Journal of Applied Mathematics and Physics, 2024, 12, 1492-1521
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Dynamics of Plate Equations with Memory Driven by Multiplicative Noise on Bounded Domains

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How to cite this paper: Bakhet, M.Y.A., Dafallah, A.A., Wang, J., Ma, Q.Z., Mosa, F.M., Mohamed, A.E., Lolika, P.O. and Chinor, M.M. (2024) Dynamics of Plate Equations with Memory Driven by Multiplicative Noise on Bounded Domains. *Journal of Applied Mathematics and Physics*, **12**, 1492-1521.
https://doi.org/10.4236/jamp.2024.124092

Received: March 9, 2024
Accepted: April 27, 2024
Published: April 30, 2024

Abstract

This article examines the dynamics for stochastic plate equations with linear memory in the case of bounded domain. We investigate the existence of solutions and bounded absorbing set by using the uniform pullback attractors on the tails estimates, and the asymptotic compactness of the random dynamical system is proved by decomposition method, and then we obtain the existence of a random attractor.

Keywords

Plate Equations, Random Attractors, Memory Term, Dynamical Systems

1. Introduction

In this paper, we investigate the existence of a random attractor for the following stochastic plate equations with linear memory and multiplicative noise on bounded domain:

$$\begin{cases} u_{tt}+\alpha u_t+\Delta^2 u+\int_0^\infty \mu(s)\Delta^2\left(u(t)-u(t-s)\right)\mathrm{d}s+f(u)=g(x)+cu\circ\dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ u(x,t)=u_0(x),\ u_t(x,t)=u_1(x),\ x\in U, t\le 0, \\ u\big|_{\partial U}=\dfrac{\partial u}{\partial n}\Big|_{\partial U}=0,\ t\ge 0. \end{cases} \tag{1.1}$$

where α and $c>0$ are positive constants and $\mu(s)\ge 0$ for every $s\in\mathbb{R}^+$,

U is an open bounded set of $\mathbb{R}^5$ with smooth boundary ∂U, $u=u(x,t)$ is a real function on $U\times[0,+\infty)$, $g\in H_0^1(U)\cap H^2(U)$ is a given external force and $W(x,t)$ is an independent two-sided real-valued wiener process on probability space $(\Omega,\mathcal{F},\mathbb{P})$, where

$$\Omega=\left\{\omega=(\omega_1,\omega_2,\cdots,\omega_m)\in C\left(\mathbb{R},\mathbb{R}^m\right):\omega(0)=0\right\},$$

is endowed with compact-open topology, $\mathbb{P}$ is the corresponding wiener measure, and $\mathcal{F}$ is the $\mathbb{P}$-completion of Borel σ-algebra on Ω. We identify $\omega(t)$ with $\left(W_1(t),W_2(t),\cdots,W_m(t)\right)$, *i.e.*

$$\omega(t)=\left(W_1(t),W_2(t),\cdots,W_m(t)\right),t\in\mathbb{R}.$$

Then, define the time shift $(\theta_t)_{t\in\mathbb{R}}$ on Ω by:

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\ t\in\mathbb{R},\omega\in\Omega.$$

The following conditions are necessary to obtain our main results.

(h_1) The memory kernel μ is assumed to satisfy the following conditions:

$$\begin{cases}\mu\in\mathbb{C}^1\left(\mathbb{R}^+\right)\cap\mathbb{L}^1\left(\mathbb{R}^+\right),\mu(s)\geq 0,\mu'(s)\leq 0,\forall s\in\mathbb{R}^+,\\ \mu'(s)+\delta\mu(s)\leq 0,\forall s\in\mathbb{R}^+,\text{ and for some }\delta>0,\end{cases}$$

and

$$\int_0^\infty\mu(s)\mathrm{d}s<\infty.$$

(h_2) The nonlinear term $f\in\mathbb{C}^1(\mathbb{R})$ with $f(0)=0$ and satisfies the following conditions:

$$\left|f'(u)\right|\leq C_1\left(1+|u|^4\right),\ \forall u\in\mathbb{R},\tag{1.2}$$

$$F(u)=\int_0^u f(s)\mathrm{d}s\geq C_2\left(|u|^6-1\right),\ \forall u\in\mathbb{R},\tag{1.3}$$

and

$$uf(u)\geq C_3\left(F(s)-1\right),\forall u\in\mathbb{R},\tag{1.4}$$

where $C_1,C_2,C_3>0$ are constants.

Following Dafermos [1], we introduce a new variable η defined by:

$$\eta(t,x,s)=u(t,x)-u(t-s,x),\tag{1.5}$$

and let $\mathfrak{R}_{\mu,2}=L_\mu^2\left(\mathbb{R}^+,H_0^2(U)\right)$ be a Hilbert space of $H_0^2(U)$-valued function on $\mathbb{R}^+$ with the inner product:

$$(\eta_1,\eta_2)_{\mu,2}=\int_0^\infty\mu(s)\left(\Delta\eta_1(s),\Delta\eta_2(s)\right)\mathrm{d}s,\forall\eta_1,\eta_2\in\mathfrak{R}_{\mu,2}.\tag{1.6}$$

Set $Z=(u,u_t,\eta)^{\mathrm{T}}$, $E=H_0^2(U)\times L^2(U)\times\mathfrak{R}_{\mu,2}$. Then, the system (1.1) is equivalent to the following initial value problem in the Hilbert space E:

$$\begin{cases}Z_t=L(Z)+N\left(Z,t,W(t)\right),\ x\in U,\ t\geq 0,\\ Z_0=\left(u_0(x),u_1(x),\eta_0(x,s)\right),\ (x,s)\in U\times\mathbb{R}^+,\end{cases}\tag{1.7}$$

where

$$\begin{cases} u(t,x) = \eta(t,x,s) = \eta(t,x,0), \ x \in \partial U, \ s \in \mathbb{R}^+, t \geq 0, \\ u(t,x) = u_0(x), \ u_t(t,x) = u_1(x), \ x \in U, \\ \eta(0,x,s) = \eta_0(x,s) = u(0,x) - u(-s,x), \ (x,s) \in U \times \mathbb{R}^+, \end{cases} \tag{1.8}$$

$$L(Z) = \begin{pmatrix} u_t \\ -\Delta^2 u - \alpha u_t - \int_0^\infty \mu(s) \Delta^2 \eta(s) \mathrm{d}s \\ u_t - \eta_s \end{pmatrix}, \tag{1.9}$$

$$N(Z,t,W(t)) = \begin{pmatrix} 0 \\ -g(u) + f(x) + cu \circ \dfrac{\mathrm{d}W(x,t)}{\mathrm{d}t} \\ 0 \end{pmatrix}, \tag{1.10}$$

$$D(L) = \left\{ Z \in E \mid \begin{matrix} u + \int_0^\infty \mu(s)\eta(s)\mathrm{d}s \in H^4(U) \cap H_0^2(U) \\ u_t \in H_0^2(U), \eta(s) \in H_\mu^1(\mathbb{R}^+, H_0^2(U)), \eta(0) = 0 \end{matrix} \right\}. \tag{1.11}$$

The stochastic plate equation is one of the fundamental stochastic partial differential equations (SPDEs) of hyperbolic type, which have been explored in [2] [3] [4] [5]. The behavior of its solutions is significantly different from those of solutions to other SPDEs.

Problem (1.1) models transversal vibration of the extensible elastic plate in a historical space, which is established based on the framework of elastic vibration by Woinowsky-Krieger [6] and Berger [7]. It can also be regarded as an elastoplastic flow equation with some kind of memory effect [1]. When $\mu = c = 0$, then (1.1) reduces to determined autonomous damped plate equation.

In recent years, there have many results on the dynamics of a variety of systems related to Equation (1.1). The hyperbolic equations with memory have been studied in [8]-[15] and references therein. For instance, Khanmamedov [16] and Yue and Zhong [2] proved the existence of global attractors for plate equations with critical exponent, [17]-[22] obtained the nonlinear damped, and Ma *et al.* [23] [24] [25] [26] [27] obtained the strongly damped. The existence of random attractors for such system in a bounded domain has been studied in [28]. Furthermore, long-time dynamics of a plate equation with memory and time delay is considered by Feng in [29], under suitable assumptions on real numbers μ_1 and μ_2, the quasi-stability property of the system is established and obtained the existence of global attractor, which has finite fractal dimension, and proved the existence of exponential attractors, defined in bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with a sufficiently smooth boundary $\partial\Omega$. Shen and Ma in [30] obtained the existence of random attractors for weakly dissipative plate equations with memory and additive noise by defining the energy functionals and using the compactness translation theorem.

Crauel *et al.* [31] [32] [33] studied the random attractors for stochastic dynamical system. Recently, many authors have established the existence of random attractors for other equations (see [34]-[45]). In Equation (1.1), there are fewer results and most previous authors have concentrated on the deterministic case, but there is no result of random attractors for Equation (1.1).

To prove the existence of random dynamical system (RDS) for short, the key step is to establish the compactness of the system. For our system (1.7), there are two essential difficulties in proving the compactness. Firstly, the critical growth condition (1.2) of f can be overcome by using the decomposition of solution and more accurate calculation. Secondly, the memory kernel itself, because there is no compact embedding in the history space, we introduce a new variable and define an extended Hilbert space, as well as combine with the compactness transform theorem.

The rest of the paper is organized as follows. In Section 2, we give the existence and uniqueness of the solutions. In Section 3, we devote to uniform estimates and the existence of bounded absorbing sets for the solutions and pullback compactness. In Section 4, the compactness of the random dynamical system is established by the decomposition of solution of the random differential equation into two parts. In Section 5, we prove the asymptotic compactness of the solutions, existence and uniqueness of a random attractor in E.

2. Preliminaries and Abstract Results

As mentioned in the introduction, our main purpose is to prove the dynamics of stochastic partial differential equations with multiplicative noise. For that matter, first, we recall some basic concepts related to random attractors for stochastic dynamical systems (see [9] [31] [32] [46] [47] [48] [49]), which are important for getting our main results. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and (X,d) be a polish space with the Borel σ-algebra $B(X)$. The distance between $x\in X$ and $B\subseteq X$ is denoted by $d(x,B)$. If $B\subseteq X$ and $C\subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B,C)=\sup_{x\in B} d(x,C)$.

Definition 2.1. $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$ is called a metric dynamical system if $\theta:\mathbb{R}\times\Omega\to\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t}=\theta_t\circ\theta_s$ for all $s,t\in\mathbb{R}$ and $\theta_0 P=P$ for all $t\in\mathbb{R}$.

Definition 2.2. A mapping $\Phi(t,\tau,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is called continuous cocycle on X over $\mathbb{R}$ and $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$, if for all $\tau\in\mathbb{R},\omega\in\Omega$ and $t,s\in\mathbb{R}^+$, the following conditions are satisfied:

1) $\Phi(t,\tau,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is a $\left(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F},\mathcal{B}(\mathbb{R})\right)$ measurable mapping.

2) $\Phi(0,\tau,\omega,x)$ is identity on X.

3) $\Phi(t+s,\tau,\omega,x)=\Phi(t,\tau+s,\theta_s\omega,x)\circ\Phi(s,\tau,\omega,x)$.

4) $\Phi(t,\tau,\omega,x):X\to X$ is continuous.

Definition 2.3. Let 2^X be the collection of all subsets of X, a set valued mapping $(\tau,\omega)\mapsto\mathcal{D}(t\,\omega):\mathbb{R}\times\Omega\mapsto 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if $\mathcal{D}(t,\omega)$ is a (usually closed) nonempty subset of X and the mapping $\omega\in\Omega\mapsto d(X,B(\tau,\omega))$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for every fixed $x\in X$ and $\tau\in\mathbb{R}$. Let $B=B(t,\omega)\in\mathcal{D}(t,\omega):\tau\in\mathbb{R},\omega\in\Omega$ is called a random set.

Definition 2.4. A random bounded set $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for p-a.e $\omega\in\Omega$,

$$\lim_{t\mapsto\infty}\mathrm{e}^{-\beta t}\,d\left(B(\theta_{-t}\omega)\right)=0,\forall\,\beta>0,$$

where

$$d(B)=\sup_{x\in B}\|x\|_X.$$

Definition 2.5. Let $\mathcal{D}$ be a collection of random subset of X and $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, then K is called an absorbing set of $\Phi\in\mathcal{D}$ if for all $\tau\in\mathbb{R},\omega\in\Omega$ and $B\in\mathcal{D}$, there exists, $T=T(\tau,\omega,B)>0$ such that:

$$\Phi\left(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)\right)\subseteq K(\tau,\omega),\forall\,t\geq T.$$

Definition 2.6. Let $\mathcal{D}$ be a collection of random subset of X, the Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for p-a.e $\omega\in\Omega$, $\left\{\Phi\left(t_n,\theta_{-t_n}\omega,x_n\right)\right\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n\mapsto\infty$ and $x_n\in B\left(\theta_{-t_n}\omega\right)$ with $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subset of X and $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied: for all $t\in\mathbb{R}^+,\tau\in\mathbb{R}$ and $\omega\in\Omega$,

1) $\mathcal{A}(\tau,\omega)$ is compact, and $\omega\mapsto d\left(x,\mathcal{A}(\omega)\right)$ is measurable for every $x\in X$.

2) $\mathcal{A}(\tau,\omega)$ is invariant, that is:

$$\Phi\left(t,\tau,\omega,\mathcal{A}(\tau,\omega)\right)=\mathcal{A}\left(\tau+t,\theta_t\omega\right),\forall\,t\geq\tau.$$

3) $\mathcal{A}(\tau,\omega)$ attracts every set in $\mathcal{D}$, that is for every $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$,

$$\lim_{t\mapsto\infty}d_X\left(\Phi\left(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)\right),\mathcal{A}(\tau,\omega)\right)=0.$$

where d_X is the Hausdorff semi-distance given by:

$$d_X(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X \text{ for any } Y\in X \text{ and } Z\in X.$$

Remark 2.8. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with wiener measure $\mathbb{P}$, the wiener shift $(\theta_t)_{t\in\mathbb{R}}$ is defined by:

$$\theta_s\omega(t)=\omega(t+s)-\omega(s),t,s\in\mathbb{R},$$

then $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$ is an ergodic metric dynamical system.

Lemma 2.9. [31] [32] Let $\mathcal{D}$ be a neighborhood-closed collection of (τ,ω)-parameterized families of nonempty subsets of X and Φ be a continuous cocycle on X over $\mathbb{R}$ and $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$. Then, Φ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is pullback $\mathcal{D}$-asymptotically compact in X and Φ has a closed, $\mathcal{F}$-measurable pullback $\mathcal{D}$-absorbing set $K\in\mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $\mathcal{A}=\mathcal{A}(\tau,\omega)$ is given:

$$\mathcal{A}(\tau,\omega)=\bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi\left(t,\tau-t,\theta_{-t}\omega,K\left(\tau-t,\theta_{-t}\omega\right)\right)},\tau\in\mathbb{R},\omega\in\Omega.$$

In this article, we will take $\mathcal{D}$ as the collection of all tempered random subsets.

Lemma 2.10. [50] For any $k>0$ and any $\phi\in H_0^1(U)\cap L^\infty(U)$, the following equality holds:

$$-\int_U\left(|\phi|^k\phi\right)\Delta\phi\mathrm{d}x=(k+1)\left(\frac{2}{k+2}\right)^2\int_U\left|\nabla|\phi|^{\frac{k+2}{2}}\right|^2\mathrm{d}x.$$

Lemma 2.11. [9] Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, the first injection being compact. Let $Y\subset L_\mu^2\left(\mathbb{R}^+,X\right)$ satisfy the following hypotheses:

1) Y is bounded in $L_\mu^2\left(\mathbb{R}^+,X_0\right)\cap H_\mu^1\left(\mathbb{R}^+,X_1\right)$.

2) $\sup_{\eta\in Y}\|\eta(s)\|_X^2\leq K_0$, $\forall s\in\mathbb{R}^+$ for some $K_0>0$.

Then, Y relatively compact in $L_\mu^2\left(\mathbb{R}^+,X\right)$.

3. Existence and Uniqueness of Solutions

From now on, assume that conditions (h_1) - (h_2) hold, the space E and the probability space $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$ are defined in Section 1. Let $A=\Delta^2$ with Neumann boundary condition on U, $D(A)=H^4(U)\cap H_0^2(U)$. We can define the powers A^ν of A for $\nu\in\mathbb{R}$. The space $V_{2\nu}=D\left(A^{\frac{\nu}{2}}\right)$ is the Hilbert space with the following inner product and norm, respectively:

$$(u,v)_{2\nu}=\left(A^{\frac{\nu}{2}}u,A^{\frac{\nu}{2}}v\right),\ \|u\|_{2\nu}^2=\left(A^{\frac{\nu}{2}}u,A^{\frac{\nu}{2}}u\right).$$

The injection $V_{\nu_1}\hookrightarrow V_{\nu_2}$ is compact if $\nu_1>\nu_2$. Then, by the generalized Poincaré inequality, there holds:

$$\|u\|_{\nu_1}^2\geq\lambda_1\|u\|_{\nu_2}^2,$$

where $\lambda_1>0$ is the first eigenvalue of A. In particular, $V_0=L^2(U)$, $V_1=H_0^1(U)$, $V_2=H_0^2(U)$, and $\left(A^{\frac{1}{4}}u,A^{\frac{1}{4}}v\right)=(\nabla u,\nabla v)$, $\forall u,v\in H_0^1(U)$. The inner product and norm in $L^2(U)$ is denoted by $(\cdot,\cdot),\|\cdot\|$, and in $H_0^2(U)$ is denoted by $((\cdot,\cdot)),\|\cdot\|_2$, respectively. By ($h_1$), the space $\mathfrak{R}_{\mu,2\nu}=L_\mu^2\left(\mathbb{R}^+,V_{2\nu}\right)$ is a Hilbert space of $V_{2\nu}$-valued function on $\mathbb{R}^+$ with the inner product and norm, respectively:

$$(\eta,\eta_1)_{\mu,2\nu}=\int_0^\infty\mu(s)\left(A^{\frac{\nu}{2}}\eta(s),A^{\frac{\nu}{2}}\eta_1(s)\right)\mathrm{d}s,\ \forall\eta,\eta_1\in V_{2\nu},\tag{3.1}$$

$$\|\eta\|_{\mu,2\nu}^2=(\eta,\eta)_{\mu,2\nu}=\int_0^\infty\mu(s)\left\|A^{\frac{\nu}{2}}\eta(s)\right\|^2\mathrm{d}s,\tag{3.2}$$

and on $\mathfrak{R}_{\mu,2\nu}$, the linear operator $-\partial_s$ has domain:

$$D(-\partial_s)=\left\{\eta\in H_\mu^1\left(\mathbb{R}^+,V_{2\nu}\right):\eta(0)=0\right\},$$
$$H_\mu^1\left(\mathbb{R}^+,V_{2\nu}\right)=\left\{\eta:\eta(s),\partial_s\eta\in L_\mu^2\left(\mathbb{R}^+,V_{2\nu}\right)\right\},$$

which generates a right-translation semigroup (see [1] [9] [13] [15] [51]).

Then, Equation (1.1) can be transformed into the following system:

$$\begin{cases}u_{tt}+\alpha u_t+\Delta^2u+\int_0^\infty\mu(s)\Delta\eta(s)\mathrm{d}s+f(u)=g(x)+cu\circ\dfrac{\mathrm{d}W}{\mathrm{d}t},\\ \eta_t=-\eta_s+u_t,\end{cases}\tag{3.3}$$

:

with the initial-boundary conditions:

$$\begin{cases}u(x,0)=u_0(x),\\ u_t(x,0)=u_1(x),\ x\in U,\\ \eta_0(x,s)=u_0(x)-u_0(x,-s),\ \forall\, x\in U,s\in\mathbb{R}^+.\end{cases}\tag{3.4}$$

The symbol C and $C_i\left(i=1,2,\cdots\right)$ are positive constants, which may change from line to line.

In this section, we show the existence, uniqueness and continuous dependence of (mild) solution of initial problem (1.7) in *E*, which generates a continuous RDS on E over $\mathbb{R}$ and $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$. For our purpose, we convert the problem (1.7) into a deterministic system with random parameters but without noise terms.

Due to Ornstein-Uhlenbeck process deducing by the Brownian motion, which holds the Itô differential equation:

$$\mathrm{d}z+\alpha z\mathrm{d}t=\mathrm{d}W(t),\tag{3.5}$$

and hence, the solution is given by:

$$z(\theta_t\omega)=-\alpha\int_\infty^0\mathrm{e}^{\alpha s}(\theta_t\omega)(s)\mathrm{d}s,\quad t\in\mathbb{R}.\tag{3.6}$$

It is known from [48] [49], the random variable $\left|z(\omega)\right|$ is tempered and there is a θ_t-invariant set $\bar{\Omega}\subseteq\Omega$ of full P measure such that for every $\omega\in\bar{\Omega}$, $t\mapsto z(\theta_t\omega)$ is continuous in t and:

$$\lim_{t\to\infty}\mathrm{e}^{-\alpha t}\left|z(\theta_{-t}\omega)\right|=0,\ \forall\,\alpha>0,\omega\in\bar{\Omega}.\tag{3.7}$$

Equation (3.6) has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process (see [31] [32] [36] [52] for more details).

For convenience, in the following, we write $\bar{\Omega}$ as Ω. Next, we need to transform the stochastic system into deterministic with a random parameter, then show that it generates a random dynamical system.

Let:

$$w(t,\omega,x)=u_t(t,\omega,x)+\varepsilon u(t,\omega,x)-cu(t,\omega,x)z(\theta_t\omega),t>0,\tag{3.8}$$

$$\varphi=\begin{pmatrix}u\\ w\\ \eta\end{pmatrix}=T_\varepsilon\begin{pmatrix}u\\ u_t\\ \eta\end{pmatrix}=T_\varepsilon Z,\ T_\varepsilon=\begin{pmatrix}0&1&0\\ \varepsilon&1&0\\ 0&0&1\end{pmatrix},\tag{3.9}$$

where

$$\varepsilon=\frac{2\alpha}{3+2\alpha k+\frac{\alpha^2}{\lambda_1}+\sqrt{\left(3+2\alpha k+\frac{\alpha^2}{\lambda_1}\right)^2-24\alpha k}}>0,\ k=\frac{\|\mu\|_{L^1(\mathbb{R}^+)}}{\delta}>0,$$

$\lambda_1(>0)$ is the smallest eigenvalue of operator A with Neumann boundary condition on U.

In this paper, we assume that:

$$|c|<\min\left\{\alpha^2\sqrt{\lambda_1}\frac{-\frac{1}{\sqrt{\pi\alpha}}\left(\frac{2C_1}{\sqrt{\lambda_1}}+3+\frac{C_1}{C_2}\right)^2+\sqrt{\frac{1}{\pi\alpha}\left(\frac{2C_1}{\sqrt{\lambda_1}}+3+\frac{C_1}{C_2}\right)^2+\frac{\sigma\left(16\varepsilon^2+\alpha\right)}{\alpha^2\sqrt{\lambda_1}}}}{16\varepsilon^2+\alpha},\right.$$

$$\alpha^2\sqrt{\lambda_1}\frac{-\frac{1+\left(2R_6(\omega)+2\right)\sqrt{\lambda_1}}{\sqrt{\lambda_1\pi\alpha}}+\sqrt{\frac{\left(1+2R_6(\omega)\sqrt{\lambda_1}+2\sqrt{\lambda_1}\right)^2}{\lambda_1\pi\alpha}+\frac{\left(16\varepsilon^2+\alpha\right)\sigma_2(\omega)}{\alpha^2\sqrt{\lambda_1}}}}{16\varepsilon^2+\alpha} \tag{3.10}$$

$$\left.\alpha^2\sqrt{\lambda_1}\frac{-\frac{1+\left(2R_6(\omega)+2\right)\sqrt{\lambda_1}}{\sqrt{\lambda_1\pi\alpha}}+\sqrt{\frac{\left(1+2R_6(\omega)\sqrt{\lambda_1}+2\sqrt{\lambda_1}\right)^2}{\lambda_1\pi\alpha}+\frac{\left(16\varepsilon^2+\alpha\right)\sigma_3}{\alpha^2\sqrt{\lambda_1}}}}{16\varepsilon^2+\alpha}\right\},$$

where $\sigma,\sigma_3>0,\sigma_2(\omega),R_6(\omega)>0$.

By (3.8) and (1.1), we can obtain the following random evolution equation:

$$\begin{cases}u_t+\varepsilon u-w=cuz(\theta_t\omega),\\ w_t-\varepsilon(\alpha-\varepsilon)u+Au+(\alpha-\varepsilon)w+\int_0^\infty\mu(s)A\eta(s)\mathrm{d}s+f(u)\\ =g(x)-cz(\theta_t\omega)\left(w-2\varepsilon u+cuz(\theta_t\omega)\right),\\ \eta_t+\eta_s+\varepsilon u-w=cuz(\theta_t\omega).\end{cases} \tag{3.11}$$

Then, the problem (3.11) is equivalent to the following determined system with random parameter in E:

$$\begin{cases}\varphi'+H(\varphi)=Q(\varphi,\theta_t\omega,t),\\ \varphi_\tau(\omega)=\left(u_0,u_1+\varepsilon u_0-cu_0z(\omega),\eta_0\right)^{\mathrm{T}},t\geq 0,\end{cases} \tag{3.12}$$

where

$$H(\varphi)=\begin{pmatrix}\varepsilon u-w\\ -\varepsilon(\alpha-\varepsilon)u+Au+(\alpha-\varepsilon)w+\int_0^\infty\mu(s)A\eta(s)\mathrm{d}s\\ \varepsilon u-w+\eta_s\end{pmatrix}=-T_\varepsilon HT_\varepsilon(\psi), \tag{3.13}$$

$$Q(\varphi,\theta_t\omega,t)=\begin{pmatrix}cuz(\theta_t\omega)\\ -cz(\theta_t\omega)\left(w-2\varepsilon u+cuz(\theta_t\omega)\right)-f(u)+g(x)\\ cuz(\theta_t\omega)\end{pmatrix}. \tag{3.14}$$

In line with [9] [53], we know that the operator L in (1.9) is the infinitesimal generator of C_0-semigroup e^{Lt} of contractions on E for $t>0$. Since $-H=T_\varepsilon LT_{-\varepsilon}$, and T_ε is an isomorphism of E, the operator $-H$ also generates a C_0-semigroup e^{-Ht} of contractions on E. By the assumptions (h_2) and the embedding relation $H_0^2(U) \hookrightarrow L^{10}(U)$, it is easy to check $Q(\varphi,t,\omega):E\to E$ is locally Lipschitz continuous with respect to φ, by the classical semigroup theory concerning the (local) existence and uniqueness solution of evolution differential equation [53], we have the following theorem.

Theorem 3.1. Assume that (h_1) - (h_3) hold. Then, for each $\omega\in\Omega$ and for any $\varphi_0\in E$, there exists $T>0$ such that (3.12) has a unique mild function $\varphi(\cdot,\omega,\varphi_0)\in C([0,T);E)$ such that $\varphi(0,\omega,\varphi_0)=\varphi(0)$ satisfies the integral equation:

$$\varphi(t,\omega,\varphi_0)=\mathrm{e}^{-Ht}\varphi_0(\omega)+\int_0^t \mathrm{e}^{-H(t-s)}Q(\varphi(s,\omega,\varphi_0),\theta_s\omega,s)\mathrm{d}s. \tag{3.15}$$

Moreover, $\varphi(t,\omega,\varphi_0)$ is jointly continuous in φ_0 and measurable in ω.

From Theorem 3.1, we know that for P-a.s. each $\omega\in\Omega$, then the following results hold for all $T>0$:

1) If $\varphi_0(\omega)\in E$ then $\varphi(\cdot,\omega,\varphi_0)\in C([0,T);E)$.

2) $\varphi(t,\omega,\varphi_0)$ is jointly continuous into t and measurable in ω.

3) The solution mapping of (3.12) satisfies the properties of Random Dynamical System.

We notice that a unique solution $\varphi(\cdot,\omega,\varphi_0)$ of (3.12) can define a continuous random dynamical system over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Hence, the solution mapping:

$$\begin{aligned}&\bar{\Phi}(t,\omega):\mathbb{R}\times\Omega\times E\mapsto E,\ t\geq 0,\\ &\varphi(0,\omega)=(u_0,v_0,\eta_0)^{\mathrm{T}}\mapsto(u(t,\omega),v(t,\omega),\eta(t,\omega))^{\mathrm{T}}=\varphi(t,\omega),\end{aligned} \tag{3.16}$$

generates a random dynamical system. Moreover,

$$\Phi(t,\omega):\varphi(0,\omega)+(0,\varepsilon z(\omega),0)^{\mathrm{T}}\mapsto\varphi(t,\omega)+(0,\varepsilon z(\theta_t\omega),0)^{\mathrm{T}}. \tag{3.17}$$

We also define the following transformation:

$$\psi_1=u,\psi_2=u_t+\varepsilon u, \tag{3.18}$$

similar to (3.12), we get that:

$$\begin{cases}\psi'+H\psi=Q(\psi,t,\omega)\\ \psi_0(\omega)=(u_0,v_0,\eta_0)^{\mathrm{T}}=(u_0,u_1+\varepsilon u_0,\eta_0)^{\mathrm{T}},\end{cases} \tag{3.19}$$

where

$$\psi=\begin{pmatrix}u\\ v\\ \eta\end{pmatrix}, \tag{3.20}$$

$$H(\psi)=\begin{pmatrix}\varepsilon u-v\\ -\varepsilon(\alpha-\varepsilon)u+Au+(\alpha-\varepsilon)v+\eta\\ \varepsilon u-v+\eta_s\end{pmatrix} \tag{3.21}$$

and

$$Q(\psi,\omega,t)=\begin{pmatrix}0\\ cvz(\theta_t\omega)-f(u)+g(x)\\ 0\end{pmatrix}.$$

It is easy to see that:

$$\Upsilon(t,\omega,Z_0)=R^{-1}_{\varepsilon,\theta_t\omega}\Phi(t,\omega)R_{\varepsilon,\theta_t\omega}:Z_0\to Z(t,\omega,Z_0),\tag{3.22}$$

and

$$\Psi(t,\omega,\psi_0)=T_\varepsilon\Upsilon T_{-\varepsilon}:\psi_\tau\to\psi(t,\omega,\psi_0),\tag{3.23}$$

are continuous RDS over $\mathbb{R}$ and $\left(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}}\right)$ associated with system (3.7) and (3.15) respectively.

We introduce the isomorphism $T_\varepsilon Y=(u,u_t,\eta)^{\mathrm{T}}$, $Y=(u,v,\eta)^{\mathrm{T}}\in E$, which has inverse isomorphism $T_{-\varepsilon}Y=(u,v-\varepsilon u,\eta)^{\mathrm{T}}$, it follows that (θ,ψ) with mapping:

$$\Psi=T_\varepsilon\Phi(t,\omega)T_{-\varepsilon}=\Psi(t,\omega)\tag{3.24}$$

is a random dynamical system from above discussion, we show that the two RDS are equivalent.

4. Random Absorbing Set

In this section, we will show the existence of a random absorbing set for the RDS $\varphi(t,\omega,\varphi_0(\omega)),t\ge0$ in the space *E*.

Lemma 4.1. Suppose that (h_1) - (h_2) hold. Then, there exists a closed tempered absorbing ball $B_0(\omega)\in\mathfrak{D}(E)$ of *E*, centered at 0 with random radius $M_0(\omega)>0$ such that for any bounded non-random set $B\in\mathfrak{D}(E)$, there exists a deterministic $t_B(\omega)>0$, such that the solution $\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))$ of (3.12) with initial value $(u_0,u_1+\varepsilon u_0-cu_0z(\omega),\eta_0)^{\mathrm{T}}\in B$ satisfies, for *P-a.s.* $\omega\in\Omega$,

$$\left\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\right\|_E^2\le M_0(\omega),\ \forall t\ge t_B(\omega),\tag{4.1}$$

that is,

$$\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subseteq B_0(\omega),\ \forall t\ge t_B(\omega).$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ of (3.12) with $\varphi(r)=(u(r),w(r),\eta_r)^{\mathrm{T}}$, we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_E^2+(H(\varphi,\varphi))_E=(Q(\varphi,\theta_t\omega,t),\varphi).\tag{4.2}$$

Similar to the proof of Lemma 2 in [54], we have:

$$\begin{aligned}(H(\varphi),\varphi)_E&=\varepsilon\|u\|_2^2-((w,u))+(\alpha-\varepsilon)\|w\|^2-\varepsilon(\alpha-\varepsilon)(u,w)+(Au,w)\\&\quad+\varepsilon(u,\eta)_{\mu,2}+\left(\int_0^\infty\mu(s)A\eta(s)\mathrm{d}s,w\right)-(w,\eta)_{\mu,2}+(\eta_s,\eta)_{\mu,2}\\&=\varepsilon\|u\|_2^2+(\alpha-\varepsilon)\|w\|^2-\varepsilon(\alpha-\varepsilon)(u,w)+\varepsilon(u,\eta)_{\mu,2}+(\eta_s,\eta)_{\mu,2}.\end{aligned}\tag{4.3}$$

Then, by using (h_1), we find that:

$$\varepsilon(u,\eta)_{\mu,2} \geq -k\varepsilon^2 \|u\|_2^2 - \frac{\delta}{4}\|\eta\|_{\mu,2}^2, \tag{4.4}$$

$$(\eta_s,\eta)_{\mu,2} \geq \frac{\delta}{2}\|\eta\|_{\mu,2}^2. \tag{4.5}$$

Applying (4.3)-(4.5), Hölder inequality, Young inequality and Poincaré inequality, we obtain that:

$$\begin{aligned}(H(\varphi),\varphi)_E &\geq \varepsilon(1-k\varepsilon)\|u\|_2^2 + (\alpha-\varepsilon)\|w\|^2 + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 - \varepsilon(\alpha-\varepsilon)(u,w)\\ &\geq \varepsilon(1-k\varepsilon)\|u\|_2^2 + (\alpha-\varepsilon)\|w\|^2 + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|u\|_2\cdot\|w\|\\ &= \frac{\varepsilon}{2}\left(\|u\|_2^2+\|w\|^2\right) + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 + \frac{\alpha}{2}\|w\|^2 + \varepsilon\left(\frac{1}{2}-k\varepsilon\right)\|u\|_2^2\\ &\quad + \left(\frac{\alpha}{2}-\frac{3\varepsilon}{2}\right)\|w\|^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|u\|_2\cdot\|w\|.\end{aligned} \tag{4.6}$$

It follows from a simple computation that:

$$\varepsilon\left(\frac{1}{2}-k\varepsilon\right)\left(\frac{\alpha}{2}-\frac{3\varepsilon}{2}\right) = \frac{\varepsilon^2\alpha^2}{4\lambda_1}. \tag{4.7}$$

Hence, combining (4.6) and (4.7), we find that:

$$(H(\varphi,\varphi))_E \geq \frac{\varepsilon}{2}\left(\|u\|_2^2+\|w\|^2\right) + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 + \frac{\alpha}{2}\|w\|^2. \tag{4.8}$$

Let us estimate the right hand side of (4.2):

$$\begin{aligned}(Q(\varphi,\theta_t\omega,t),\varphi) &= \left((cuz(\theta_t\omega),u)\right) - (cwz(\theta_t\omega),w) + (2c\varepsilon uz(\theta_t\omega),w)\\ &\quad - (c^2uz^2(\theta_t\omega),w) - (f(u),w) + (g(x),w) + (cuz(\theta_t\omega),\eta)_{\mu,2}.\end{aligned} \tag{4.9}$$

By the Cauchy-Schwartz inequality, we find that:

$$\left((cuz(\theta_t\omega),u)\right) \leq |c||z(\theta_t\omega)|\|u\|_2^2, \tag{4.10}$$

$$(cwz(\theta_t\omega),w) \leq |c||z(\theta_t\omega)|\|w\|^2, \tag{4.11}$$

$$(2c\varepsilon uz(\theta_t\omega),w) \leq \frac{8\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_1}}\|u\|_2^2 + \frac{\alpha}{8}\|w\|^2, \tag{4.12}$$

$$c^2z^2(\theta_t\omega)|(u,w)| \leq \frac{c^2}{2\sqrt{\lambda_1}}|z(\theta_t\omega)|^2\left(\|u\|_2^2+\|w\|^2\right), \tag{4.13}$$

$$(g(x),w) \leq \frac{2}{\alpha}\|g(x)\|^2 + \frac{\alpha}{8}\|w\|^2, \tag{4.14}$$

$$(cuz(\theta_t\omega),\eta)_{\mu,2} \leq \frac{|c||z(\theta_t\omega)|}{2}\left(\|u\|_2^2+\|\eta\|_{\mu,2}^2\right). \tag{4.15}$$

Then, we estimate nonlinear term (4.9), by (h_2) and the Hölder inequality, we get that:

$$\begin{aligned}\left(f(u),w\right)&=\left(f(u),u_t+\varepsilon u-cuz(\theta_t\omega)\right)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_U F(u)\mathrm{d}x+\varepsilon\left(f(u),u\right)-cz(\theta_t\omega)\left(f(u),u\right).\end{aligned}\tag{4.16}$$

Applying (1.2)-(1.4), we have:

$$\begin{aligned}&\left|cz(\theta_t\omega)\left(f(u),u\right)\right|\\&\le C_1|c||z(\theta_t\omega)|\int_U\left(|u|^2+|u|^6\right)\mathrm{d}x\\&\le\frac{C_1}{C_2}|c||z(\theta_t\omega)|\int_U\left(F(u)+C_2\right)\mathrm{d}x+C_1|c||z(\theta_t\omega)|\|u\|^2\\&\le\frac{C_1}{C_2}|c||z(\theta_t\omega)|\int_U F(u)\mathrm{d}x+C_1|U||c||z(\theta_t\omega)|+\frac{C_1|c||z(\theta_t\omega)|}{\sqrt{\lambda_1}}\|u\|_2^2;\end{aligned}\tag{4.17}$$

$$\varepsilon\left(f(u),u\right)\ge\varepsilon C_3\int_U F(u)\mathrm{d}x-\varepsilon C_3|U|.\tag{4.18}$$

Thus, due to (4.16)-(4.18), we obtain that:

$$\begin{aligned}\left(f(u),w\right)\ge&\frac{\mathrm{d}}{\mathrm{d}t}\tilde F(u)-\varepsilon C_3\tilde F(u)+\varepsilon C_3|U|-\frac{C_1}{C_2}|c||z(\theta_t\omega)|\tilde F(u)\\&-C_1|U||c||z(\theta_t\omega)|-\frac{C_1|c||z(\theta_t\omega)|}{\sqrt{\lambda_1}}\|u\|_2^2,\end{aligned}\tag{4.19}$$

where $\tilde F(u)=\int_U F(u)\mathrm{d}x$.

Collecting (4.2), (4.8), (4.19) and (4.9)-(4.15), we show that:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_E^2+2\tilde F(u)\right)+\frac{\varepsilon}{2}\left(\|u\|_2^2+\|w\|^2\right)+\frac{\delta}{4}\|\eta\|_{\mu,2}^2+\frac{\alpha}{2}\|w\|^2+\varepsilon C_3\tilde F(u)\\&\le\frac{C_1}{C_2}|c||z(\theta_t\omega)|\tilde F(u)+\frac{|c||z(\theta_t\omega)|}{2}\|\eta\|_{\mu,2}^2+\left(|c||z(\theta_t\omega)|+\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_1}}\right)\|w\|^2\\&+\left(\frac{C_1|c||z(\theta_t\omega)|}{\sqrt{\lambda_1}}+\frac{3|c||z(\theta_t\omega)|}{2}+\frac{8\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_1}}+\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_1}}\right)\|u\|_2^2\\&+\varepsilon C_3|U|+C_1|U||c||z(\theta_t\omega)|+\frac{2}{\alpha}\|g(x)\|^2,\end{aligned}$$

choose $\sigma=\min\left\{\varepsilon,\frac{\delta}{2},\varepsilon C_3\right\}$. Due to $\|\varphi\|_E^2=\left(\|u\|_2^2+\|w\|^2+\|\eta\|_{\mu,2}^2\right)$, then we have the following equivalent system:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_E^2+2\tilde F(u)\right)+\sigma\left(\varphi_E^2+2\tilde F(u)\right)\\&\le\frac{2C_1}{C_2}|c||z(\theta_t\omega)|\tilde F(u)+|c||z(\theta_t\omega)|\|\eta\|_{\mu,2}^2+\left(2|c||z(\theta_t\omega)|+\frac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}\right)\|w\|^2\\&+\left(\frac{2C_1|c||z(\theta_t\omega)|}{\sqrt{\lambda_1}}+3|c||z(\theta_t\omega)|+\frac{16\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_1}}+\frac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}\right)\|u\|_2^2\\&+2C_3|U|+2C_1|U||c||z(\theta_t\omega)|+\frac{4}{\alpha}\|g(x)\|^2,\end{aligned}$$

where

$$\rho(t,\theta_t\omega)=\sigma-\left(\frac{2C_1|c||z(\theta_t\omega)|}{\sqrt{\lambda_1}}+3|c||z(\theta_t\omega)|+\frac{16\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_1}}\right.$$
$$\left.+\frac{c^2|z(\theta_t\omega)|^2}{\sqrt{\lambda_1}}+\frac{C_1}{C_2}|c||z(\theta_t\omega)|\right). \tag{4.20}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)+\rho(t,\theta_t\omega)\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)$$
$$\le 2C_3|U|+2C_1|U||c||z(\theta_t\omega)|+\frac{4}{\alpha}\|g(x)\|^2. \tag{4.21}$$

So, applying Gronwall's Lemma to (4.21), we have:

$$\|\varphi(t,\omega,\varphi_0(\omega))\|_E^2+2\tilde{F}(u)$$
$$\le \mathrm{e}^{-\int_0^t\rho(s,\theta_s\omega)\mathrm{d}s}\left(\|\varphi_0(\omega)\|_E^2+2\tilde{F}(u_0)\right)+2C_1|U||c|\int_0^t\mathrm{e}^{-\int_s^t\rho(\tau,\theta_\tau\omega)\mathrm{d}\tau}|z(\theta_s\omega)|\mathrm{d}s \tag{4.22}$$
$$+\left(2C_3|U|+\frac{4}{\alpha}\|g(x)\|^2\right)\int_0^t\mathrm{e}^{-\int_s^t\rho(\tau,\theta_\tau\omega)\mathrm{d}\tau}\mathrm{d}s.$$

Substitiuting ω by $\theta_{-t}\omega$, from (4.22), we have:

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2+2\tilde{F}(u)$$
$$\le \mathrm{e}^{-\int_0^t\rho(s-t,\theta_{s-t}\omega)\mathrm{d}s}\left(\|\varphi_0(\theta_{-t}\omega)\|_E^2+2\tilde{F}(u_0)\right)+2C_1|U||c|\int_0^t\mathrm{e}^{-\int_s^t\rho(\tau-t,\theta_{\tau-t}\omega)\mathrm{d}\tau}|z(\theta_{s-t}\omega)|\mathrm{d}s$$
$$+\left(2C_3|U|+\frac{4}{\alpha}\|g(x)\|^2\right)\int_0^t\mathrm{e}^{-\int_s^t\rho(\tau-t,\theta_{\tau-t}\omega)\mathrm{d}\tau}\mathrm{d}s \tag{4.23}$$
$$\le \mathrm{e}^{-\int_{-t}^0\rho(s,\theta_s\omega)\mathrm{d}s}\left(\|\varphi_0(\theta_{-t}\omega)\|_E^2+2\tilde{F}(u_0)\right)+2C_1|U||c|\int_{-t}^0\mathrm{e}^{-\int_s^0\rho(\tau,\theta_\tau\omega)\mathrm{d}\tau}|z(\theta_s\omega)|\mathrm{d}s$$
$$+\left(2C_3|U|+\frac{4}{\alpha}\|g(x)\|^2\right)\int_{-t}^0\mathrm{e}^{-\int_s^0\rho(\tau,\theta_\tau\omega)\mathrm{d}\tau}\mathrm{d}s.$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic, it follows from (3.2) and the ergodic theorem that:

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^0|z(\theta_r\omega)|\mathrm{d}r=E(|z(\theta_r\omega)|)=\frac{1}{\sqrt{\pi\alpha}}, \tag{4.24}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^0|z(\theta_r\omega)|^2\mathrm{d}r=E\left(|z(\theta_r\omega)|^2\right)=\frac{1}{2\alpha}. \tag{4.25}$$

From (4.24) and (4.25), we know that there exists $T_1(\omega)>0$ such that for any $t\ge T_1(\omega)$,

$$\int_{-t}^0|z(\theta_r\omega)|\mathrm{d}r<\frac{1}{\sqrt{\pi\alpha}}t,\ \int_{-t}^0|z(\theta_r\omega)|^2\mathrm{d}r<\frac{1}{2\alpha}t. \tag{4.26}$$

Next, we need to obtain that for any $s\le -T_1$,

$$\mathrm{e}^{\int_0^s\rho(\tau,\theta_\tau\omega)\mathrm{d}\tau}\le \mathrm{e}^{\frac{\sigma}{2}s}. \tag{4.27}$$

Indeed, by (4.26), we have:

$$\int_0^s \left[\sigma - \left(\frac{2C_1 |c| |z(\theta_\tau \omega)|}{\sqrt{\lambda_1}} + 3|c||z(\theta_\tau \omega)| + \frac{16\varepsilon^2 |c|^2 |z(\theta_\tau \omega)|^2}{\alpha \sqrt{\lambda_1}} + \frac{c^2 |z(\theta_\tau \omega)|^2}{\sqrt{\lambda_1}} + \frac{C_1}{C_2} |c||z(\theta_\tau \omega)| \right) \right] \mathrm{d}\tau$$

$$> \sigma s - |c| \frac{4C_1}{\sqrt{\lambda_1 \pi \alpha}} s - |c| \frac{6}{\sqrt{\pi \alpha}} s - |c|^2 \frac{8\varepsilon^2}{\alpha^2 \sqrt{\lambda_1}} s - |c|^2 \frac{1}{2\alpha \sqrt{\lambda_1}} s - |c| \frac{2C_1}{C_2 \sqrt{\pi \alpha}} s$$

$$= -\frac{16\varepsilon^2 + \alpha}{2\alpha^2 \sqrt{\lambda_1}} |c|^2 s - \frac{2}{\sqrt{\pi \alpha}} \left(\frac{2C_1}{\sqrt{\lambda_1}} + 3 + \frac{C_1}{C_2} \right) |c| s + \sigma s.$$

In order to obtain (4.27), for any $s \le -T_1$, there holds:

$$\frac{16\varepsilon^2 + \alpha}{2\alpha^2 \sqrt{\lambda_1}} |c|^2 + \frac{1}{\sqrt{\pi \alpha}} \left(\frac{2C_1}{\sqrt{\lambda_1}} + 3 + \frac{C_1}{C_2} \right) |c| - \frac{\sigma}{2} < 0.$$

Solving this quadratic inequality, we find that:

$$|c| < \alpha^2 \sqrt{\lambda_1} \frac{ -\frac{1}{\sqrt{\pi \alpha}} \left(\frac{2C_1}{\sqrt{\lambda_1}} + 3 + \frac{C_1}{C_2} \right)^2 + \sqrt{ \frac{1}{\pi \alpha} \left(\frac{2C_1}{\sqrt{\lambda_1}} + 3 + \frac{C_1}{C_2} \right)^2 + \frac{\sigma (16\varepsilon^2 + \alpha)}{\alpha^2 \sqrt{\lambda_1}} } }{16\varepsilon^2 + \alpha}.$$

Since $|z(\theta_t \omega)|$ is tempered, it follows from (4.27) that the following integral is bounded:

$$\begin{aligned} R_1(\omega) &= 2C_1 |U||c| \int_{-t}^0 \mathrm{e}^{-\int_s^0 \rho(\tau, \theta_\tau \omega) \mathrm{d}\tau} |z(\theta_s \omega)| \mathrm{d}s \\ &\quad + \left(2C_3 |U| + \frac{4}{\alpha} \|g(x)\|^2 \right) \int_{-t}^0 \mathrm{e}^{-\int_s^0 \rho(\tau, \theta_\tau \omega) \mathrm{d}\tau} \mathrm{d}s. \end{aligned} \tag{4.28}$$

According to (1.3)-(1.5), we have:

$$\begin{aligned} C_2 \|u\|_1^6 - C_2 |U| &\le \int_U F(u) \mathrm{d}x \le \frac{1}{C_3} \int_U f(u) u \mathrm{d}x + |U| \\ &\le \frac{1}{C_3} \int_U \left(1 + |u|^5 \right) |u| \mathrm{d}x + |U| \\ &\le \frac{1}{C_3} \left(\int_U |u| \mathrm{d}x + \int_U |u|^6 \mathrm{d}x \right) + |U| \\ &\le C \left(|U| + \|u\|_1^6 \right). \end{aligned} \tag{4.29}$$

It follows from theorem 3.1 and $\varphi_0(\theta_{-t}\omega) \in B_0(\theta_{-t}\omega)$ that:

$$\lim_{t \to +\infty} \mathrm{e}^{-\int_{-t}^0 \rho(\tau, \theta_\tau \omega) \mathrm{d}\tau} \left[\|\varphi_0(\theta_{-t}\omega)\|_E^2 + 2\tilde{F}(u_0) \right] = 0. \tag{4.30}$$

Combining with (4.27)-(4.30), there exits $t_B(\omega) \ge T_1$ such that for all $t \ge t_B(\omega)$,

$$\|\varphi(t, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega))\|_E^2 \le C(1 + R_1(\omega)) = M_0(\omega).$$

Then, we complete the proof. □

5. Decomposition of Solutions

In order to obtain regularity estimates later, as in [52], we decompose the equations (3.3). At first, we will give the following decomposition on nonlinearity

$f(u)=f_1(u)+f_2(u)$ and $f_1, f_2 \in C^1(\mathbb{R})$ satisfies the following conditions:

$$\begin{cases} 1. \left|f_1(s)\right| \le C\left(|s|+|s|^5\right), \ \forall s \in \mathbb{R}, \\ 2. \ f_1(s)\cdot s \ge 0, \ \ \forall s \in \mathbb{R}, \\ 3. \exists k_1 \ge 1, k_1 F_1(s) - C \le s f_1(s), \ \ \forall s \in \mathbb{R}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} 1. \left|f_2'(s)\right| \le C\left(1+|s|^p\right), \ f_2(0)=0, \ \forall s \in \mathbb{R}, \ p<4, \\ 2. \exists k_2 \ge 1, k_2 F_2(s) - C \le s f_2(s), \ \ \forall s \in \mathbb{R}, \\ 3. \tilde{C}_2\left(|s|^6 - 1\right) \le F_2(s), \ \ \forall s \in \mathbb{R}, \end{cases} \tag{5.2}$$

where $F_i(s)=\int_0^s f_i(r)\,\mathrm{d}r, (i=1,2)$, $C, \tilde{C}_2 > 0$ are constants.

We decompose the solution $\varphi=(u,w,\eta)^{\mathrm{T}}$ of the system (3.12) into the two parts:

$$\varphi=\varphi_L+\varphi_N,$$

where $\varphi_L=(u_L,w_L,\eta_L)$, $\varphi_N=(u_N,w_N,\eta_N)$ solve the following equations, respectively:

$$\begin{cases} \varphi_L' + H(\varphi_L) + Q_1(\varphi_L) = 0, \\ \varphi_L(0,\omega) = (u_0, u_1+\varepsilon u_0, \eta_0)^{\mathrm{T}}, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \varphi_N' + H(\varphi_N) + Q_2(\varphi,\varphi_L) = \tilde{Q}_2(\varphi,\theta_t\omega), \\ \varphi_N(0,\omega) = (0, -cu_0 z(\omega), 0)^{\mathrm{T}}, \end{cases} \tag{5.4}$$

where

$$Q_1(\varphi_L)=\begin{pmatrix} 0 \\ f_1(u_L) \\ 0 \end{pmatrix}, \ Q_2(\varphi,\varphi_L)=\begin{pmatrix} 0 \\ f(u)-f_1(u_L) \\ 0 \end{pmatrix},$$
$$\tilde{Q}_2(\omega)=\begin{pmatrix} cu_N z(\theta_t\omega) \\ -cz(\theta_t\omega)\left(w_N - 2\varepsilon u_N + cu_N z(\theta_t\omega)\right) + g(x) \\ cu_N z(\theta_t\omega) \end{pmatrix}. \tag{5.5}$$

To prove the existence of a compact random attractor for the random dynamical system Φ, we need to get the solutions of systems (5.3) and (5.4), which one decays exponentially and another is bounded in higher regular space. In order to get the regularity estimate, we will prove some priori estimates for the solutions of systems (5.3) on $U\times[0,\infty]$ as follows.

Lemma 5.1. For any P-a.e. $\omega\in\Omega, t\ge 0$, there exists $M_1(\omega)>0$ such that the solution $\varphi_L=(u_L,w_L,\eta_L)^{\mathrm{T}}$ of (5.3) with initial data $\varphi_L(0,\omega)=(u_0,u_1+\varepsilon u_0,\eta_0)^{\mathrm{T}}=\varphi_0(\theta_{-t}\omega)+(0,cu_0 z(\theta_{-t}\omega),0)^{\mathrm{T}}\in B_0(\theta_{-t}\omega)$ satisfies:

$$\left\|\varphi_L\left(t,\theta_{-t}\omega,\varphi_L(0,\theta_{-t}\omega)\right)\right\|_E^2 \le M_1(\omega). \tag{5.6}$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ of (5.3) with $\varphi_L=(u_L,w_L,\eta_L)^{\mathrm{T}}$, we show that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_L\|_E^2+\left(H(\varphi_L),\varphi_L\right)_E+\left(Q_1(\varphi_L),\varphi_L\right)=0, \tag{5.7}$$

Similar to the proof of (4.8), we obtain that:

$$\left(H(\varphi_L),\varphi_L\right)_E\geq\frac{\varepsilon}{2}\left(\|u_L\|_2^2+\|w_L\|^2\right)+\frac{\alpha}{2}\|w_L\|^2+\frac{\delta}{4}\|\eta_L\|_{\mu,2}^2. \tag{5.8}$$

Now, we estimate the third term of (5.7). According to $(5.1)_3$, we get:

$$\begin{aligned}\left(f_1(u_L),w_L\right)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_U F_1(u_L)\mathrm{d}x+\varepsilon\int_U f_1(u_L)\cdot u_L\mathrm{d}x\\&\geq\frac{\mathrm{d}}{\mathrm{d}t}\int_U F_1(u_L)\mathrm{d}x+k_1\varepsilon\int_U F_1(s)\mathrm{d}x-\varepsilon C|U|.\end{aligned} \tag{5.9}$$

Thus, it follows from (5.7)-(5.10) and (5.3) that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\right)+\sigma_L\left(\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\right)\leq 2\varepsilon C|U|, \tag{5.10}$$

where $\sigma_L=\min\left\{\varepsilon,\frac{\delta}{2},k_1\varepsilon\right\}$. By Gronwall's Lemma to (5.10), we have:

$$\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\leq \mathrm{e}^{-\sigma_L t}\left(\|\varphi_L(0,\omega)\|_E^2+2\tilde{F}_1(u_0)\right)+\frac{2\varepsilon C|U|}{\sigma_L}. \tag{5.11}$$

According to (5.1), we have:

$$\begin{aligned}0&\leq\int_U F(u)\mathrm{d}x\leq\frac{1}{k_1}\int_U f(u)u\mathrm{d}x+\frac{C}{k_1}|U|\leq\frac{1}{k_1}\int_U\left(1+|u|^5\right)|u|\mathrm{d}x+\frac{C}{k_1}|U|\\&\leq\frac{C}{k_1}\left(\int_U|u|\mathrm{d}x+\int_U|u|^6\mathrm{d}x\right)+\frac{C}{k_1}|U|\leq C\left(|U|+\|u\|_1^6\right).\end{aligned} \tag{5.12}$$

Combining (5.11)-(5.12) with $\varphi_L(0,\theta_{-t}\omega)\in B_0(\theta_{-t}\omega)$, we get:

$$\begin{aligned}&\left\|\varphi_L\left(t,\theta_{-t}\omega,\varphi_L(0,\theta_{-t}\omega)\right)\right\|_E^2\\&\leq\mathrm{e}^{-\sigma_L t}\left(\|\varphi_L(0,\theta_{-t}\omega)\|_E^2+2C\left(|U|+\|u_0\|_1^6\right)\right)+\frac{2\varepsilon C|U|}{\sigma_L}\\&=M_1(\omega).\end{aligned} \tag{5.13}$$

So, the proof is completed. □

Lemma 5.2. For any P-a.e. $\omega\in\Omega, t\geq 0$, there exists $M_2(\omega)>0,\sigma_1(\omega)\geq 0$ such that the solution $\varphi_L=(u_L,w_L,\eta_L)^{\mathrm{T}}$ of (5.3) with initial data $\varphi_L(0,\omega)=(u_0,u_1+\varepsilon u_0,\eta_0)^{\mathrm{T}}\in B_0(\theta_{-t}\omega)$ satisfies:

$$\left\|\varphi_L\left(t,\theta_{-t}\omega,\varphi_L(0,\theta_{-t}\omega)\right)\right\|_E^2\leq M_2(\omega)\mathrm{e}^{-\sigma_1(\omega)t},\ t\geq 0. \tag{5.14}$$

Proof. We consider (5.1), (5.7) and similar to Lemma 5.1, we conclude that:

$$0\leq\tilde{F}_1(u_L)\leq C\left(\|u_L\|^2+\|u_L\|_{L^6(U)}^6\right). \tag{5.15}$$

Applying interpolation inequality, we have:

$$\begin{aligned}\|u_L\|_{L^6}^6 &\le \|u_L\|\cdot\|u_L\|_{L^{10}}^5 \le \|u_L\|_{L^{10}}^4\left(\|u_L\|\cdot\|u_L\|_{L^{10}}\right)\\ &\le \|u_L\|_{L^{10}}^4\left(\frac{1}{2}\|u_L\|^2+\frac{1}{2}\|u_L\|_{L^{10}}^2\right)\le C\|u_L\|_2^4\left(\|u_L\|^2+\|u_L\|_2^2\right).\end{aligned} \tag{5.16}$$

Hence, combining (5.15)-(5.16) with Lemma 5.1, we find that there exists $M_3(\omega)>0$, such that:

$$\|u_L\|_2^2 \ge \frac{1}{M_3(\omega)}\tilde{F}_1(u_L). \tag{5.17}$$

Due to (5.7)-(5.8), $(5.1)_2$ and (5.17), we can obtain the following result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\right)+\frac{\varepsilon}{2}\left(\|u_L\|_2^2+\|w_L\|^2\right)+\frac{\delta}{2}\|\eta_L\|_{\mu,2}^2+\frac{\varepsilon}{2M_3(\omega)}\tilde{F}_1(u_L)\le 0,$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\right)+\sigma_1(\omega)\left(\|\varphi_L\|_E^2+2\tilde{F}_1(u_L)\right)\le 0, \tag{5.18}$$

where $\sigma_1(\omega)=\min\left\{\frac{\varepsilon}{2},\frac{\delta}{2},\frac{\varepsilon}{4M_3(\omega)}\right\}$.

By applying Gronwall's inequality to (5.18), it yields:

$$\begin{aligned}\left\|\varphi_L\left(t,\theta_{-t}\omega,\varphi_L(0,\theta_{-t}\omega)\right)\right\|_E^2 &\le \left(\|\varphi_L(0,\theta_{-t}\omega)\|_E^2+\tilde{F}_1(u_0)\right)\mathrm{e}^{-\sigma_1(\omega)t}\\ &\le \left(\|\varphi_L(0,\theta_{-t}\omega)\|_E^2+C\left(|U|+\|u\|_1^6\right)\right)\mathrm{e}^{-\sigma_1(\omega)t}\\ &= M_2(\omega)\mathrm{e}^{-\sigma_1(\omega)t}.\end{aligned} \tag{5.19}$$

Then, the proof is completed. □

Next, we estimate the component φ_N in (5.4).

Lemma 5.3. For any P-a.e. $\omega\in\Omega, t\ge 0$, there exists $\sigma_2(\omega)>0$ such that the solution $\varphi_N=(u_N,w_N,\eta_N)^{\mathrm{T}}$ of (5.4) with initial data $\varphi_N(0,\omega)=\left(0,-cu_N z(\theta_t\omega,0),\eta_0\right)^{\mathrm{T}}\in B_0(\omega)$ satisfies:

$$\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2+\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|^2+\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2\le \mathrm{e}^{\frac{\sigma_2(\omega)}{2}t}P_1(\theta_{-t}\omega), \tag{5.20}$$

where

$$\nu=\min\left\{\frac{1}{8},\frac{4-p}{4}\right\},\ \forall\, 0\le p<4, \tag{5.21}$$

and $P_1(\theta_{-t}\omega)$ is increasing function.

Proof. By (4.1), (5.6) and $\varphi_N=\varphi-\varphi_L$, there exists a random variable $R_3(\omega)>0$ such that:

$$\max\left\{\left\|\varphi\left(r,\theta_{-t}\omega,\varphi(0,\theta_{-t}\omega)\right)\right\|_E,\left\|\varphi_N\left(\left(r,\theta_{-t}\omega,\varphi_N(0,\theta_{-t}\omega)\right)\right)\right\|_E\right\}\le R_1(\omega),\ r\ge -t. \tag{5.22}$$

Taking the inner product of $(\cdot,\cdot)_E$ of (5.4) with $\left(A^\nu\varphi_N, A^\nu w_N, A^\nu\eta_N\right)^{\mathrm{T}}$, we find that:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+\left(H\left(\varphi_N\right),A^{\nu}\varphi_N\right)_E\\
&=\left(g\left(x\right),A^{\nu}w_N\right)-\left(cz\left(\theta_t\omega\right)w_N,A^{\nu}w_N\right)+\left(2\varepsilon cu_N z\left(\theta_t\omega\right),A^{\nu}w_N\right)\\
&\quad-\left(f\left(u\right)-f_1\left(u_L\right),A^{\nu}w_N\right)-\left(c^2z^2\left(\theta_t\omega\right)u_N,A^{\nu}w_N\right)\\
&\quad+\left(\left(cu_N z\left(\theta_t\omega\right),A^{\nu}u_N\right)\right)+\left(cu_N z\left(\theta_t\omega\right),A^{\nu}\eta_N\right)_{\nu,2}
\end{aligned} \tag{5.23}$$

In later calculations, we will use the following embedding relations:

$$H^{2\nu}\hookrightarrow L^{\frac{10}{5-4\nu}},H^{2-2\nu}\hookrightarrow L^{\frac{10}{1+4\nu}},H^{2+2\nu}\hookrightarrow L^{\frac{10}{1-4\nu}},L^{10}\hookrightarrow L^{\frac{10p}{4-4\nu}}\left(\frac{10p}{4-4\nu}<10\right). \tag{5.24}$$

Similar to the proof of (4.8), we deduce that:

$$\left(H\left(\varphi_N\right),A^{\nu}\varphi_N\right)_E\ge\frac{\varepsilon}{2}\left(\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2+\left\|A^{\frac{\nu}{2}}w_N\right\|^2\right)+\frac{\alpha}{2}\left\|A^{\frac{\nu}{2}}w_N\right\|^2+\frac{\delta}{4}\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2. \tag{5.25}$$

Next, we will deal with the right-hand side of (5.23). Using (4.5)-(4.10) and (5.21), we get:

$$\left(\left(cu_N z\left(\theta_t\omega\right),A^{\nu}u_N\right)\right)\le\left|c\right|\left|z\left(\theta_t\omega\right)\right|\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2, \tag{5.26}$$

$$\left(cw_N z\left(\theta_t\omega\right),A^{\nu}w_N\right)\le\left|c\right|\left|z\left(\theta_t\omega\right)\right|\left\|A^{\frac{\nu}{2}}w_N\right\|^2, \tag{5.27}$$

$$\left(2c\varepsilon u_N z\left(\theta_t\omega\right),A^{\nu}w_N\right)\le\frac{8\varepsilon^2\left|c\right|^2\left|z^2\left(\theta_t\omega\right)\right|}{\alpha\sqrt{\lambda_1}}\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2+\frac{\alpha}{8}\left\|A^{\frac{\nu}{2}}w_N\right\|^2, \tag{5.28}$$

$$\left(c^2u_Nz^2\left(\theta_t\omega\right),A^{\nu}w_N\right)\le\frac{\left|c\right|^2\left|z\left(\theta_t\omega\right)\right|^2}{2\sqrt{\lambda_1}}\left(\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2+\left\|A^{\frac{\nu}{2}}w_N\right\|^2\right), \tag{5.29}$$

$$\left(g\left(x\right),A^{\nu}w_N\right)\le\frac{2}{\alpha}\left\|A^{\frac{\nu}{2}}g\left(x\right)\right\|^2+\frac{\alpha}{8}\left\|A^{\frac{\nu}{2}}w_N\right\|^2, \tag{5.30}$$

$$\left(cu_Nz\left(\theta_t\omega\right),A^{\nu}\eta_N\right)_{\mu,2}\le\frac{\left|c\right|\left|z\left(\theta_t\omega\right)\right|}{2\sqrt{\lambda_1}}\left(\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2+\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2\right). \tag{5.31}$$

For the nonlinear term, we have:

$$\begin{aligned}
\left(f\left(u\right)-f_1\left(u_L\right),A^{\nu}w_N\right)&=\left(f_2\left(u_L\right),A^{\nu}w_N\right)+\left(f\left(u\right)-f\left(u_L\right),A^{\nu}w_N\right)\\
&=\left(f_2\left(u_L\right),A^{\nu}\left(u_{Nt}+\varepsilon u_N-cu_Nz\left(\theta_t\omega\right)\right)\right)\\
&\quad+\left(f\left(u\right)-f\left(u_L\right),A^{\nu}\left(u_{Nt}+\varepsilon u_N-cu_Nz\left(\theta_t\omega\right)\right)\right).
\end{aligned}$$

Firstly, we deal with the term:

$$\begin{aligned}
&\left(f_2\left(u_L\right),A^{\nu}u_{Nt}+\varepsilon A^{\nu}u_NA^{\nu}u_N\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(f_2\left(u_L\right),A^{\nu}u_N\right)+\varepsilon\left(f_2\left(u_L\right),A^{\nu}u_N\right)-\left(f_2'\left(u_L\right)u_{Lt},A^{\nu}u_N\right),
\end{aligned} \tag{5.32}$$

by $(5.2)_1$, (5.24) and Lemma 5.1, we have:

$$
\begin{aligned}
&\left|\left(f_2'(u_L)u_{Lt}, A^\nu u_N\right)\right| \\
&\le C\int_U \left(1+|u_L|^p\right)|u_{Lt}|\left|A^\nu u_N\right| \mathrm{d}x \\
&\le C\left(\int_U \left|A^\nu u_N\right|^{\frac{10}{1+4\nu}} \mathrm{d}x\right)^{\frac{1+4\nu}{10}} \cdot \left(\int_U \left(1+|u_L|^p\right)^{\frac{10}{4-4\nu}} \mathrm{d}x\right)^{\frac{4-4\nu}{10}} \cdot \left(\int_U |u_{Lt}|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\left\|A^{\frac{\nu}{2}}u_N\right\|_2 \cdot \left(1+\|u_L\|_2^p\right)\cdot \|u_{Lt}\| \\
&\le R_2(\omega)+\frac{\varepsilon}{8}\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2,
\end{aligned} \tag{5.33}
$$

and

$$
\begin{aligned}
&\left|\left(f_2(u_L), cA^\nu u_N z(\theta_t\omega)\right)\right| \\
&\le C|c|\left|z(\theta_t\omega)\right|\int_U \left(1+|u_L|^p\right)|u_L|\left|A^\nu u_N\right| \mathrm{d}x \\
&\le C|c|\left|z(\theta_t\omega)\right|\left(\int_U \left(1+|u_L|^p\right)^{\frac{10}{4-4\nu}} \mathrm{d}x\right)^{\frac{4-4\nu}{10}} \cdot \left(\int_U |u_L|^2 \mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_U \left|A^\nu u_N\right|^{\frac{10}{1+4\nu}} \mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C|c|\left|z(\theta_t\omega)\right|\left(1+\|u_L\|_2^p\right)\cdot \|u_L\| \cdot \left\|A^{\frac{\nu}{2}}u_N\right\|_2 \\
&\le R_3(\omega)|c|^2\left|z(\theta_t\omega)\right|^2+\frac{\varepsilon}{8}\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2.
\end{aligned} \tag{5.34}
$$

Secondly, we consider the following term:

$$
\begin{aligned}
&\left(f(u)-f(u_L), A^\nu u_{Nt}+\varepsilon A^\nu u_N\right) \\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(f(u)-f(u_L), A^\nu u_N\right)+\varepsilon\left(f(u)-f(u_L), A^\nu u_N\right) \\
&\quad -\left(f'(u)u_t-f'(u_L)u_{Lt}, A^\nu u_N\right) \\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(f(u)-f(u_L), A^\nu u_N\right)+\varepsilon\left(f(u)-f(u_L), A^\nu u_N\right) \\
&\quad -\left(f'(u)u_{Lt}-f'(u_L)u_{Lt}, A^\nu u_N\right)-\left(f'(u)u_{Nt}, A^\nu u_N\right).
\end{aligned} \tag{5.35}
$$

According to (1.2), (5.24), Lemma 4.1, 5.1 and $w_N(t,\omega,x)=u_{Nt}(t,\omega,x)+\varepsilon u_N(t,\omega,x)-cu_N z(\theta_t\omega)$, we obtain that:

$$
\begin{aligned}
&\left|\left(f'(u)u_{Lt}-f'(u_L)u_{Lt}, A^\nu u_N\right)\right| \\
&\le C\int_U |u_{Lt}|\cdot|u_N|\cdot\left(1+|u|^3+|u_L|^3\right)\left|A^\nu u_N\right| \mathrm{d}x \\
&\le C\left(\int_U |u_{Lt}|^2 \mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_U |u_N|^{\frac{10}{1-4\nu}} \mathrm{d}x\right)^{\frac{1-4\nu}{10}} \cdot \left(\int_U \left(1+|u|^3+|u_L|^3\right)^{\frac{10}{3}} \mathrm{d}x\right)^{\frac{3}{10}} \\
&\quad \cdot \left(\int_U \left|A^\nu u_N\right|^{\frac{10}{1+4\nu}} \mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le R_4(\omega)\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2,
\end{aligned} \tag{5.36}
$$

$$\begin{aligned}
&\left|\left(f'(u)u_{Nt},A^{\nu}u_N\right)\right| \\
&\le C_1\int_U |u_{Nt}|\cdot\left(1+|u|^4\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C_1\left(\int_U |u_{Nt}|^{\frac{10}{5-4\nu}}\,\mathrm{d}x\right)^{\frac{5-4\nu}{10}}\cdot\left(\int_U\left(1+|u|^4\right)^{\frac{10}{4}}\mathrm{d}x\right)^{\frac{4}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|\cdot\left(1+\|u\|_2^4\right)\cdot\left\|A^{\frac{\nu}{2}}u_N\right\|_2 \le R_5(\omega)\left\|A^{\frac{\nu}{2}}u_N\right\|_2\cdot\left(\left\|A^{\frac{\nu}{2}}u_N\right\|+\varepsilon\right),
\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}
&\left|\left(f(u)-f(u_L),cA^{\nu}u_N z(\theta_t\omega)\right)\right| \\
&\le C|c||z(\theta_t\omega)|\int_U |u_N|\cdot\left(1+|u|^4+|u_L|^4\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C|c||z(\theta_t\omega)|\left(\int_U |u_N|^{\frac{10}{5-4\nu}}\,\mathrm{d}x\right)^{\frac{5-4\nu}{10}}\cdot\left(\int_U\left(1+|u|^4+|u_L|^4\right)^{\frac{10}{4}}\mathrm{d}x\right)^{\frac{4}{10}} \\
&\quad\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C|c||z(\theta_t\omega)|\left\|A^{\frac{\nu}{2}}u_N\right\|\left\|A^{\frac{\nu}{2}}u_N\right\|_2\left(1+\|u\|_2^4+\|u_L\|_2^4\right) \\
&\le R_6(\omega)|c||z(\theta_t\omega)|\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2.
\end{aligned} \tag{5.38}$$

In addition, by (1.2), $(5.2)_1$, (5.24) and Lemma 5.1, we find that:

$$\begin{aligned}
&\left|\left(f_2(u_L),A^{\nu}u_N\right)\right| \\
&\le C\int_U |u_L|\cdot\left(1+|u_L|^p\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C\left(\int_U |u_L|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_U\left(1+|u_L|^p\right)^{\frac{10}{4-4\nu}}\mathrm{d}x\right)^{\frac{4-4\nu}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C\|u_L\|\cdot\left\|A^{\frac{\nu}{2}}u_N\right\|_2\cdot\left(1+\|u_L\|_2^p\right) \\
&\le R_7(\omega)\left\|A^{\frac{\nu}{2}}u_N\right\|_2,
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
&\left|\left(f(u)-f(u_L),A^{\nu}u_N\right)\right| \\
&\le C\int_U |u_N|\cdot\left(1+|u|^4+|u_L|^4\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C\left(\int_U |u_N|^{\frac{10}{5-4\nu}}\,\mathrm{d}x\right)^{\frac{5-4\nu}{10}}\cdot\left(\int_U\left(1+|u|^4+|u_L|^4\right)^{\frac{10}{4}}\mathrm{d}x\right)^{\frac{4}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C\left\|A^{\frac{\nu}{2}}u_N\right\|\left(1+\|u\|_2^4+\|u_L\|_2^4\right)\left\|A^{\frac{\nu}{2}}u_N\right\|_2
\end{aligned}$$

$$\leq R_8(\omega)\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2. \tag{5.40}$$

Thus, combining with (5.23)-(5.40), we can show that:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^\nu u_N\right)\right)\\&\leq\left(\sigma_2(\omega)+\frac{1+2R_6(\omega)\sqrt{\lambda_1}+2\sqrt{\lambda_1}}{\sqrt{\lambda_1}}|c||z|+\frac{16\varepsilon^2+\alpha}{\alpha^2\sqrt{\lambda_1}}|c|^2|z|^2\right)\\&\cdot\left(\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^\nu u_N\right)\right)+R_9(\omega)+2C|c|^2|z|^2+C\|g\|_1^2,\end{aligned} \tag{5.41}$$

that is,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^\nu u_N\right)\right)\\&-\rho_1(t,\theta_t\omega)\left(\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^\nu u_N\right)\right)\\&\leq R_9(\omega)+2C|c||z|+C\|g\|_1^2,\end{aligned} \tag{5.42}$$

where

$$\rho_1(t,\theta_t\omega)=\sigma_2(\omega)+\frac{1+2R_6(\omega)\sqrt{\lambda_1}+2\sqrt{\lambda_1}}{\sqrt{\lambda_1}}|c||z|+\frac{16\varepsilon^2+\alpha}{\alpha\sqrt{\lambda_1}}|c|^2|z|^2. \tag{5.43}$$

Let

$$Y(t)=\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^\nu u_N\right).$$

By Gronwall's inequality to (5.42), we have:

$$\begin{aligned}Y(t)\leq{}&\mathrm{e}^{\int_0^t\rho_1(s,\theta_s\omega)\mathrm{d}s}Y(0)+2C|c|^2\int_0^t\mathrm{e}^{\int_s^t\rho_1(\tau,\theta_\tau\omega)\mathrm{d}\tau}|z(\theta_s\omega)|^2\,\mathrm{d}s\\&+\left(R_9(\omega)+C\|g(x)\|_1^2\right)\int_0^t\mathrm{e}^{\int_s^t\rho_1(\tau,\theta_\tau\omega)\mathrm{d}\tau}\mathrm{d}s.\end{aligned}$$

Similar to the proof of Lemma 4.1, we have:

$$\begin{aligned}&Y\left(t,\theta_{-t}\omega,\varphi(0,\theta_{-t}\omega)\right)\\&\leq\mathrm{e}^{\int_{-t}^0\rho_1(s,\theta_s\omega)\mathrm{d}s}Y\left(0,\theta_{-t}\omega,\varphi(0,\theta_{-t}\omega)\right)+2C|c|^2\int_{-t}^0\mathrm{e}^{\int_s^0\rho_1(\tau,\theta_\tau\omega)\mathrm{d}\tau}|z(\theta_s\omega)|^2\,\mathrm{d}s\\&+\left(R_{10}(\omega)+C\|g(x)\|_1^2\right)\int_{-t}^0\mathrm{e}^{\int_s^0\rho_1(\tau,\theta_\tau\omega)\mathrm{d}\tau}\mathrm{d}s.\end{aligned} \tag{5.44}$$

Due to (4.26) again, there exists $\sigma_2(\omega)>0$ such that:

$$\mathrm{e}^{\int_s^0\rho_1(\tau,\theta_\tau\omega)\mathrm{d}\tau}\leq\mathrm{e}^{-\frac{\sigma_2(\omega)}{2}s}, \tag{5.45}$$

where c satisfies (4.10). Thus, it follows from (5.39)-(5.40) and (5.43)-(5.46) that:

$$\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2+\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|^2+\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2\le e^{\frac{\sigma_2(\omega)}{2}t}P_1(\theta_{-t}\omega), \tag{5.46}$$

where $P(\theta_{-t}\omega)$ is increasing function. Then, the proof is finished. □

Lemma 5.4. Assume that (h_2) holds. For any $\iota>0$, there exists C_ι and $M_\iota(\omega)$, such that $\mathbb{P}$-*a.s.* $u_t(s)=u_1(s)+u_2(s)$,

$$\|u_1(s)\|_{2+2\nu}\le K_\iota(\omega),\forall -t\le s\le 0, \tag{5.47}$$

$$\int_{r_1}^{r_2}\|u_2\|_2^2\,ds\le \iota(r_2-r_1)+C_\iota,\forall -t\le r_1\le r_2\le 0. \tag{5.48}$$

Proof. Combining (5.14), (5.20) with the technique used in [55], we can finish the proof. □

Lemma 5.5. Assume that (h_1) - (h_2) and (5.1)-(5.2) hold. There exists a random radius $M_3(\omega)$ such that for P-a.e. $\omega\in\Omega$, the solution $\varphi_N(t,\theta_{-t}\omega,\varphi_N(0,\theta_{-t}\omega))$ of (5.4) satisfies:

$$\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2+\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|^2+\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2\le M_3(\omega), \tag{5.49}$$

where ν is given in (5.21).

Proof. Taking the inner product of $(\cdot,\cdot)_E$ of (5.4) with $\left(A^\nu\varphi_N,A^\nu w_N,A^\nu\eta_N\right)^{\mathrm{T}}$, we find that:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^{\frac{\nu}{2}}u_N\right)\right]+\left(H(\varphi),\varphi\right)_E+\varepsilon\left(f(u)-f_1(u_L),A^{\frac{\nu}{2}}u_N\right)\\
&-\left(\left[f_1'(u)-f_1'(u_L)\right]u_t,A^{\frac{\nu}{2}}u_N\right)-\left(f_1'(u_L)u_{Nt},A^{\frac{\nu}{2}}u_N\right)-\left(f_2'(u)u_t,A^{\frac{\nu}{2}}u_N\right)\\
&=\left(g(x),A^\nu w_N\right)-\left(cz(\theta_t\omega)w_N,A^\nu w_N\right)+\left(2\varepsilon cu_Nz(\theta_t\omega),A^\nu w_N\right)\\
&\quad-\left(c^2z^2(\theta_t\omega)u_N,A^\nu w_N\right)+\left(\left(cu_Nz(\theta_t\omega),A^\nu u_N\right)\right)\\
&\quad+\left(cu_Nz(\theta_t\omega),A^\nu\eta_N\right)_{\nu,2}-\left(f(u)-f_1(u_L),cA^\nu u_Nz(\theta_t\omega)\right)
\end{aligned} \tag{5.50}$$

First, we deal with the nonlinearity in (5.50). Applying (5.1), (5.6), (5.22) and Hölder's inequality, we have:

$$\begin{aligned}
&\left|\left(f_1'(u)u_t-f_1'(u_L)u_t,A^\nu u_N\right)\right|\\
&\le C\int_U|u_1+u_2|\cdot|u_N|\cdot\left(1+|u|^3+|u_L|^3\right)\left|A^\nu u_N\right|\mathrm{d}x\\
&\le C\left(\int_U|u_2|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_U|u_N|^{\frac{10}{1-4\nu}}\mathrm{d}x\right)^{\frac{1-4\nu}{10}}\cdot\left(\int_U\left(1+|u|^3+|u_L|^3\right)^{\frac{10}{3}}\mathrm{d}x\right)^{\frac{3}{10}}\\
&\quad\cdot\left(\int_U\left|A^\nu u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}}+C\left(\int_U|u_1|^{\frac{10}{1-4\nu}}\mathrm{d}x\right)^{\frac{1-4\nu}{10}}\cdot\left(\int_U|u_N|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad\cdot\left(\int_U\left(1+|u|^3+|u_L|^3\right)^{\frac{10}{3}}\mathrm{d}x\right)^{\frac{3}{10}}\cdot\left(\int_U\left|A^\nu u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}}\\
&\le R_{10}(\omega)\left(\|u_2\|_2+\left\|A^{\frac{1+\nu}{2}}u_1\right\|\right)\left\|A^{\frac{\nu}{2}}u_N\right\|_2^2,
\end{aligned} \tag{5.51}$$

$$
\begin{aligned}
&\left|\left(f_1'(u_L)u_{Nt}, A^{\nu}u_N\right)\right| \\
&\le C_1\int_U |u_{Nt}|\cdot\left(1+|u_L|^4\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C_1\left(\int_U |u_{Nt}|^{\frac{10}{5-4\nu}}\mathrm{d}x\right)^{\frac{5-4\nu}{10}}\cdot\left(\int_U\left(1+|u_L|^4\right)^{\frac{10}{4}}\mathrm{d}x\right)^{\frac{4}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \qquad (5.52)\\
&\le C\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|\cdot\left(1+\|u_L\|_2^4\right)\cdot\left\|A^{\frac{\nu}{2}}w_N\right\|_2 \le R_{11}(\omega)\left\|A^{\frac{\nu}{2}}w_N\right\|_2\cdot\left(\left\|A^{\frac{\nu}{2}}w_N\right\|+\varepsilon\right) \\
&\le R_{11}(\omega)\left\|A^{\frac{\nu}{2}}w_N\right\|_2^2+\frac{\alpha}{4}\left\|A^{\frac{\nu}{2}}w_N\right\|^2+R_{12}(\omega)\left\|A^{\frac{\nu}{2}}w_N\right\|_2,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left(f_2'(u)u_t, A^{\nu}u_N\right)\right| \\
&\le C\int_U |u_1+u_2|\cdot\left(1+|u|^p\right)\left|A^{\nu}u_N\right|\mathrm{d}x \\
&\le C\left(\int_U |u_1|^{10}\mathrm{d}x\right)^{\frac{1}{10}}\cdot\left(\int_U\left(1+|u|^p\right)^{\frac{10}{4-4\nu}}\mathrm{d}x\right)^{\frac{4-4\nu}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{5+4\nu}}\mathrm{d}x\right)^{\frac{5+4\nu}{10}} \\
&\quad +C\left(\int_U |u_2|^{2}\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_U\left(1+|u|^p\right)^{\frac{10}{4-4\nu}}\mathrm{d}x\right)^{\frac{4-4\nu}{10}}\cdot\left(\int_U\left|A^{\nu}u_N\right|^{\frac{10}{1+4\nu}}\mathrm{d}x\right)^{\frac{1+4\nu}{10}} \\
&\le C\left(\|u_2\|_2+\left\|A^{\frac{1}{2}}u_1\right\|\right)\cdot\left(1+\|u\|_2^p\right)\left\|A^{\frac{\nu}{2}}u_N\right\| \\
&\le C\left(\|u_2\|_2+\left\|A^{\frac{1+\nu}{2}}u_1\right\|\right)\cdot\left(1+\|u\|_2^p\right)\left\|A^{\frac{\nu}{2}}u_N\right\|_2 \qquad (5.53)\\
&\le R_{13}(\omega)\left(\|u_2\|_2+\left\|A^{\frac{1+\nu}{2}}u_1\right\|\right)\left\|A^{\frac{\nu}{2}}u_N\right\|_2.
\end{aligned}
$$

By (5.26)-(5.31), (5.34), (5.38) and (5.50)-(5.53), we obtain that:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}(t)+\left(\rho_3(\omega,\theta_t\omega)-R_{14}(\omega)\|u_2\|_2^2\right)\tilde{Y}(t) \\
&\le\frac{4}{\alpha}\left\|A^{\frac{\nu}{2}}g\right\|^2+2C|c|^2\left|z(\theta_t\omega)\right|^2+R_{15}(\omega),
\end{aligned} \qquad (5.54)
$$

where

$$\sigma_3=\min\left\{\frac{\varepsilon}{2},\frac{\delta}{2}\right\},$$

$$\tilde{Y}(t)=\left\|A^{\frac{\nu}{2}}\varphi_N\right\|_E^2+2\left(f(u)-f_1(u_L),A^{\frac{\nu}{2}}u_N\right)$$

$$\rho_3(\omega,\theta_t\omega)=\sigma_3-\frac{16\varepsilon^2+\alpha}{\alpha\sqrt{\lambda_1}}|c|^2\left|z(\theta_t\omega)\right|^2-\frac{2\sqrt{\lambda_1}+1+2C\sqrt{\lambda_1}}{\sqrt{\lambda_1}}|c|\left|z(\theta_t\omega)\right|.$$

By Gronwall's inequality to (5.54), we get that:

$$\begin{aligned}\tilde{Y}(t) &\le e^{-\int_0^t\left(\rho_3(s,\theta_s\omega)-R_{14}(\omega)\|u_2(s)\|_2^2\right)ds}\tilde{Y}(0)+2C|c|\int_0^t e^{-\int_s^t\left(\rho_3(\tau,\theta_\tau\omega)-R_{14}(\omega)\|u_2(\tau)\|_2^2\right)d\tau}\left|z(\theta_s\omega)\right|^2 ds \\ &+\left(R_{15}(\omega)+C\|g(x)\|_1^2\right)\int_0^t e^{-\int_s^t\left(\rho_3(\tau,\theta_\tau\omega)-R_{14}(\omega)\|u_2(\tau)\|_2^2\right)d\tau}ds \\ &= e^{-\int_0^t\left(\rho_3(s,\theta_s\omega)-R_{14}(\omega)\|u_2(s)\|_2^2\right)ds}\tilde{Y}(0)+M_4(\omega),\end{aligned} \tag{5.55}$$

where

$$M_4(\omega)=2C|c|\int_0^t e^{-\int_s^t\left(\rho_3(\tau,\theta_\tau\omega)-R_{14}(\omega)\|u_2(\tau)\|_2^2\right)d\tau}\left|z(\theta_s\omega)\right|^2 ds +\left(R_{15}(\omega)+C\|g(x)\|_1^2\right)\int_0^t e^{-\int_s^t\left(\rho_3(\tau,\theta_\tau\omega)-R_{14}(\omega)\|u_2(\tau)\|_2^2\right)d\tau}ds.$$

For any $\iota>0$ such that $\iota<\dfrac{\sigma_3}{2R_{13}(\omega)}$ is so small,

$$e^{\int_0^s R_{14}(\omega)\|u_2(\tau)\|_2^2 d\tau}\le e^{\frac{1}{4}\sigma_3 s}e^{R_{16}(\omega)}.$$

Next, similar to the proof of Lemma 4.1, we know that:

$$e^{\int_0^s \rho_3(\tau,\theta_\tau\omega)d\tau}\le e^{\frac{\sigma_3}{2}s}, \tag{5.56}$$

where c satisfies (4.10).

Hence, combining (5.39)-(5.40) with tempered $\left|z(\theta_t\omega)\right|$, we obtain that:

$$\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2+\left\|A^{\frac{\nu}{2}}u_{Nt}\right\|^2+\left\|A^{\frac{\nu}{2}}\eta_N\right\|_{\mu,2}^2\le e^{-\frac{\sigma_3}{2}t}\tilde{Y}(0)+M_4(\omega)\le M_3(\omega).$$

Then, the proof is completed. □

6. Random Attractors

In this section, we establish the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (3.12) on $\mathbb{R}^5$, that is, by Lemma 4.1, Φ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness and then implies the existence of a unique $\mathcal{D}$-random attractor. Next, due to decomposition of solutions, we shall prove the $\mathcal{D}$-pullback asymptotic compactness of Φ (see [56] [57]).

Since $\omega\in\Omega, t\ge 0$, we get:

$$\eta_N\left(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega),s\right) =\begin{cases}u_N\left(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)\right)-u_N\left(t-s,\theta_{-t}\omega,\varphi_0(\theta_{-t+s}\omega)\right), & s\le t,\\ u_N\left(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)\right), & t\le s;\end{cases} \tag{6.1}$$

$$\eta_{N_s}\left(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega),s\right)=\begin{cases}u_{N_t}\left(t-s,\theta_{-t+s}\omega,\varphi_0(\theta_{-t+s}\omega)\right), & 0\le s\le t,\\ 0, & t\le s.\end{cases} \tag{6.2}$$

Denote $\tilde{B}(\omega)$ as:

$$\tilde{B}(\omega)=\overline{\bigcup_{\varphi_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)}\bigcup_{t\ge 0}\eta_N\left(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega),s\right)}, s\in\mathbb{R}^+,\omega\in\Omega,t\ge 0, \tag{6.3}$$

is the solution of (3.12), where $\varphi=(u,w,\eta)^{\mathrm{T}}$. Next, it follows from Lemma 5.5

and (6.1)-(6.2) that:

$$\max\left\{\left\|\eta_{Ns}\left(t,\theta_t\omega,\varphi_0\left(\theta_{-t}\omega\right),s\right)\right\|_{\mu,2\nu}^2,\left\|\eta_N\left(t,\theta_{-t}\omega,\varphi_0\left(\theta_{-t}\omega\right),s\right)\right\|_{\mu,2\nu+2}^2\right\} \leq M_3(\omega),\ \forall s\geq 0, \tag{6.4}$$

which implies $\tilde{B}(\omega)$ is bounded in $L_\mu^2\left(\mathbb{R}^+,V_{2\nu+2}\right)\cap H_\mu^2\left(\mathbb{R}^+,V_{2\mu}\right)$. Also, by Lemmas 4.1, 5.5 and (6.2), there holds:

$$\sup_{\eta\in\tilde{B}(\omega),s\geq 0}\left\|\Delta\eta(s)\right\|^2=\sup_{t\geq}\ \sup_{\varphi_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)}\left\|\Delta\eta_N\left(t,\theta_{-t}\omega,\varphi_0\left(\theta_{-t}\omega\right),s\right)\right\|^2\leq 2M_0(\omega). \tag{6.5}$$

By (h_1) and (6.5), for any $\eta\in\tilde{B}(\omega)$, we find that:

$$\left\|\eta(s)\right\|_{\mu,2}^2=\int_0^{+\infty}\mu(s)\left\|\Delta\eta(s)\right\|^2\mathrm{d}s\leq 2R_1(\omega)\int_0^{+\infty}\mathrm{e}^{-\delta s}\mathrm{d}s\leq\frac{2M_0(\omega)}{\delta}, \tag{6.6}$$

which shows that $\tilde{B}(\omega)\subset L_\mu^2\left(\mathbb{R}^+,H_0^2(U)\right)$ is bounded. It follows from Lemma 2.11 that the set $\tilde{B}(\omega)$ is relatively compact in $L_\mu^2\left(\mathbb{R}^+,H_0^2(U)\right)$. Next, we investigate the main result about the existence of a random attractor for Random Dynamical System Φ.

Lemma 6.1. Assume that (h_1) - (h_2) hold. Then, for any $t\geq 0,\omega\in\Omega$, the RDS Φ associated with (3.12) possesses a uniformly $\mathfrak{D}(E)$-attracting set $\Lambda(\omega)\subset E$ and possesses a $\mathfrak{D}(E)$-random attractor $\mathcal{A}(\omega)\subseteq\Lambda(\omega)\cap B_0(\omega)$.

Proof For any $t\geq 0,\omega\in\Omega$, as Lemma 5.5, let $B_\nu(\omega)$ be the closed ball in $V_{2+2\nu}\times V_{2\nu}$ of radius $\sqrt{M_5(\omega)}$. Set:

$$\Lambda(\omega)=B_\nu(\omega)\times\tilde{B}(\omega). \tag{6.7}$$

Then, $\Lambda(\omega)\in\mathfrak{D}(E)$. Because $V_{2+2\nu}\times V_{2\nu}\hookrightarrow H_0^2(U)\times L^2(U)$ is compact, and $B_\nu(\omega)$ is compact in $H_0^2(U)\times L^2(U)$. At the same time, $\tilde{B}(\omega)$ is compact in $\mathfrak{R}_{\mu,2}$, then $\Lambda(\omega)$ is compact in *E*. Next, we prove the following attraction property of $\Lambda(\omega)$: for every $B(\omega)\in\mathfrak{D}(E)$,

$$\lim_{t\to\infty}d_E\left(\Phi\left(t,\theta_{-t}\omega,B\left(\theta_{-t}\omega\right)\right),\Lambda(\omega)\right)=0. \tag{6.8}$$

Indeed, firstly, according to Lemma 4.1, there exists closed, tempered and measurable absorbing set $B_0(\omega)$ such that $B\in\mathfrak{D}(E)$, for any $t_B(\omega)\geq 0$,

$$\varphi\left(t,\theta_{-t}\omega,B\left(\theta_{-t}\omega\right)\right)\subseteq B_0(\omega),\ \forall t\geq t_B(\omega). \tag{6.9}$$

Moreover, let:

$$B_1(\omega)=\bigcup_{t\geq t(\omega,B_0)}\Phi\left(t,\theta_{-t}\omega\right)B_0\left(\theta_{-t}\omega\right).$$

Assume that $t\geq t_B(\omega)$ and $t_0=t-t_B(\omega)>t_{B_0}(\omega)>0$. Making use of the property 3) of Φ and (6.9), we deduce that:

$$\begin{aligned}\varphi\left(t,\theta_{-t}\omega,B\left(\theta_{-t}\omega\right)\right)&=\varphi\left(t_0+t_B(\omega),\theta_{-t_0-t_B(\omega)}\omega,B\left(\theta_{-t_0-t_B(\omega)}\omega\right)\right)\\&=\varphi\left(t_0,\theta_{-t_0}\omega,\varphi\left(t_B(\omega),\theta_{-t_0-t_B(\omega)}\omega,B\left(\theta_{-t_0-t_B(\omega)}\omega\right)\right)\right)\\&\subseteq\varphi\left(t_0,\theta_{-t_0}\omega,B_0\left(\theta_{-t_0}\omega\right)\right)\subseteq B_1(\omega).\end{aligned} \tag{6.10}$$

For any $t\geq t_B(\omega)+t_{B_0}(\omega)$, choose

$\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) \in \varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))$, where $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$. Due to (6.10) and Lemma 5.1, we have:

$$\varphi_N(t,\theta_{-t}\omega,\varphi_N(0,\theta_{-t}\omega)) = \varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) - \varphi_L(t,\theta_{-t}\omega,\varphi_l(0,\theta_{-t}\omega)) \in \Lambda(\omega).$$

So, according to Lemma 5.2, we find that for $t \ge t_B(\omega) + t_{B_0}(\omega)$,

$$\inf_{\psi\in\Lambda(\omega)} \left\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) - \psi\right\|_E^2 \le \left\|\varphi_L(t,\theta_{-t}\omega,\varphi_L(0,\theta_{-t}\omega))\right\|_E^2 \le M_2(\omega)\mathrm{e}^{-\sigma_1(\omega)t},$$

that is,

$$\operatorname{dist}(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega),\Lambda(\omega)) \to 0,\ t\to+\infty$$

Thus, (6.8) holds. Therefore, applying Lemma 2.11 and Theorem 4.1, we obtain that the RDS Φ possesses a $\mathfrak{D}(E)$-pullback random attractors $\mathfrak{A}(\omega) \subseteq \Lambda(\omega) \cap B_0(\omega)$.

Then, the proof is completed. □

Acknowledgement

The authors would like to thank anonymous reviewers for their helpful comments that improved the presentation of this work. Mohamed Y. A. Bakhet acknowledges and thanks the financial support from the University of Juba, South Sudan.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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