Journal of Applied Mathematics and Physics, 2024, 12, 1522-1540
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Momentum as Translations at Conformal Infinity

Richard James Petti[1], **Jacob Luke Graham**[2]

[1]Arlington, MA, USA
[2]Newton, MA, USA
Email: rjpetti@gmail.com, jlg462@cornell.edu, jakegraham7777@gmail.com

How to cite this paper: Petti, R.J. and Graham, J.L. (2024) Momentum as Translations at Conformal Infinity. *Journal of Applied Mathematics and Physics*, **12**, 1522-1540.
https://doi.org/10.4236/jamp.2024.124093

Received: February 11, 2024
Accepted: April 27, 2024
Published: April 30, 2024

Abstract

Although General Relativity is the classic example of a physical theory based on differential geometry, the momentum tensor is the only part of the field equation that is not derived from or interpreted with differential geometry. This work extends General Relativity and Einstein-Cartan theory by augmenting the Poincaré group with projective (special) conformal transformations, which are translations at conformal infinity. Momentum becomes a part of the differential geometry of spacetime. The Lie algebra of these transformations is represented by vectorfields on an associated Minkowski fiber space. Variation of projective conformal scalar curvature generates a 2-index tensor that serves as linear momentum in the field equations of General Relativity. The computation yields a constructive realization of Mach's principle: local inertia is determined by local motion relative to mass at conformal infinity in each fiber. The vectorfields have a cellular structure that is similar to that of turbulent fluids.

Keywords

Projective Symmetry, Conformal Symmetry, Momentum, General Relativity, Einstein-Cartan Mach's Principle

1. Introduction

This work extends General Relativity (GR) and Einstein-Cartan (EC) theory by augmenting the structure group with projective (special) conformal transformations, which are translations at conformal infinity [1]. The Lie algebra of the structure group is represented by vectorfields on an associated Minkowski affine fiber space. Metricity is preserved at the origin of each Minkowski fiber because projective conformal connection coefficients vanish at the origin of each fiber.

Variation of projective conformal scalar curvature generates a 2-index tensor that serves as the momentum in the field equations of GR. Momentum gains an interpretation as part of the differential geometry of spacetime, whereas gravitational theory normally treats momentum as an exogeneous term without an interpretation in differential geometry.

The computation yields a constructive realization of Mach's principle: local inertia is determined by local motion relative to mass at conformal infinity in each fiber.

1.1. The Relation of This Work to General Relativity and Einstein-Cartan Theory

The structure group of GR is the Lorentz group. The structure group of EC is the Poincaré group with an affine Minkowski space as the associated fiber. The structure group of this work is the Poincaré group augmented by including the projective conformal transformations.

The basic computations to derive the field equations in this work are analogous to those in GR and EC

1) Compute the scalar curvature, which in this work contains projective conformal terms.

2) Define action as the integral of scalar curvature. Variation of the action with respect to the frame field on spacetime yields the gravitational field equations. Variation of the projective conformal terms in the action generates a 2-tensor that enters as the momentum tensor in the field equations of GR and EC.

In GR and EC, the momentum tensor is exogenous to the differential geometry: it is not derived from the geometry nor does it have an interpretation in differential geometry. In this work, the momentum tensor is derived from the differential geometry and is part of the geometry. This difference makes the computations more complicated than in GR or EC. The computations are segregated in **Appendix**, so they do not obscure the simple analogy with GR and EC. The computations are performed using computer algebra software, which provides a complete and verifiable record of the computations and contains many text comments that provide a roadmap through the computations.

1.2. The Relation of This Work to Conformal Field Theory

Some sources define the term "Conformal Field Theory" (CFT) as shorthand for "Conformal Quantum Field Theory" (CQFT). Other sources include classical conformal field theory such as the results herein. Most of the research in CFT today is CQFT: the mathematical structure is usually based on infinite dimensional Lie algebras in finite dimensional spacetimes that include holomorphic and meromorphic functions. The usual application domains of CFT are phase changes, critical points, and boundary effects in materials, and high-energy quantum field theory. CFT is usually restricted to massless fields because the global conformal structure of CFT is not compatible with mass terms [2] [3]. This work has no such restriction, because the conformal symmetry is local, contained in the connection form.

The conformal structure in this work focuses on momentum tensors in classical gravitational theories. The only algebra is the Poincaré group of EC augmented by classical projective conformal transformations. This approach distills a small part of CFT to provide an answer to basic questions in gravitational theory: What is the geometric interpretation of momentum in classical gravitational theories, and does momentum have an interpretation in differential geometry?

This work is accessible to experts in gravitational theory and differential geometry with no background in CFT.

The remainder of this article has six parts.

- Section 2: Affine connections on diffeomorphism bundles.
- Section 3: Lagrangian and field equations.
- Section 4: Linear momentum as a Noether current.
- Section 5: Mach's principle.
- Section 6: Flow of projective conformal vectorfields.
- Section 7: Summary.
- **Appendix**: Computer algebra computation of projective conformal action and its variation.

2. Affine Connections on Diffeomorphism Bundles

2.1. Affine Connections on Linear Minkowski Bundles

2.1.1. Bundles

Ξ = smooth spacetime manifold of dimension dim, coordinates $\{\xi^\mu, \mu = 1\ldots\text{dim}\}$.

$T\Xi$ = tangent bundle of Ξ. $T_\xi\Xi$ = tangent fiber over $\xi \in \Xi$.

X = Minkowski associated fiber with affine coordinates $\{x^i, i = 1\ldots\text{dim}\}$, fixed origin point o in each fiber X_ξ = fiber over $\xi \in \Xi$. X is a flat affine manifold that supports translations, not a vector space.

TX = linear tangent bundle of the affine space X. T_xX = tangent fiber over point $x \in X$.

SO(p, q) = standard orthogonal group acting on X; (p + q = dim).

g_{ij} = Minkowski metric on X with constant components in Minkowski coordinates.

H = Poincaré structure group Affine(p, q), p + q = dim. Lie(H) = Lie algebra of H.

P = principal fiber bundle with base manifold Ξ and structure group H.

2.1.2. Connection 1-Forms

For a base manifold of dimension dim, the structure group H = Affine(p, q) and Lie(H) have dimension dim(dim + 1)/2. The basis of Lie(H) can be chosen to consist of dim independent infinitesimal translations and dim(dim − 1)/2 independent infinitesimal rotations in the associated representation space $X = R^{\text{dim}}$.

ω = affine connection 1-form on P.

$K_\mu{}^i(\xi)$ = translational connection coefficients on X and a frame field on Ξ. $K_\mu{}^i(\xi)$ is invertible with inverse $K_i{}^\mu(\xi)$.

$B_\mu{}^i{}_j(\xi)$ = rotational connection coefficients. Metricity requires that the symmetric form $B_{\mu(ij)} = 0$.

When moving in ξ^μ direction in spacetime, $B_\mu{}^i{}_j(\xi)\, x^j$ defines a rotation centered at $K_\mu{}^i(\xi) \in X$.

$\Gamma_\mu{}^i(\xi, x) = K_\mu{}^i(\xi) + B_\mu{}^i{}_j(\xi)\, x^j$ = connection 1-form at $\xi \in \Xi$ and $x \in X$, in vectorfield form.

Viewing $\Gamma_\mu{}^i(\xi, x)$ as a vectorfield on X is isomorphic to the conventional view in terms of translational and rotational connection coefficients. The vectorfield view provides better insight when projective conformal transformations are included, and more so when working with general diffeomorphism bundles. (An example of a diffeomorphism bundle is one whose associated fiber is a space of constant curvature with no preferred origin point.)

$g_{\mu\nu} = K_\mu{}^i(\xi)\, g_{ij}\, K_\nu{}^j(\xi)$ = metric on spacetime Ξ pulled back from X by the frame field $K_\mu{}^i(\xi)$.

2.2. Affine Connection Augmented by Projective Conformal Transformations

The structure group augmented by projective conformal transformations is denote by HC and its Lie algebra as Lie(HC).

Each projective conformal transformation can be defined as a composition of three simpler transformations.

- A conformal inversion on each Minkowski fiber maps the origin $o \in X$ to conformal infinity and vice versa.
- A translation $C_\mu{}^i$ at origin $o \in X$.
- A second conformal inversion that maps the origin $o \in X$ to conformal infinity and vice versa.

As a vectorfield, this transformation is a translation at conformal infinity, without introducing points at infinity.

The connection 1-form $\Gamma_\mu{}^i(\xi, x)$, including projective conformal transformations, in coordinate form is:

$$\Gamma_\mu{}^i(\xi, x) = K_\mu{}^i(\xi) + B_\mu{}^i{}_j(\xi)\, x^j + C_\mu{}^k(\xi)\left(x^2 \delta_k{}^i - 2x_k x^i\right) \quad \text{(Apx, A-2)} \qquad (1)$$

In general, each independent vectorfield that is added to Lie(H) adds a term in the connection 1-form.

The geometry discussed in this work is precisely this case: the affine structure includes projective conformal transformations and the associated fiber has a preferred origin point. The connection forms and curvature tensors have only two explicit indices.

$$\Gamma_\mu{}^i(\xi, x),\quad R^i_{\mu\nu}(\xi, x) \qquad (2)$$

The model includes a preferred origin point because all local geometry and physics occurs at the point x = 0 ∈ X. For example, the curvature that is relevant for gravitation is $R_{\mu\nu}i$ (ξ, x = 0).

One might ask why the affine fiber space X is retained if all local physics occurs at x = 0. This question has several answers.

- A linear connection retains the linear fiber space in order to represent rotational transformations.
- An affine connection retains the affine fiber X to represent the translational transformations.
- In the theory of conformal structure of quantum wave mechanics, the connection coefficient on the fiber space X contains the physics of the Klein-Gordon field in the fiber [4]. This enables the differential geometry to incorporate the Klein-Gordon equation. Without the fiber space, the geometry cannot contain the physics of the KG field.
- The theory herein retains the associated Minkowski fiber X to represent the projective conformal inversions without introducing any points at conformal infinity. The projective conformal vectorfield on fiber contains the information needed to model conformal inversion.

The computational rule for covariant derivatives and curvature in this theory is to treat the fiber coordinate x as the third index, or more precisely, the partial derivative with respect to x acts as the third explicit index.

Covariant derivates in Riemannian geometry have the form:

$$\nabla_\mu x^i(\xi) = \partial_\mu x^i(\xi) + \Gamma^i_{\mu j}(\xi) x^j(\xi) \tag{3}$$

whereas in this theory:

$$\nabla_\mu x^i(\xi) = \partial_\mu x^i(\xi) + \Gamma^i_\mu(\xi, x(\xi)) \tag{4}$$

In local bundle coordinates, curvature in Riemannian geometry is:

$$R^i_{\mu\nu j}(\xi) = \partial_\mu \Gamma^i_{\nu j}(\xi) - \partial_\nu \Gamma^i_{\mu j}(\xi) + \Gamma^i_{\mu k}(\xi)\Gamma^k_{\nu j}(\xi) - \Gamma^i_{\nu k}(\xi)\Gamma^k_{\mu j}(\xi) \tag{5}$$

In this theory, the fiber index j:

$$\begin{aligned} R^i_{\mu\nu}(\xi, x) = {} & \partial_\mu \Gamma_\nu{}^i(\xi, x) - \partial_\nu \Gamma_\mu{}^i(\xi, x) + \left(\partial_{x^j}\right)\left(\Gamma_\mu{}^i(\xi, x)\right)\Gamma_\nu{}^j(\xi, x) \\ & - \left(\partial_{x^j}\right)\left(\Gamma_\nu{}^i(\xi, x)\right)\Gamma_\mu{}^j(\xi, x) \end{aligned} \tag{6}$$

The formula in Equation (6) is implemented in **Appendix** in Section 1 for connection coefficients and in Section 2.1 for the full curvature tensor.

2.3. Curvature Tensor in Coordinate Form

Local physics uses vectors, connection coefficients, and curvature only at the origin x = 0 ∈ X. The projective conformal symmetries affect curvature, but not the connection coefficients, at x = 0.

We compute the projective conformal scalar curvature in seven steps.

Step 1: Compute the full curvature in vectorfield form, denoted full_curv_vect in (Apx, c4). It is not displayed because it has 38 terms, it is not formatted for viewing, and it is used only to compute full curvature at 0 ∈ X.

Step 2: Compute the full curvature in 4-index form denoted $\text{full\_curv}_{\mu\nu j}{}^{i}$, (in (Apx, c5). This is not displayed.

Step 3: Compute the full curvature at x = 0, denoted $\text{full\_curv\_x0}_{\mu\nu j}{}^{i}(\xi\ 0)$ in (Apx, c6).

Step 4: Compute projective conformal curvature at x = 0, denoted proj_conf_cur_x0, in (Apx, d8).

Step 5: Compute projective conformal Ricci curvature at x = 0, denoted proj_conf_riccicurv_x0 in (Apx, d9).

Step 6: Compute projective conformal scalar curvature at x = 0, denoted proj_conf_scalar_curv_x0 in (Apx, d10).

Step 7: Simplify the projective conformal scalar curvature at x=0 to get a single term:

The result of this computation is:

$$\text{projective conformal scalar curvature at } x = 0 \text{ is } -2(\dim - 1)K_i{}^{\mu}C_{\mu}{}^{i} \quad \text{(Apx, d13)} \tag{7}$$

If dim = 4, then

$$\text{projective conformal scalar curvature at } x = 0 \text{ and } \dim = 4 \text{ is } -6K_i{}^{\mu}C_{\mu}{}^{i} \tag{8}$$

2.4. The Role of Connections and Fiber Bundles in Physical Theories

In field theories based on parallel translation and curvature—hence connections on fiber bundles—the fiber space represents a perfectly symmetric vacuum spacetime, whose symmetry is not broken by the presence of matter. For example, GR uses a linear Minkowski space X with a fixed origin, and EC uses an affine space as the associated fiber. A further extension of the role for the fiber space is that the cosmological constant Λ should be viewed as the scalar curvature (times a constant) of a fiber that is a space of constant curvature [5]. To add a field that is not naturally derived from differential geometry, it is necessary to introduce an action term with no differential geometric origin or interpretation whose sole justification is that it produces the desired field equations. In gravitational theory, momentum is the most important case of a field that is introduced via an ad-hoc term in the Lagrangian that is not rooted in the geometry, whose sole justification is that variation yields desired terms in the field equations. In summary, the main argument for using connections on fiber bundles in physics is to separate the universal perfect symmetry of spacetime from local fields that break the perfect symmetry.

A common complaint about using connections in physical theories is that the principal bundle introduces many "unnecessary" frames that complicate computations. Even this inconvenience is hinting at structure: the vacuum is like an idealized version of a perfect single crystal modeled by the fiber; spacetime is like a polycrystalline material, where grain boundaries bind the perfect crystals together into a composite medium, like a geometric connection binds to together the high symmetry fibers to create a manifold with broken symmetries.

3. Lagrangian and Field Equations

The Lagrangian in GR, EC and this projective conformal theory is the scalar curvature of the connection. This includes Riemannian curvature, Riemann-Cartan curvature (including affine torsion to model intrinsic angular momentum), and projective conformal curvature. The terms in these three types of curvature are numerous, as are the interaction terms among the three types of curvature.

The main objective is to focus on the projective conformal terms in the scalar curvature. Therefore, the Lagrangian used in the variational computation contains only terms that include some projective conformal curvature. For example, consider a satellite in a spacetime that is far from any other source of gravitational curvature. This simplification amounts to assuming that the gravitational field is weak compared to fields arising from the momentum tensor of the satellite. This strategy is used to model many isolated objects in GR, for example Schwarzschild or Kerr black holes. This assumption greatly simplifies the computations.

The projective conformal terms in the connection and curvature also represent mass at infinity or a great distance. However, it is represented by a highly uniform vectorfield in local fiber spaces, so it does not represent the particular features of nearby matter.

3.1. The Variation of Action

The action due to the projective conformal field is 1/2 projective scalar curvature.

$$\text{action} = -(\dim - 1) K_i^{\ \mu} C_\mu^{\ i} \sqrt{|\det(g)|} \tag{9}$$

If dim = 4, $\text{action} = -3K_i^{\ \mu} C_\mu^{\ i} \sqrt{|\det(g)|}$, then

$$\delta\text{action}/\delta K_i^{\ \mu} = -3\left(\left(K_j^{\ \nu} C_\nu^{\ j}\right) K_i^{\ \mu} + C_i^{\ \mu}\right)\sqrt{|\det(g)|} \quad (\text{Apx, d20}) \tag{10}$$

3.2. Express Momentum P_i^{μ} in Terms of Conformal Field C_i^{μ} and Vice-Versa

The momentum P_i^{μ} can be expressed in term of the conformal field C_i^{μ}.

$$P_i^{\mu} = -(\dim - 1)\left(K_i^{\ \mu}\left(K_j^{\ \nu} C_\nu^{\ j}\right) + C_i^{\mu}\right)/(2\kappa) \quad (\text{Apx, d22}) \tag{11}$$

If dim = 4, this is:

$$P_i^{\mu} = -3\left(K_i^{\ \mu}\left(K_j^{\ \nu} C_\nu^{\ j}\right) + C_i^{\mu}\right)/(2\kappa) \quad (\text{Apx, d23}) \tag{12}$$

Given the momentum tensor P_i^{μ}, Equation (13) provides a way to specify the projective conformal coefficient C_i^{μ}.

$$C_i^{\ \mu} = \frac{(\dim - 1)\left(2K_i^{\ \mu}\kappa \text{PinvK} + (\dim - 1) K_i^{\ \mu} \text{CinvK} - 2\dim P_i^{\mu}\kappa\right)}{(\dim - 1)\dim} \quad (\text{Apx, d26}) \tag{13}$$

4. Linear Momentum as a Noether Current

In physical theories based on fiber bundles, Noether currents are associated with symmetry operations in the structure group of a theory and its Lie algebra.

GR derives linear momentum as the Noether current of spacetime translations, using Lie derivatives in spacetime as the translations. This formulation of momentum is not really a Noether current on a fiber bundle because the symmetry from which it is derived is not contained in the structure group.

EC derives linear momentum as the Noether current of spacetime translations and intrinsic angular momentum as the Noether current of translation operators in the Poincaré group, which is the structure group. EC includes exchange of intrinsic and orbital angular momentum because its linear momentum tensor is nonsymmetric. Linear momentum has a geometric interpretation that is basically the same as in GR, but the translation operator is included in the structure group. Intrinsic angular momentum has a geometric interpretation as infinitesimal translational holonomy around closed loops in spacetime [6].

The projective conformal theory presented here replaces the ad-hoc action term for linear momentum with the projective conformal term in the scalar curvature.

- These terms provide a geometric interpretation of momentum as translations at conformal infinity.

-The factor x^2 in Equation (1) ensures that projective conformal curvature does not appear in EC.

-Momentum has been the only term in gravitational field equations that has no such interpretation.

5. Mach's Principle

5.1. The History of Mach's Principle

In 1883, Ernst Mach published the conjecture now called Mach's principle [7]. It is an imprecise hypothesis that matter at a distance determines local linear inertia and moment of inertia in rotational mechanics. Mach felt that all matter in the universe should contribute to the local concept that matter is "not accelerating" and "not rotating" [8] [9].

Early discussions of Mach's Principle focused on rotational inertia, most clearly in the Lense-Thirring effect which is a relativistic correction to the precession of a gyroscope near a large rotating mass.

Mach's principle guided Einstein in development of GR, as expressed in a letter Einstein wrote to Mach [10].

> "…inertia originates in a kind of interaction between bodies … If one rotates a heavy shell of matter] relative to the fixed stars about an axis going through its center, a Coriolis force arises in the interior of the shell."

5.2. A Constructive Realization of Mach's Principle

A simple version of Mach's Principle is that momentum of an object is determined by motion of local matter relative to mass at conformal infinity. This work derives a computational statement of Mach's Principle: variation of the action with respect to the frame field ${K_\mu}^i$ yields a 2-index tensor in the field equations that serves as the momentum tensor in GR. This work derives the relationship between the momentum tensor P_i^{μ} and the projective conformal tensor C_{μ}^{i}(Apx, d22).

6. The Flow of Projective Conformal Vectorfields

This section introduces a geometric approach to the flow of projective conformal vectorfields. Graphs of the integral curves of projective conformal vectorfields show that these curves have a cell structure.

Figure 1 shows the projective conformal vectorfield in 2D in which C_{μ}^{i} is pointed in the $+x^{i}$ direction. Changing the direction of C merely rotates the image in3-space. The vectorfield is zero at the origin.

Figure 2 shows the projective conformal vectorfield in 3D in which C_{μ}^{i} is pointed in the +z (upward) direction. The vectorfield forms a flow moving from the +z direction to the −z direction. The flow does not rotate about the z axis.

Both figures show that the tubes formed by the integral curves are compressed as the flow lines approach the center of the cell. After leaving the central region of the cell, the integral curves cycle back to join the integral curves that are entering the tube at the other end. The direction of C determines the longest axis of each tube. This pattern forms finite ellipsoidal or tubular cells that fill the flat Minkowski fiber space.

One speculation about future applications of projective conformal gauge theory may be worth mentioning. The similarity between projective conformal vectorfields and flow vectorfields and flows of turbulent fluids suggests that projective conformal geometry may have a role in introducing stochastic behavior into the interior structure of momentum fields.

This cell structure is similar to the cell structure of simple turbulent fluid flows. In turbulent flows, the long axis of the cell is the mean direction and magnitude

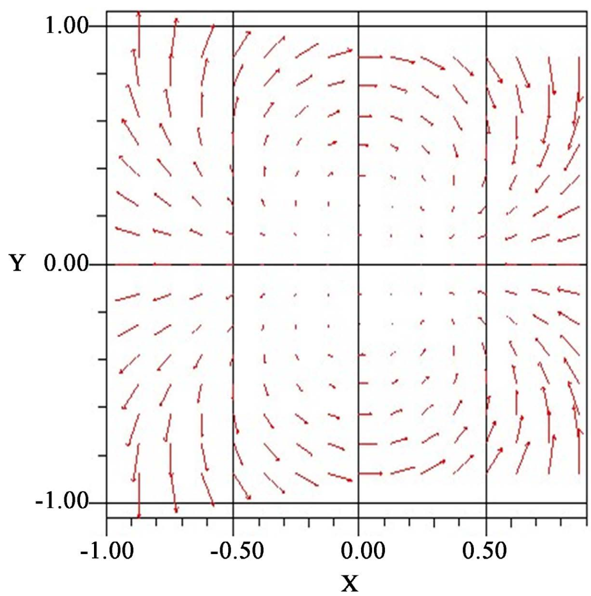

Figure 1. Vectorfield in 2D, C pointing in +x direction.

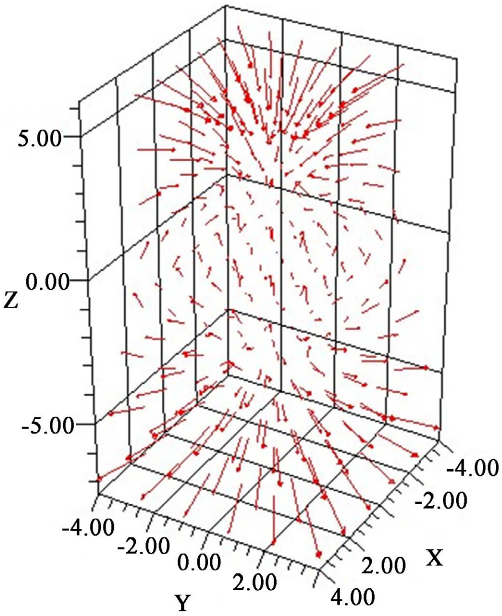

Figure 2. Vectorfield in 3D, C pointing in +z direction.

of the fluid flow; the flow along this line is mostly laminar flow. The width of each cell captures the variance of the flow [11].

7. Summary

Although General Relativity is the classic example of a physical theory based on differential geometry, the momentum tensor is the only part of the field equation that is not derived from or interpreted with differential geometry. This work extends GR to include projective conformal transformations, which are translations at conformal infinity. Momentum becomes a part of the differential geometry of spacetime. The geometry is a gauge theory for projective conformal transformations. This extension has two effects.

1) Variation of the projective conformal term in the scalar curvature generates a tensor in the field equations that can be interpreted as linear momentum. The linear momentum tensor is a part of the differential geometry. Momentum is the only term in Einstein's equations that has lacked an interpretation in terms of differential geometry.

2) This work provides a constructive realization of Mach's principle: linear momentum arises from translation relative to an inertial frame at infinity.

Statements and Declarations

The authors and family had no funding for this research.

The authors and family have no management or advisory positions in any scientific research or publishing organizations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Penrose, R. (1968) Structure of Spacetime, Conformal Infinity. *Battelle Rencontres—Lectures in Mathematics and Physics*, **1967**, 171-186.

[2] Atkins, J. (2021) Introduction to Conformal Field Theory in Two Dimensions: With Application to the Free Fermions. University of Rochester, Rochester, NY. https://www.sas.rochester.edu/mth/undergraduate/honorspaperspdfs/jeremyatkins2021.pdf

[3] McAvity, D.M. and Osborn, H. (1993) Energy Momentum Tensor in Conformal Field Theories near a Boundary. *Nuclear Physics B*, **406**, 655-680. https://arxiv.org/abs/hep-th/9302068 https://doi.org/10.1016/0550-3213(93)90005-A

[4] Petti, R.J. (2022) Conformal Structure of Quantum Wave Mechanics. *International Journal of Geometric Methods in Modern Physics*, **19**, Article ID: 2250174. https://arxiv.org/abs/2208.10969 https://doi.org/10.1142/S0219887822501742

[5] Petti R.J. (1977) Some Aspects of the Geometry of First Quantized Theories. II. *General Relativity and Gravitation*, **8**, 887-903. https://doi.org/10.1007/BF00759238

[6] Petti, R.J. (2022) Derivation of Einstein-Cartan Theory from General Relativity. arXiv: 1301.1588. http://arxiv.org/abs/1301.1588

[7] Mach, E. (2014) The Science of Mechanics. Cambridge University Press, Cambridge. (1883 in German, 1893 in English)

[8] Wald, R.M. (1984) General Relativity. University of Chicago Press, Chicago, 9, 71, 89, 187, 319.

[9] Wikipedia. Mach's Principle. https://en.wikipedia.org/wiki/Mach%27s_principle

[10] Misner, C., Thorne, K. and Wheeler, J. (1973) Gravitation. Scott Foresman and Co., Glenview, 544.

[11] Basic Turbulent Flow. https://www.mit.edu/course/1/1.061/www/dream/SEVEN/SEVENTHEORY.PDF

[12] Macsyma, Inc. (1996) Macsyma Mathematics and System Reference Manual. 16th Edition, Macsyma, Inc., Arlington, MA.

Appendix: The Computer Algebra Computation of Projective Conformal Action and Its Variation

This **Appendix** contains a computer algebra computation of the scalar curvature of a connection on an associated fiber bundle that includes projective conformal transformations.

- The structure group is the Lorentz group, augmented with projective conformal transformations.
- The associated fiber is a flat affine space with fixed origin point and a Lorentzian metric.
- Coordinates on the fiber are denoted x^i with lower case Roman letters as indices. Coordinates on the base space are denoted ξ^μ with Greek letters and indices.
- The connection coefficients are translational ($K_\mu{}^i$), rotational ($B_\mu{}^i{}_j$), and projective conformal ($C_\mu{}^i$). Affine connection coefficients normally with indices $\Gamma_\mu{}^i{}_j$, and $B_\mu{}^i{}_j$, but Macsyma writes these as $\Gamma_{\mu j}{}^i$ and $B_{\mu j}{}^i$.
- The Lagrangian of the theory is the scalar curvature at the fixed origin of the associated fiber.

Computations are performed using Macsyma 2.4.1a [12]. Macsyma's tensor simplification functions are not strong enough to perform all necessary simplifications automatically. In these situations, carefully validated manual simplifications are used.

For readers who are not familiar with Macsyma code, many comments are included to explain what each command line does, and sometimes what each command does.

We use connections on an affine bundle over spacetime, where:

- Ξ = spacetime manifold with local coordinates $\{\xi^\mu\}$ (Greek indices).
- X = Minkowski affine space with Cartesian coordinates $\{x^j\}$ (Roman indices).
- H = Structure group is a subset of the group of diffeomorphisms of X.
- L(H) = Lie algebra of the structure group H consists of smooth vectorfields on X.

Initialize.

a) Turn off frame fields and frame brackets.

b) Declare "x" constant. Macsyma cannot distinguish x^j as a fiber coordinate.

c) Define contraction properties for the connection coefficients.

d) $K_\mu{}^i(x)$ = fundamental 1-form = isomorphism from spacetime vectors to vectors on TX; $K_i{}^\mu$ = inverse of $K_\mu{}^i$ and $K_i{}^\mu$ are the only isomorphisms between spacetime and fiber vectors.

e) $B_{\mu j}i(\xi, x)$= affine rotational connection coefficients. x acts as a continuous lower index. Macsyma is programmed to write $B_\mu{}^i{}_j$ as $B_{\mu j}{}^i$.

f) gf_{ij} = fiber metric; symmetric, constant in fiber coordinates x^j; gf can raise and lower (lower case Roman) fiber indices.

Compute vectorfield representation of GAM:

- One (Greek) spacetime index.

- One (Roman) upper fiber index.
- One continuous coordinate x as lower fiber index.

Note: x^{fl} acts like a continuous third index in $Gam_{\mu}{}^{i}$.

(loadprint: false, init_itensor(), imetric(g), ratfac:true, declare(x, constant),
declare_isymmetry(\b, 3, [anti(2,3)]),
(c1) declare(gf, constant), define_icontraction(gf), define_icontraction(gf, gf, kdelta),
define_icontraction(\k, \k, kdelta), declare_isymmetry(gf, 2, [sym(1,2)]),
declare_isymmetry(gf, 2, [sym(1,2)]))$.

A1. Connection Coefficients

Define connection coefficients GAM in terms of connection coefficients K, B, C. K, B, C are constants in Minkowski fiber coordinates x.

- $Gam_{\mu}{}^{i}(\xi,x)$ = total connection coefficients, expressed as vectorfields on X.
- $K_{\mu}{}^{i}(\xi)$ = translational connection coefficients (the frame field).
- $B_{\mu j}{}^{i}(\xi,x)$ = rotational connection coefficients on X. In simple affine case (no projective conformal term), $B_{\mu j}{}^{i}(\xi,x)\, x^{j}$ is a vectorfield on X.
- $C_{\mu}{}^{i}(\xi,x)$ = Projective conformal connection coefficients.
- gfi j = constant Minkowski metric on affine fiber X.

Compute vectorfield representation of GAM:

- One (Greek) spacetime index.
- One (Roman) upper fiber index.
- One continuous coordinate x as lower fiber index.

Note: x^{fl} acts lika a continuous third index in $Gam_{\mu}{}^{i}$.

(loadprint: false, init_itensor(), imetric(g), ratfac:true, declare(x, constant),
declare_isymmetry(\b, 3, [anti(2,3)]),
(c2) declare(gf, constant), define_icontraction(gf), define_icontraction(gf, gf, kdelta),
define_icontraction(\k, \k, kdelta), declare_isymmetry(gf, 2, [sym(1,2)]),
declare_isymmetry(gf, 2, [sym(1,2)]))$.

(d2) $$Gam_{mu}{}^{i} = \left(x_{fl}x^{fl}kdelta_{k}{}^{i} - 2x^{i}x_{k}\right)C_{mu}{}^{k} + K_{mu}{}^{i} + x^{fl}B_{mufl}{}^{i}$$

Compute 3-index connection coefficient $Gam_{\mu j}{}^{i}$ = diff($Gam_{\mu}{}^{i}\, x^{j}$).

This is 3-index connection coefficient including projective conformal vectorfield.

Derivative by xj converts variable xj to the 4th index needed in index contraction.

The main goal of Section 2 below is to compute full curvature in 3-index vectorfield form:

$$R_{\mu\nu}{}^{i}(\xi,x) = dGam_{\mu j}{}^{i}Gam_{\nu j}{}^{i} - dGam_{\nu j}{}^{i}Gam_{\mu j}{}^{i}.$$

(remcomps(d\gam), ivariation(\gam([mu, @i]), x([@j])),
(c3) icontract(ratexpand(%%)), factorsum(%%), components(d\gam([mu, j, @i]), %%),
ishow('d\gam([mu, j, @i]) = d\gam([mu, j, @i]))).

(d3) $$dGam_{muj}{}^{i} = B_{muj}{}^{i} - 2\left(x^{i}C_{muj} + x_{\%2}kdelta_{j}{}^{i}C_{mu}{}^{\%2}\right) + 2x_{j}C_{mu}{}^{i}$$

A2. Full Curvature

A2.1. The Vectorfield Form of Full Curvature

Compute full curvature Rmni(x, x) as a vectorfield on X.

- The vectorfield form expresses the connection form as an element of the Lie algebra of the group of diffeomorshisms of X.
- Icontract(ratexpand(...)) removes kronecker deltas.
  Ratsimp(..., \c) replaces structure removed by ratexpand.

```
        (remcomps(full_curv_vect), \gam([nu, @i], mu) − \gam([mu, @i], nu)
      + d\gam([mu, j, @i])*\gam([nu,@j]) − d\gam([nu, j, @i])*\gam([mu,@j]),
(c4)    icontract(rename(ratexpand(%%))), ratsimp(%%, \c),
        components(full_curv_vect([mu, nu, @i]), %%),
        ishow('full_curv_vect([mu, nu, @i]) = full_curv_vect([mu, nu, @i])))$.
```

A2.2. Full Curvature as 4-Index Tensor

Compute conventional 4-index version Rmn ji (x, x).

- Diff(full_curv_vectmni (x, x) converts xj to discrete lower index j.
- Display is large. It is used only to compute Rmnji at x = 0. So omit display.

```
      (remcomps(full_curv),   ivariation(full_curv_vect([mu, nu, @i]), x([@j])),
(c5)  itenform(%%), collectterms(icontract(ratexpand(%%)), \c),
      components(full_curv([mu, nu, j, @i]), %%),
      ishow('full_curv([mu, nu, j, @i]) = full_curv([mu, nu, j, @i])))$.
```

Compute full_curv_x0(ξ, x) at x = 0 in X.

Local physics depends only on tensors at the origin x = 0 in X.

a) Set variable x = 0 and simplify occurrences of $C_\mu{}^i$.

B) Display Full_curv_x0(ξ, x=0).

```
    (remcomps(full_curv_x0), tmp: flush(full_curv([mu, nu, j, @i]), x), ratsimp(%%, \c),
(c6) components(full_curv_x0([mu, nu, j, @i]), %%),
    ishow('full_curv_x0([mu, nu, j, @i]) = full_curv_x0([mu, nu, j, @i])))
```

$$
\begin{aligned}
\text{full\_curv\_x0}_{\text{mu nu j}}{}^{\text{i}} = {} & B_{\text{nuj,mu}}{}^{\text{i}} + B_{\text{mu\%2}}{}^{\text{i}} B_{\text{nuj}}{}^{\text{\%2}} + 2C_{\text{mu\%2}}{}^{\text{i}} * K_{\text{nuj}} + 2K_{\text{mu}}{}^{\text{i}} C_{\text{nuj}} \\
& - 2C_{\text{muj}} K_{\text{nu}}{}^{\text{i}} - 2K_{\text{muj}} C_{\text{nu}}{}^{\text{i}} - B_{\text{muj}} * B_{\text{nu\%2}}{}^{\text{i}} \\
& - 2\text{kdelta}_{\text{j}}{}^{\text{i}} C_{\text{mu\%1}} K_{\text{nu}}{}^{\text{\%1}} + 2\text{kdelta}_{\text{j}}{}^{\text{i}} K_{\text{mu}}{}^{\text{\%1}} C_{\text{nu\%1}} - B_{\text{muj,nu}}{}^{\text{i}}
\end{aligned} \quad \text{(d6)}
$$

To check vectorfield methods, reconstruct conventional Riemannian curvature.

```
        (remcomps(non_conf_curv_x0),
(c7) map(lambda([zz], if freeof(\c, zz) then zz else 0),
ratexpand(full_curv_x0([mu, nu, j, @i]))),
        components(non_conf_curv_x0([mu, nu, j, @i]), %%),
        ishow('non_conf_curv_x0 = non_conf_curv_x0([mu, nu, j, @i])))
```

$$
\text{(d7)} \quad \text{non\_conf\_curv\_x0} = B_{\text{nu j,mu}}{}^{\text{i}} + B_{\text{mu \%2}}{}^{\text{i}} B_{\text{nu j}}{}^{\text{\%2}} - B_{\text{mu j}}{}^{\text{\%2}} B_{\text{nu\%2}}{}^{\text{i}} - B_{\text{mu j,nu}}{}^{\text{i}}
$$

A3. Projective Conformal Curvature

Compute projective conformal curvature at x = 0 by retaining only terms containing C.

```
(remcomps(projconf_curv_x0),
map(lambda([zz], if freeof(\c, zz) then 0 else zz),
ratexpand(full_curv_x0([mu, nu, j, @i]))),
```

(c8) ratsimp((%%), \c), subst(k, %1, %%), map(lambda([zz], collectterms(zz, kdelta)), %%),
components(projconf_curv_x0([mu, nu, j, @i]), %%),
ishow('projconf_curv_x0([mu, nu, j, @i]) = projconf_curv_x0([mu, nu, j, @i]))).

(d8)
$$\mathrm{projconf\_curv\_x0}_{mu\,nu\,j}{}^{i} = 2\Big(\mathrm{kdelta}_j{}^i * \big(K_{mu}{}^k C_{nuk} - C_{muk} K_{nu}{}^k\big) + C_{mu}{}^i K_{nuj} + K_{mu}{}^i C_{nu\,j} - C_{muj} K_{nu}{}^i - K_{muj} C_{nu}{}^i\Big)$$

Compute projective conformal Ricci curvature at x = 0.

a) Factor, then expand proj_conf_curv_x0.

b) projconf_riccicurv_x0$_{\mu}{}^{j}$ = (projconf_curv_x0$_{\mu\nu k}{}^{i}$) ($K_i{}^{\mu}$)

(remcomps(projconf_riccicurv_x0), projconf_curv_x0([mu, nu, j, @i])*\k([i, @nu]),

(c9) ratexpand(%%), factor(%%),
components(projconf_riccicurv_x0([mu, j]), %%),
ishow('projconf_riccicurv_x0([mu, j]) = projconf_riccicurv_x0([mu, j]))).

(d9)
$$\mathrm{projconf\_riccicurv\_x0}_{mu\,j} = -2K_i{}^{nu}\Big(\mathrm{kdelta}_j{}^i C_{muk} K_{nu}{}^k - \mathrm{kdelta}_j{}^i K_{mu}{}^k C_{nuk} - C_{mu}{}^i K_{nuj} - K_{mu}{}^i C_{nuj} + C_{muj} K_{nu}{}^i + K_{muj} C_{nu}{}^i\Big)$$

Compute projective conformal scalar curvature without manual simplifications.

(c10) (remcomps(projconf_scalarcurv_x0),
projconf_riccicurv_x0([mu, j])*gf([@j, @m])*\k([m, @mu]),
components(projconf_scalarcurv_x0([]), %%),
ishow('projconf_scalarcurv_x0([]) = projconf_scalarcurv_x0([]))).

(d10)
$$\mathrm{projconf\_scalarcurv\_x0}_{mu\,j} = -2K_i{}^{nu} gf^{jm} K_m^{mu}\Big(\mathrm{kdelta}_j{}^i C_{muk} K_{nu}{}^k - \mathrm{kdelta}_j{}^i K_{mu}{}^k C_{nuk} - C_{mu}{}^i K_{nuj} - K_{mu}{}^i C_{nuj} + C_{muj} K_{nu}{}^i + K_{muj} C_{nu}{}^i\Big)$$

Simplify projconf_scalarcurv_x0.

Macsyma cannot do the operations below, so apply manual simplifications of tensor indices.

- Contract spacetime indices with metric $g_{\mu\nu}$ and fiber indices with metric gf_{ij}.
- Rename indices while preserving distinction between base and fiber indices.

The computation is done in six steps.

Label steps a1, a2, a3, a4, a5, a6. Verify each operation separately.

- a1: $C_{\mu}{}^{m}$ replaces $gf^{jm} * \mathrm{kdelta}_j{}^i * K_i{}^{\nu} * K_{\nu}{}^{k} * C_{\mu}{}^{k}$.

- a2: $C_\nu{}^i$ replaces kdelta$_j{}^i$ * gf$_j{}^m$ * $K_m{}^\mu$ * $K_\mu{}^k$ * $C\nu^k$.
- a3: Kim replaces $K_i{}^\nu$ gf$_j{}^m$ * $K_m{}^\mu$ * $K\nu^j$.
- a4: $K_m{}^\nu$ * $C_\nu{}^m$ replaces $K_i{}^\nu$ * gf^{jm} * $K_m{}^\mu$ * $K_\mu{}^i$ * $C_\nu{}^j$.
- a5: $K_i{}^\mu$ * $C_\mu{}^i$ replaces $K_i{}^\nu$ * gf^{jm} * $K_m{}^\mu$ * $K_\nu{}^i$ * $C_\mu{}^j$.
- a6: dim replaces gf^{jm} * $K_m{}^\mu$ * K_{mj}.

```
block([a1, a2, a3, a4, a5, a6], remcomps(projconf_scalarcurv_x0_test),
     projconf_riccicurv_x0([mu, j])*gf([@j, @m])*\k([m, @mu]), ratexpand(%%),
    ratsubst(a1*\c([mu, @m]), gf([@j, @m]) * kdelta([j, @i]) * \k([i, @nu]) * \k([nu, @k])
       * \c([mu, k]), %%),
    ratsubst(a2*\c([nu, @i]), kdelta([j, @i]) * gf([@j,@m]) * \k([m, @mu]) * \k([mu, @k])
       * \c([nu, k]), %%),
(c11)    ratsubst(a3*\k([i, @mu]), \k([i, @nu])*gf([@j, @m])*\k([m, @mu])*\k([nu,
j]), %%),
     ratsubst(a4*\k([m,@nu]) * \c([nu, @m]),
       \k([i, @nu])*gf([@j, @m])*\k([m, @mu])*\k([mu, @i])*\c([nu, j]), %%),
     ratsubst(a5*\k([i, @mu])*\c([mu, @i]),
       \k([i, @nu])*gf([@j, @m])*\k([m, @mu])*\k([nu, @i])*\c([mu, j]), %%),
     ratsubst(a6*dim, gf([@j,@m])* \k([m, @mu])*\k([mu, j]), %%),
     components(projconf_scalarcurv_x0_test([]), %%),
     ishow('projconf_scalarcurv_x0_test([]) = projconf_scalarcurv_x0_test([])))
```

projconf_scalarcurv_x0_test

(d11) $$= 2a4K_m{}^{nu}C_{nu}{}^m + (2a2 - 2a6\,dim)K_i{}^{nu}C_{nu}{}^i - 2a1K_m{}^{mu}C_{mu}{}^m + (2a3 - 2a5)K_i{}^{mu}C_{mui}$$

Compute projconf_scalarcurv, including manual simplifications.

```
(remcomps(projconf_scalarcurv_x0),
   projconf_riccicurv_x0([mu, j])*gf([@j, @m])*\k([m, @mu]), ratex-
pand(%%),
   ratsubst(\c([mu, @m]), gf([@j, @m])*kdelta([j, @i]) * \k([i, @nu])* \k([nu,
@k])
      *\c([mu, k]), %%),
   ratsubst(\c([nu, @i]), kdelta([j, @i]) * gf([@j, @m]) * \k([m, @mu]) *
\k([mu, @k])
      * \c([nu, k]), %%),
   ratsubst(\k([i, @mu]), \k([i, @nu])*gf([@j, @m])*\k([m, @mu])
(c12)    *\k([nu, j]), %%),
   ratsubst(\k([m,@nu])* \c([nu, @m]),
     \k([i, @nu]) * gf([@j, @m]) * \k([m, @mu]) * \k([mu, @i]) * \c([nu,
j]), %%),
   ratsubst(\k([i, @mu]) * \c([mu, @i]),
     \k([i, @nu])*gf([@j, @m])* \k([m, @mu])* \k([nu, @i]) * \c([mu,
j]), %%),
   ratsubst(dim, gf([@j,@m])* \k([m, @mu])*\k([mu, j]), %%),
  components(projconf_scalarcurv_x0([]), %%),
```

ishow('projconf_scalarcurv_x0([]) = projconf_scalarcurv_x0([]))).

(d12) $\text{projconf\_scalarcurv\_x0} = 2K_{m}{}^{nu}C_{nu}{}^{m} + (2-2\,\text{dim})K_{i}{}^{nu}C_{nu}{}^{i} - 2K_{m}{}^{mu}C_{mu}{}^{m}$

Manually simplify the result to −2 (dim − 1) $K_i{}^{\mu}$ $C_{\mu}{}^{i}$.

```
    (remcomps(projconf_scalarcurv_x0),
(c13)    components(projconf_scalarcurv_x0([]), −2 * (dim − 1) * \k([i, @mu]) * \c([mu, @i])),
    ishow('projconf_scalarcurv_x0 = projconf_scalarcurv_x0([]))).
```

(d13) $\text{projconf\_scalarcurv\_x0} = -2(\text{dim}-1)K_{i}{}^{mu}C_{mu}{}^{i}$

For dim = 4, projconf_scalarcurv_x0 = −6 $K_i{}^{\mu}$ $C_{\mu}{}^{i}$.

```
    (remcomps(projconf_scalarcurv_x0_4d),
(c14)    subst(dim=4, projconf_scalarcurv_x0([])),
    components(projconf_scalarcurv_x0_4d([]), %%),
    ishow('projconf_scalarcurv_x0_4d([]) = projconf_scalarcurv_x0_4d([])))
```

(d14) $\text{projconf\_scalarcurv\_x0\_4d} = -6K_{i}{}^{mu}C_{mu}{}^{i}$

A4. The Variation of Projective Conformal Action

Definitions:

- projconf_Lagr = 1/2 projconf_scalarcurv_x0.
- projconf_action = integral(projconf_Lagr*dvolume).

A4.1. The Variation of Volume Form by $K_{\mu}{}^{i}$

Spacetime volume factor = $\sqrt{|\det(g)|}$

```
(c15) (volume_g: sqrt(abs(det(g))), ishow('volume_g = volume_g))
```

(d15) $\text{volume\_g} = \sqrt{|\det(g)|}$

Vary volume factor by spacetime metric $g_{\alpha\beta}$ to get δ (sqrt(|det(g)|))/δ $g_{\alpha\beta}$

```
(c16) (vary_volume_g: ivariation(volume_g, g([alpha,beta])),
    ishow('vary_volume_g = vary_volume_g))
```

(d16) $\text{vary\_volume\_g}^{\beta\,\alpha} = \dfrac{g^{\beta\,\alpha}\sqrt{|\det(g)|}}{2}$

Vary spacetime metric $g_{\alpha\beta} = K_{\alpha}{}^{a}\, gf_{ab}\, K_{\beta}{}^{b}$ with respect to $K_{\mu}{}^{i}$ to get $\delta(K_{\alpha}{}^{a}\, gf_{ab}\, K_{\beta}{}^{b})/\delta K_{\mu}{}^{i}$.

Because $\delta(K_{\alpha}{}^{a}\, gf_{ab}\, K_{\beta}{}^{b})/\delta K_{\mu}{}^{i}$ includes a factor 2, variation of action yields 2 $G_i{}^{\mu}$ − 2 k $P_i{}^{\mu}$ = 0.

```
    (ivariation(\k([alpha, @a])*gf([a, b])*\k([beta, @b]), \k([mu, @i])),
(c17)    vary_gf: icontract(%%),
    /* ratsubst(mu, alpha,, %%), ratsubst(nu, beta, %%), */
    ishow('vary_gf = vary_gf))
```

(d17) $\text{vary\_gf} = \text{kdelta}_{alpha}{}^{\mu}K_{i\,beta} + \text{kdelta}_{beta}{}^{\mu}K_{i\,alpha}$

Use chain rule (multiply previous two results) to get δvolume_factor/$\delta K_i{}^{\mu}$

```
    (vary_volume_g * vary_gf,
```

(c18) vary_volume_gf: icontract(ratexpand(%%)),
ishow('vary_volume_gf = vary_volume_gf))

(d18) $$\text{vary\_volume\_gf} = \sqrt{|\det(g)|}K_i^{\mu}$$

A4.2. The Variation of projconf_Lagr with Respect to $K_\mu{}^i$

a) Assume $C_\mu{}^i$ is independent of $K_\mu{}^i$.
b) Set projconf_Lagr = 1/2 projconf_scalarcurv_x0, but with dummy indices.
(Macsyma needs contracted dummy indices for some later operations.
"a" is a lower dummy index, "b" is upper dummy index on K.)

c) Compute δ(projconf_Lagr)/δ$K_\mu{}^i$

(remcomps(projconf_\lagr), components(projconf_\lagr([]), −(dim − 1) * \k([a, @b]) * \c([b, @a])),
remcomps(vary_projconf_\lagr),
(c19) block([%a,%b], components(vary_projconf_\lagr([%a, @%b]),
icontract(ivariation(projconf_\lagr([]), \k([%b, @%a]))))),
ishow('vary_projconf_\lagr([i, @mu]) = vary_projconf_\lagr([i, @mu])))

(d19) $$\text{vary\_projconf\_Lagr}_i{}^{\mu} = -(\dim - 1)C_i{}^{\mu}$$

A.4.3. The Variation of projconf_action Wrt $K_\mu{}^i$

a) Define projconf_action = projconf_Lagr * sqrt(|det(g)|)
b) Compute dprojconf_action/dKmi
Notation: KinvP = Kim Pmi, KinvC = Kim Cmi , CinvK = Cim Kmi , PinvK = Pim Kmi
"inv" means preceding variable has upper and lower indices inverted (Greek on top).
(remcomps(vary_projconf_action),
components(vary_projconf_action([i, @mu]),
(c20) factor(ratsubst(\kinv\c, \k([a, @b])*\c([b, @a]), vary_projconf_\lagr([i, @mu]) * sqrt(abs(det(g))))
+ projconf_\lagr([]) * vary_volume_gf)),
ishow('vary_projconf_action([i, @mu]) = vary_projconf_action([i, @mu])))

(d20) $$\text{vary\_projconf\_action}_i{}^{\mu} - (\dim - 1)\sqrt{|\det(g)|}\left(K_a{}^b C_b{}^a K_i{}^{\mu} + C_i{}^{\mu}\right)$$

A5. Relation between Momentum $P_i{}^\mu$ and Conformal Field $C_i{}^\mu$

Einstein's field equation is $G_\mu - k\, P_\mu = 0$, where
- (Greek kappa) is the gravitational constant. When dim = 4, □ = 8 π G/c^4.
- G = universal gravitational constant.
- P = momentum tensor.

With frame fields (lower case Roman indices), the field equation becomes

$G_i{}^{\mu} - k\ P_i{}^{\mu} = 0.$

A5.1. Express Momentum P in Terms of K and C

Express momentum $P_i{}^{\mu}$ in terms of conformal field $C_i{}^{\mu}$.

a) Variational principle defines □ $P_i{}^{\mu} = -\delta$projconf_action/$\delta K_i{}^{\mu}$.

b) Divide by another factor of 2 because $\delta g/\delta K = 2$ g.

(factor(vary_projconf_action([i, @mu])),

(c21) \p_from_\c: (p([i, @mu]) = %%/volume_g/2/kappa),
ishow(\p_from_\c))

(d21) $$P_i^{\mu} = -\frac{(\dim - 1)\left(K_a{}^b C_b{}^a K_i{}^{\mu} + C_i{}^{\mu}\right)}{2\kappa}$$

Manually replace $K_a{}^b\ C_b{}^a$ with KinvC. ("a" is lower dummy index and "b" an upper dummy index for K).

(c22) (\p_from_\c: \p([I, @mu]) = −(dim − 1) * (\kinv\c * \k([i, @mu]) + \c([I, @mu]))/(2 * kappa)
ishow(\p_from_\c))

(d22) $$P_i^{\mu} = -\frac{(\dim - 1)\left(K_i{}^{\mu}\mathrm{KinvC} + C_i{}^{\mu}\right)}{2\kappa}$$

A5.2. Express Conformal Field C in Terms of K and P

Solve equation P_from_C for P in terms of C – except for term KinvC.

(c23) (almost_\c_from_\p: first(linsolve(\p_from_\c, \c([i, @mu]))),
ishow(almost_\c_from_\p))

(d23) $$C_i^{\mu} = -\frac{(\dim - 1)K_i{}^{\mu}\mathrm{KinvC} + 2P_i{}^{\mu}\kappa}{\dim - 1}$$

Contract equation above with $K_{\mu}{}^{i}$. then multiply by (dim − 1)

(c24) trace_\c_from_\p: (dim-1) * \cinv\k = −((dim − 1) * dim * \kinv\c + 2*\pinv\k * kappa)

(d24) $$(\dim - 1)\mathrm{CinvK} = -2\kappa\ \mathrm{PinvK} - (\dim - 1)\dim \mathrm{KinvC}$$

Solve previous equation for KinvC.

(c25) \kinv\c_solution: factor(first(linsolve(trace_\c_from_\p, \kinv\c)))

(d25) $$\mathrm{KinvC} = -\frac{2\kappa\ \mathrm{PinvK} + \dim \mathrm{CinvK} - \mathrm{CinvK}}{(\dim - 1)\dim}$$

Substitute previous expression for KinvC into C_from_P.

This yields equation for conformal field C in terms of K and momentum tensor P.

(c26) (\c_from_\p: ratsimp(subst(\kinv\c_solution, almost_\c_from_\p)),
ishow(\c_from_\p))

(d26) $$C_i^{\mu} = \frac{2K_i{}^{\mu}\kappa\ \mathrm{PinvK} + (\dim - 1)K_i{}^{\mu}\mathrm{CinvK} - 2\dim P_i{}^{\mu}\kappa}{(\dim - 1)\dim}$$