC remains unsolved for general planar graphs G with $\Delta = 6$, it is known to hold for some special cases, as follows.

Theorem 1. Let G be a planar graph with $\Delta = 6$. Then G is 8-totally-colorable if one of the following conditions holds.

(1) G contains no adjacent triangles (see [10]).

(2) G contains no chordal k-cycles for some $k \in \{4,5,6\}$ (see [11]).

(3) For every vertex x of G, there is an integer $k_x \in \{3,4,5,6,7,8\}$, such that x is not in any k_x-cycle (see [12]).

(4) $v_5^4 + 2\left(v_5^{5^+} + v_6^4\right) + 3v_6^5 + 4v_6^{6^+} < 24$, where v_n^k denotes the number of vertices of degree n that lie on k distinct triangles in G (see [13]).

(5) G contains no any subgraph isomorphic to a 4-fan (see [14]).

(6) G is claw-free (see [15]).

(7) G is recursive maximal (see [16]).

In this paper, we obtain the following result.

Theorem 2. Let G be a planar graph with $\Delta = 6$. If every 7-cycle of G contains at most two chords, then G is 8-totally-colorable.

On the other hand, Shen *et al.* [17] conjectured that every planar graph is (Δ + 1)-totally-colorable, for $4 \le \Delta \le 8$. This first result was given in [18] for $\Delta \ge 14$, which was finally improved to $\Delta \ge 9$ in [19]. Some related results can be found in [20]-[30].

Now, we define some more definitions and notations. A vertex of degree k, at most k or at least k is called a k*-vertex*, k^-*-vertex* or a k^+*-vertex*, respectively. A k*-face* (resp., k^-*-face*, k^+*-face*) is a face of degree k (resp., at most k, at least k). A face $f = (v_1, v_2, \cdots, v_k)$ with consecutive vertices $v_1, v_2, \cdots, v_k$ along its boundary is called a $(d(v_1), d(v_2), \cdots, d(v_k))$*-face*. Denote by $f_k(v)$ the number of k-faces incident with v, by $f_b(v)$ the number of $(4,5,6)$-faces incident with v.

2. Proof of Theorem 2

Let $G = (V, E, F)$ be a minimal counterexample to Theorem 2, such that $|V| + |E|$ is minimum. Thus every proper subgraph of G admits an 8-total-coloring. Embed G into the plane, and denoted face set of G by F. We first show some structure properties of G.

Lemma 3. ([10]) (1) G is 2-connected.

(2) G contains no edge uv with $\min\{d(u), d(v)\} \le 3$ and $d(u) + d(v) \le 8$.

By Lemma 3(1), if f is a face of G, then the boundary of f is a cycle. By Lemma 3(2), $\delta \ge 3$ and if v is 3-vertex of G, then the three neighbors of v are all 6-vertices.

Lemma 4. ([13]) (1) G contains no $(3,6,6)$-triangle.

(2) G contains no $(4,4,6)$-triangle.

(3) G contains no $(4,5^-,5^-)$-triangle.

Lemma 5. ([13] [14]) If (v_1,v_2,v_3) is a triangle in *G*, where $d(v_1)=4$, $d(v_2)=5$ and $d(v_3)=6$, then

(1) both v_1v_2 and v_1v_3 are only incident with one triangle, *i.e.* (v_1,v_2,v_3).

(2) v_2v_3 is only incident with one $(4,5,6)$-triangle, *i.e.* (v_1,v_2,v_3).

(3) v_3 is not adjacent to any 3-vertex.

Lemma 6. ([13]) (1) G contains no 4-face incident with more than one 3-vertex.

(2) G contains no 5-face incident with more than one 3-vertex.

We shall complete the proof of Theorem 2 by using the discharging method. According to the Euler's formula $|V|-|E|+|F|=2$, we have

$$\sum_{v\in V}(2d(v)-6)+\sum_{f\in F}(d(f)-6)=-12<0.$$

For each $x\in V\cup F$, we define $ch(x)$ to be the initial charge of x. In particular, $ch(x)=2d(x)-6$ ($x\in V$), $ch(x)=2d(x)-6$ ($x\in F$). Obviously, $\sum_{x\in V\cup F}ch(x)=-12$. Then, we will apply the following discharging rules to reassign a new charge denoted by $ch'(x)$. If we can get $ch'(x)\ge 0$, then we obtain an obvious contradiction to $0\le\sum_{x\in V\cup F}ch'(x)=-12$, and complete the proof.

For a face $f=(v_1,v_2,\cdots,v_k)$ of *G*, we use

$$(d(v_1),d(v_2),\cdots,d(v_k))\to(c_1,c_2,\cdots,c_k)$$

to denote that v_i sends c_i to f for $i=1,2,\cdots,k$. Define the discharging rules that we need as follows.

R1. For $f=(v_1,v_2,v_3,v_4,v_5)$, we have

$$(3,6,4^+,4^+,6)\to\left(0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),$$

$$(4^+,4^+,4^+,4^+,4^+)\to\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right).$$

R2. For $f=(v_1,v_2,v_3,v_4)$, we have

$$(3,6,4^+,6)\to\left(0,\frac{3}{4},\frac{1}{2},\frac{3}{4}\right),$$

$$(4^+,4^+,4^+,4^+)\to\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

R3. For any 5-vertex v and $f=(v_1,v_2,v_3)$, let $\eta(v)=\left(ch(v)-\frac{1}{2}f_4(v)-\frac{1}{4}f_5(v)\right)\Big/f_3(v)$. In addition,

$$(4,5,6)\to\left(\frac{1}{2},\eta(v_2),\frac{5}{2}-\eta(v_2)\right),$$

$$(4,6,6)\to\left(\frac{1}{2},\frac{5}{4},\frac{5}{4}\right),$$

$$(5,5,5)\to(\eta(v_1),\eta(v_2),\eta(v_3)),$$

$$(5,5,6) \to (\eta(v_1), \eta(v_2), 3-\eta(v_1)-\eta(v_2)),$$

$$(5,6,6) \to \left(\eta(v_1), \frac{3-\eta(v_1)}{2}, \frac{3-\eta(v_1)}{2}\right),$$

$$(6,6,6) \to (1,1,1).$$

For every vertex v and every face f, we will check that $ch'(v) \geq 0$ and $ch'(f) \geq 0$. Denote by $c(v \to f)$ the total charge from v to f.

Let $f \in F$. If f is a 6^+-face, then $ch'(f) = ch(f) = d(f) - 6 \geq 0$. If f is a 5-face, then $ch'(f) \geq -1 + \min\left\{4 \times \frac{1}{4}, 5 \times \frac{1}{5}\right\} = 0$ by Lemma 3, Lemma 6(2) and R1. If f is a 4-face, then $ch'(f) \geq -2 + \min\left\{2 \times \frac{3}{4} + \frac{1}{2}, 4 \times \frac{1}{2}\right\} = 0$ by Lemma 3, Lemma 6(1) and R2.

Suppose $f = (v_1, v_2, v_3)$ is a 3-face. If (v_1, v_2, v_3) is not a $(5,5,5)$-face, then $ch'(f) \geq -3 + 3 = 0$ by Lemma 3, Lemma 4 and R3. Now we consider the case that (v_1, v_2, v_3) is a $(5,5,5)$-face, that is, v_1, v_2 and v_3 are all 5-vertices. By R3, v_i transfers at least $\frac{4}{5}$ to each 3-face incident with v_i $(i = 1,2,3)$. By R1 and R2, v_i transfers at most $\frac{1}{2}$ to each 4^+-face incident with v_i $(i = 1,2,3)$. Thus if $f_3(v_i) \leq 3$, then $c(v_i \to f) \geq \left(4 - 2 \times \frac{1}{2}\right) \Big/ 3 = 1$ $(i = 1,2,3)$, we have $ch'(f) \geq -3 + 3 \times 1 = 0$. Otherwise, without loss of generality (WLOG), assume $f_3(v_1) \geq 4$, it follows that both v_2 and v_3 must be incident with at least two 5^+-faces, *i.e.* $f_{5^+}(v_2) \geq 2$ and $f_{5^+}(v_3) \geq 2$. Therefore,

$$c(v_i \to f) \geq 4 - 2 \times \frac{1}{4} \Big/ 3 = \frac{7}{6} \quad (i = 2,3),$$

we have $ch'(f) \geq -3 + \frac{4}{5} + \frac{7}{6} + \frac{7}{6} = \frac{2}{15} > 0$.

Let $v \in V$. If v is a 3-vertex, then $ch'(v) = ch(v) = 2d(v) - 6 = 0$. If v is a 4-vertex or 5-vertex, then $ch'(v) \geq 0$ according to the above discharging rules.

Suppose v is a 6-vertex. Let $v_1, v_2, \cdots, v_6$ be all the neighbors of v and $f_1, f_2, \cdots, f_6$ be all the faces incident with v, where f_m is incident with v_m, v_{m+1}, and $m \in \{1, 2, \cdots, 6\}$. In general, each subscript of this paper is taken modulo 6. First, we give two lemmas.

Lemma 7. If (v_i, v_{i+1}, v) and (v_j, v_{j+1}, v) are two triangles in *G*, where $d(v_i) = 4$, $d(v_{i+1}) = 5$ and $d(v) = 6$, then both v_j and v_{j+1} are 5^+-vertices.

Proof. By Lemma 3 and Lemma 4, we have min $\{d(v_j), d(v_{j+1})\} \geq 4$ and max $\{d(v_j), d(v_{j+1})\} \geq 5$. Assume to be contrary that $d(v_j) = 4$ or $d(v_{j+1}) = 4$. WLOG, suppose that $d(v_j) = 4$. Let $N(v_i) = \{v_{i+1}, v, u_1, u_2\}$ and $N(v_{i+1}) = \{v_i, v, u_3, u_4, u_5\}$. We only need to consider the case that $v_j \neq v_i$ and $v_{j+1} \neq v_{i+1}$ by Lemma 5, see Figure 1(a).

The proof consists of two steps. First, we show that *G* has a partial 8-total-coloring in which only v_i is not colored. Second, we show how to extend it to an 8-total-coloring of *G*. If φ is a (partial) 8-total-coloring of *G*, $v \in V$, let $S(v)=\{\varphi(vx) \mid x \in N(v)\}$, $S[v]=S(v)\cup\{\varphi(e)\}$. Consider a proper subgraph $G'=G-v_iv_{i+1}$ of *G*, G' has a 8-total-coloring φ:

$$V(G')\cup E(G')\to L=\{1,2,\cdots,8\}.$$

Erase the colors on v_i. Clearly, $|S(v_i)|=3$, $|S[v_{i+1}]|=5$ and $|S[v]|=7$. Let $L\setminus S[v]=\{\alpha\}$. If $S(v_i)\cap S[v_{i+1}]\neq\varnothing$, then there are at most 7 forbidden colors for v_iv_{i+1}, so v_iv_{i+1} can be properly colored. Thus we can assume that $S(v_i)\cap S[v_{i+1}]=\varnothing$ (*i.e.* $S(v_i)\cup S[v_{i+1}]=L$), so $\alpha\in S(v_i)$ or $\alpha\in S[v_{i+1}]$. If $\alpha\in S(v_i)$, then we color v_iv_{i+1} with $\varphi(vv_{i+1})$, and recolor vv_{i+1} with α. Otherwise, we color v_iv_{i+1} with $\varphi(vv_i)$, and recolor vv_i with α. Hence, *G* has such an 8-total-coloring.

Suppose now that f is a partial 8-total-coloring of *G* in which only v_i is uncolored. If we cannot extend f to *G*, then there are exactly 8 forbidden colors for v_i. WLOG, suppose that $f(u_1)=1$, $f(u_2)=2$, $f(v_{i+1})=3$, $f(v)=4$, $f(v_iu_1)=5$, $f(v_iu_2)=6$, $f(v_iv_{i+1})=7$, $f(vv_i)=8$, see **Figure 1(b)**. Also let $L\setminus S[v]=\{\alpha\}$. Obviously, $\{1,2,3,4\}\subseteq(S[v_{i+1}]\cap S[v])$. Otherwise, we can choose a color $\beta\in\{1,2,3,4\}\setminus S[v_{i+1}]$ or $\beta\in\{1,2,3,4\}\setminus S[v]$, recolor v_iv_{i+1} or v_iv with β, color v_i with 7 or 8. Note that $f(vv_{i+1})\in\{1,2,5,6\}$.

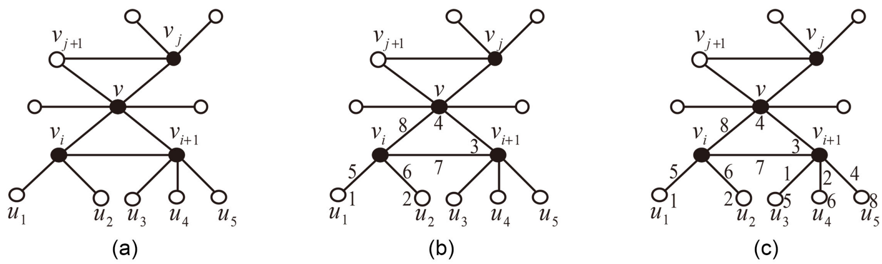

Figure 1. Configurations for the proof of Lemma 7, where vertices marked by • have no other neighbors.

Suppose $f(vv_{i+1})\in\{1,2\}$. If $7\notin S[v]$ or $8\notin S[v_{i+1}]$, then we exchange $f(v_iv_{i+1})$ and $f(vv_{i+1})$, or $f(vv_i)$ and $f(vv_{i+1})$, color v_i with 7 or 8. Otherwise, $S[v_{i+1}]=\{1,2,3,4,7,8\}$ and $\{1,2,3,4,7,8\}\subseteq S[v]$, it follows that $S[v_{i+1}]\subseteq S[v]$, we can recolor vv_{i+1} with α, v_iv_{i+1} with $f(vv_{i+1})$, color v_i with 7. Suppose $f(vv_{i+1})\in\{5,6\}$. WLOG, assume $f(vv_{i+1})=5$ (*i.e.* $S[v_{i+1}]=\{1,2,3,4,5,7\}$ and $\{1,2,3,4,5,8\}\subseteq S[v]$). Then $\{6,8\}\subseteq\{f(u_3),f(u_4),f(u_5)\}$, for otherwise, we can recolor v_{i+1} with 6 or 8, color v_i with 3. Note that $\alpha\in\{6,7\}$.

If $\alpha=7$ (*i.e.* $S[v]=\{1,2,3,4,5,6,8\}$), then $5\in\{f(u_3),f(u_4),f(u_5)\}$. Otherwise, we can recolor vv_{i+1} with 7, v_{i+1} with 5, v_iv_{i+1} with 3, and color v_i with 7. Now we exchange $f(vv_i)$ and $f(v_iv_{i+1})$, recolor v_{i+1} with 7, color v_i with 3. In the rest of this proof we assume that $\alpha=6$, then $S[v]=\{1,2,3,4,5,7,8\}$ and $\{f(vv_j),f(vv_{j+1})\}\cap\{5,6,8\}=\varnothing$, this situation is very complicated. First, we have $5\in\{f(u_3),f(u_4),f(u_5)\}$, for otherwise, we can recolor vv_{i+1} with 6, v_{i+1} with 5, and color v_i with 3. WLOG, suppose that $f(v_{i+1}u_3)=1$, $f(v_{i+1}u_4)=2$, $f(v_{i+1}u_5)=4$, $f(u_3)=5$, $f(u_4)=6$, $f(u_5)=8$, see Figure 1(c). Second, we have $\{5,6,8\}\subseteq S[v_j]$, for otherwise, we can choose a color $\beta\in\{5,6,8\}\setminus S[v_j]$, recolor vv_j with β, vv_i with $f(vv_j)$, color v_i with 8 (here, we need to further recolor vv_{i+1} with 6 when $\beta=5$, and recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$). Similarly, $\{5,6,8\}\subseteq S[v_{j+1}]$. Let $f(v_jv_{j+1})=\gamma$. In terms of γ, there are the following two cases.

Case 1. $\gamma\in\{5,6,8\}$.

Then $f(vv_j)\in S[v_{j+1}]$. Otherwise, we can exchange $f(vv_j)$ and $f(v_jv_{j+1})$, recolor vv_i with $f(vv_j)$, color v_i with 8 (here, we need to further recolor vv_{i+1} with 6 when $\gamma=5$, and recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$). Similarly, $f(vv_{j+1})\in S[v_j]$. Thus $S[v_j]\subseteq S[v_{j+1}]$, we can choose a color $\beta\in L\setminus S[v_{j+1}]$, recolor v_jv_{j+1} with β, vv_j with γ, vv_i with $f(vv_j)$, color v_i with 8 (here, we need to further recolor vv_{i+1} with 6 when $\gamma=5$, and recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$).

Case 2. $\gamma\notin\{5,6,8\}$.

Then $f(v_j)\in\{5,6,8\}$. If $\gamma\in\{f(u)\,|\,u\in N(v_j)\}$, then we choose $\beta\in\{1,2,3,4,7\}\setminus\{f(u)\,|\,u\in N(v_j)\}$, recolor v_j with β, vv_j with $f(v_j)$, vv_i with $f(vv_j)$, color v_i with 8 (here, we need to further recolor vv_{i+1} with 6 when $f(v_j)=5$, and recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$). Otherwise, $\gamma\notin\{f(u)\,|\,u\in N(v_j)\}$. If $f(vv_j)\in S[v_{j+1}]$, then $S[v_j]\subseteq S[v_{j+1}]$, we choose a color $\beta\in L\setminus S[v_{j+1}]$, recolor v_jv_{j+1} with β, v_j with γ, vv_j with $f(v_j)$, vv_i with $f(vv_j)$, color v_i with 8 (here, we need to further recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$). Otherwise, $f(vv_j)\notin S[v_{j+1}]$. Now we recolor v_jv_{j+1} and vv_i with $f(vv_j)$, v_j with γ, vv_j with $f(v_j)$, color v_i with 8 (here, we need to further recolor vv_{i+1} with 6 when $f(v_j)=5$, and recolor v_iv_{i+1} with 3 and v_{i+1} with 7 when $f(vv_j)=7$).

All of the above show that G is 8-totally-colorable, a contradiction. □

Lemma 8. Suppose $f_i=(v_i,v_{i+1},v)$ is a triangle with $d(v)=6$, we have

(1) If $d(v_i)=4$ and $d(v_{i+1})=5$, then $c(v\to f_i)\le\frac{13}{8}$.

(2) If $d(v_i)=d(v_{i+1})=5$, then $c(v\to f_i)\le\frac{31}{30}$.

Proof. (1) Note that $f_3(v_{i+1})\le 4$ by Lemma 5(1), we have $c(v_{i+1}\to f_i)\ge\left(4-\frac{1}{2}\right)\Big/4=\frac{7}{8}$ by R4, so $c(v\to f_i)\le\frac{5}{2}-\frac{7}{8}=\frac{13}{8}$.

(2) If $f_3(v_i)\le 3$ and $f_3(v_{i+1})\le 3$, then $c(v_i\to f_i)\ge\left(4-2\times\frac{1}{2}\right)\Big/3=1$ and $c(v_{i+1}\to f_i)\ge\left(4-2\times\frac{1}{2}\right)\Big/3=1$, so $c(v\to f_i)\le 3-1-1=1$. Otherwise, WLOG, assume $f_3(v_i)\ge 4$. Then $f_{5^+}(v_{i+1})\ge 2$ by the choice of G, we have

$$c(v_{i+1}\to f_i)\ge\left(4-2\times\frac{1}{4}\right)\Big/3=\frac{7}{6},$$

so $c(v \to f_i) \le 3 - \frac{4}{5} - \frac{7}{6} = \frac{31}{30}$. Since $\max\left\{1, \frac{31}{30}\right\} = \frac{31}{30}$, $c(v \to f_i) \le \frac{31}{30}$. □

Now, we begin to show that $ch'(v) \ge 0$ for 6-vertex v. Note that $f_b(v)$ is 0 or 1 by Lemma 7, $f_3(v) \le 4$ by the choice of *G*. In terms of $f_3(v)$, there are the following four cases.

Case 1. $f_3(v) \le 1$. Note that v transfers at most $\frac{13}{8}$ to each 3-face incident with v by Lemma 8, at most $\frac{3}{4}$ to each 4^+-face incident with v by R1 and R2. So $ch'(v) \ge 6 - f_3(v) \times \frac{13}{8} - (6 - f_3(v)) \times \frac{3}{4} = \frac{3}{2} - \frac{7}{8} f_3(v) > 0$.

Case 2. $f_3(v) = 2$. If $f_b(v) = 1$, then another 3-face is a $(5^+, 5^+, 6)$-triangle by Lemma 7, it follows that $ch'(v) \ge 6 - \frac{13}{8} - \frac{31}{30} - 4 \times \frac{3}{4} = \frac{41}{120} > 0$. Otherwise, $f_b(v) = 0$, then v transfers at most $\frac{5}{4}$ to each 3-face incident with v by Lemma 8 and R2. Hence, $ch'(v) \ge 6 - 2 \times \frac{5}{4} - 4 \times \frac{3}{4} = \frac{1}{2} > 0$.

Case 3. $f_3(v) = 3$. If $f_b(v) = 1$, then

$$ch'(v) \ge 6 - \frac{13}{8} - 2 \times \frac{31}{30} - 3 \times \frac{3}{4} = \frac{7}{120} > 0.$$

Otherwise, $f_b(v) = 0$, we have $ch'(v) \ge 6 - 3 \times \frac{5}{4} - 3 \times \frac{3}{4} = 0$.

Case 4. $f_3(v) = 4$. Note that either $f_4(v) = f_{6^+}(v) = 1$ or $f_{5^+}(v) = 2$. If $f_b(v) = 1$, then $ch'(v) \ge 6 - \frac{13}{8} - 3 \times \frac{31}{30} - \max\left\{\frac{3}{4}, 2 \times \frac{1}{4}\right\} = \frac{21}{40} > 0$. Otherwise, $f_b(v) = 0$, we have $ch'(v) \ge 6 - 4 \times \frac{5}{4} - \max\left\{\frac{3}{4}, 2 \times \frac{1}{4}\right\} = \frac{1}{4} > 0$.

3. Conclusion

In conclusion, the proof of the Theorem 2 is now complete.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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