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Infinitely Many Solutions and a Ground-State Solution for Klein-Gordon Equation Coupled with Born-Infeld Theory

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Abstract

In this paper, we intend to consider a kind of nonlinear Klein-Gordon equation coupled with Born-Infeld theory. By using critical point theory and the method of Nehari manifold, we obtain two existing results of infinitely many high-energy radial solutions and a ground-state solution for this kind of system, which improve and generalize some related results in the literature.

Keywords

Klein-Gordon Equation, Born-Infeld Theory, Infinitely Many Solutions, Ground-State Solution, Critical Point Theory

1. Introduction and Main Results

In this paper, we intend to consider the following Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(u), & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\beta > 0$, $\omega > 0$, u and ϕ are unknowns, $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and f satisfies some superlinear conditions. The Born-Infeld electromagnetic theory [1] [2] was first put up as a nonlinear correction of the Maxwell theory to solve the infiniteness issue in the classical electrodynamics of point particles (see [3]). The fundamental concept was to change classical theory simply, so that it adhered to the notion of finiteness and did not have physical quantities of infinities. Due to its importance in the theory of superstrings and membranes, Born-Infeld nonlinear electromagnetism has attracted a lot of attention from theoretical physicists and mathematicians (see [4] [5]). For more physical applications, please refer to [6] [7].

In recent years, some researchers considered the Klein-Gordon equation coupled with Born-Infeld theory by using variational methods. We recall some of them as follows.

In [8], d'Avenia and Pisani studied the following kind of Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u+\left(m^2-w^2\right)u-\left(2\omega+\phi\right)\phi u=\left|u\right|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

when $p\in\left(4,6\right)$ and $0<\omega<m$, they obtained some existing results of infinitely many radially symmetric solutions for system (1.2). After this, Mugnai [6] covered the range $2<p<4$ provided $0<\omega<\left|m\right|\sqrt{\frac{p}{2}-1}$. Replacing $\left|u\right|^{p-2}u$ by $\left|u\right|^{p-2}u+\left|u\right|^{2^*-2}u$, where $2^*:=6$ is Sobolev exponent in $\mathbb{R}^3$, Teng and Zhang [9] studied the following Klein-Gordon equation coupled with Born-Infeld theory with critical Sobolev exponent:

$$\begin{cases} -\Delta u+\left(m^2-w^2\right)u-\left(2\omega+\phi\right)\phi u=\left|u\right|^{p-2}u+\left|u\right|^{2^*-2}u, & \text{in } \mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

they admits a nontrivial solution for problem (1.3) when $m>\omega>0$ or $\left(\frac{1}{2}-\frac{1}{p}\right)m^2>\frac{1}{2}\omega^2$.

Chen and Li [10] added a perturbation $h\left(x\right)$ to the nonlinear term of problem (1.3) and removed the term $\left|u\right|^{2^*-2}u$, by using critical point theory, they obtained two different solutions, under one of the following conditions:

1) $\left|m\right|>\omega>0$, $4<p<6$; 2) $\sqrt{\frac{p}{2}-1}\left|m\right|>\omega>0$, $2<p\le 4$

In [11], Chen and Song considered the case of nonlinear terms with concave and convex, and got the existence of multiple solutions for the following problem:

$$\begin{cases} -\Delta u+a\left(x\right)u-\left(2\omega+\phi\right)\phi u=\lambda k\left(x\right)\left|u\right|^{q-2}u+g\left(x\right)\left|u\right|^{p-2}, & \text{in } \mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $1<q<2<p<6$, *a*, *k*, and *g* satisfy some appropriate assumptions.

In [12], He, Li, Chen and O'Regan investigated the following kind of Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u+\left(m^2-w^2\right)u-\left(2\omega+\phi\right)\phi u=\mu\left|u\right|^{p-2}u+\left|u\right|^{2^*-2}u, & \text{in } \mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

they showed that problem (1.5) has at least a nontrivial radial ground-state solution, under one of the following conditions:

1) $p\in\left(4,6\right)$ and $m>\omega>0$ for $\mu>0$;

2) $p \in (3,4]$ and $m > \omega > 0$ for sufficient large $\mu > 0$;

3) $p \in (2,3]$ and $\sqrt{(p-2)(4-p)}|m| > \omega > 0$ for sufficient large $\mu > 0$.

Wen, Tang and Chen [13] studied the following kind of Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

they obtained infinitely many solutions and a least energy solution for problem (1.6) under different assumptions on V and f. In [14], Zhang and Liu proved the existence of infinitely many sign-changing solutions to the problem (1.2), when $|m| > \omega > 0$, $4 \le p < 6$ or $\sqrt{\frac{p}{2} - 1}|m| > \omega > 0$, $2 < p < 4$. Other related studies on the Klein-Gordon equation or Klein-Gordon-Maxwell equation can be seen in [15]-[28].

Motivated by the above works, in this paper, to certify the boundedness of Palais-Smale sequence for case of $2 < u < 6$, we use Pohožaev identity of (1.1). By applying the ideas employed by Ref. [12], we find a Palais-Smale sequence $\{u_n\}$ of energy functional of problem (1.1) at level c_1, where $c_1 > 0$ is mountain pass level defined later by (1.1). Then, the boundedness of $\{u_n\}$ can be certified by some delicate analyses. By using critical point theory and the method of Nehari manifold, we obtain two existing results of infinitely many high-energy radial solutions and a ground-state solution (which is the solution with the smallest energy among all the solutions) for the system (1.1) and we have improved the range of ω, which improve and generalize some related results in the literature.

In this paper, we make the following assumptions:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ is a radial function, which satisfies $V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$. And there is a constant $r > 0$ such that:

$$\lim_{|y|\to+\infty} \text{meas}\left(\left\{x \in \mathbb{R}^3 : |x-y| \le r, V(x) \le M\right\}\right) = 0, \forall M > 0,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure.

(V2) $(\nabla V(x), x) \ge 0$ for all $x \in \mathbb{R}^3$ and there exists $\theta \in [0,1)$ such that $(\nabla V(x), x) \le \frac{\theta}{2|x|^2}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$.

(F1) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^5} = 0$.

(F2) $\lim_{|t|\to 0} \frac{f(t)}{|t|} = 0$.

(F3) There exists $\mu > 2$ such that $f(t)t \ge \mu F(t) \ge 0$ for all $t \in \mathbb{R}$, where $F(t) := \int_0^t f(s)\mathrm{d}s$.

(F4) $f(-t) = -f(t)$, for all $t \in \mathbb{R}$.

Now, we present two main results:

Theorem 1.1. *Assume that* (V1), (V2), (F1) - (F4) *hold. If the following condition holds*:

1) $\mu \in [3,6)$; *or* 2) $\mu \in (2,3)$ *and* $\omega \in \left(0, \sqrt{(\mu-2)(4-\mu)V_0} / (3-\mu)\right)$, *then, problem* (1.1) *possesses infinitely many solutions* $\{(u_n, \phi_n)\} \subset E \times D_r^{1,2}(\mathbb{R}^3)$ *satisfying*:

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \left[V(x) - (2\omega + \phi)\phi\right]u^2\right)\mathrm{d}x$$
$$-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla \phi|^2\,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla \phi|^4\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x \to \infty.$$

Theorem 1.2. *Assume that* (V1), (V2), (F1) - (F3) *hold. If the following condition holds*:

1) $\mu \in [3,6)$; *or* 2) $\mu \in (2,3)$ *and* $\omega \in \left(0, \sqrt{(\mu-2)(4-\mu)V_0} / (3-\mu)\right)$, then, the problem (1.1) has a ground-state solution.

Remark 1.3. *We consider the variable potential V and generalized nonlinearity f, which brings some difficulties such as the proof of boundedness of Palais-Smale sequence* ((*PS*)*-sequence for short*). *To conquer the boundedness of* (*PS*)*-sequence, we use some analytical methods. Besides, when* $\mu \in [3,6)$, *we do not need any restriction on* ω, *and when* $\mu \in (2,3)$, *we get a more delicate range for* ω. *Hence, Theorem* 1.1 *and Theorem* 1.2 *can be seen as improvements of the relative results in the literature. To the best of our knowledge, similar results for this kind of Klein-Gordon equation coupled with Born-Infeld theory by using analytical methods in this paper can not be found.*

The rest of this paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, we give out the proofs of Theorem 1.1 and Theorem 1.2. We denote C_i as different positive constants.

2. Preliminaries

Henceforth, the following notations will be used.

$\langle\,,\rangle$	denote dual inner products between workspaces.
$\rightharpoonup$	denote weak convergence.
$dist(x,y)$	denote Euclidean distance between *x* and *y*.
∂S	denote boundary of *S*.
a.e.	almost everywhere.
$\mathbb{R}^N$	denote N-dimensional Euclidean space.
$X := Y$	denote define *X* as *Y*.
$C, C_1, C_2, \cdots$	denote various positive constants.

$B_r(x) := \{y \in \mathbb{R} : |y - x| < r\}$, $\forall u \in H^1(\mathbb{R}^N)$, $r > 0$.

$u_t(x) := u(x/t)$, $\forall u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $t > 0$.

$D(\mathbb{R}^3)$ denote the complete space of $C_0^\infty(\mathbb{R}^3)$.

$D_r(\mathbb{R}^3) := \{u \in D(\mathbb{R}^3) : u(x) = u(|x|)\}$.

$D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$.

$D_r^{1,2}(\mathbb{R}^3) := \{u \in D^{1,2}(\mathbb{R}^3) : u(x) = u(|x|)\}$.

$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$.

$H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}$.

$\|u\|_D = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} |\nabla u|^4 \,\mathrm{d}x\right)^{\frac{1}{4}}$.

$\|u\|_{D_r^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$.

$\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s \,\mathrm{d}x\right)^{1/s}$, $1 < s < \infty$.

We define:

$$E := \left\{ u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x < \infty \right\}.$$

Then, E is a Hilbert space with the inner product:

$$(u,v)_E = \int_{\mathbb{R}^3} \left[\nabla u \nabla v + V(x)uv\right]\mathrm{d}x$$

and the norm $\|u\| := \|u\|_E = (u,u)_E^{1/2}$. By (V1), (V2) and Poincaré inequality, we see that $E \hookrightarrow H_r^1(\mathbb{R}^3)$ is continuous. Then, for $p \in [2,6]$, there exists $r_p > 0$ such that:

$$\|u\|_{L^p} := \left(\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le r_p \|u\|_E, \; u \in E. \tag{2.1}$$

Apparently, we know that a solution $(u,\phi) \in E \times D_r^{1,2}(\mathbb{R}^3)$ for the system (1.1) is a critical point of the energy functional $J : (u,\phi) \in E \times D_r^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as:

$$\begin{aligned} J(u,\phi) = &\frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x)u^2 - (2\omega + \phi)\phi u^2\right]\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\ &- \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla \phi|^2 \,\mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi|^4 \,\mathrm{d}x. \end{aligned} \tag{2.2}$$

We need the following lemma to reduce the functional J in the only variable u.

Lemma 2.1. [12] *For any* $u \in H^1(\mathbb{R}^3)$, *we have*:

1) *There exists a unique* $\phi = \phi_u \in D(\mathbb{R}^3)$, *which solves equation*:

$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2;$$

2) $-\omega \le \phi_u \le 0$ *on the set* $\{x : u(x) \ne 0\}$;

3) *If u is radially symmetric, then* ϕ_u *is also radially symmetric*;

4) $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$ *, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D_r(\mathbb{R}^3)$.

From the second equation in system (1.1), we get:

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla \phi|^2 \,\mathrm{d}x + \frac{\beta}{4\pi}\int_{\mathbb{R}^3} |\nabla \phi|^4 \,\mathrm{d}x = -\int_{\mathbb{R}^3} \left(\omega\phi_u + \phi_u^2\right)u^2 \,\mathrm{d}x. \tag{2.3}$$

From Lemma 2.1, we rewrite $J(u,\phi)$ as the functional $I(u) : E \to \mathbb{R}$ as follows:

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2-(2\omega+\phi_u)\phi_u u^2\right]\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x$$
$$-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x. \tag{2.4}$$

By (2.3) and (2.4), we obtain:

$$\begin{aligned}I(u)&=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2+\phi_u^2u^2\right]\mathrm{d}x-\int_{\mathbb{R}^3}\left(\omega\phi_u+\phi_u^2\right)u^2\,\mathrm{d}x\\&\quad-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x\\&=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2+\phi_u^2u^2\right]\mathrm{d}x+\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x\\&\quad+\frac{3\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x\\&=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2-\omega\phi_u u^2\right]\mathrm{d}x+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x.\end{aligned} \tag{2.5}$$

For any $u,\tilde{v}\in E$, we have:

$$\langle I'(u),\tilde{v}\rangle=\int_{\mathbb{R}^3}\left[\nabla u\nabla\tilde{v}+V(x)u\tilde{v}-(2\omega+\phi_u)\phi_u u\tilde{v}\right]\mathrm{d}x-\int_{\mathbb{R}^3}f(u)\tilde{v}\mathrm{d}x. \tag{2.6}$$

For $\lambda\in[1/2,1]$, we define the family of functionals $I_\lambda:E\to\mathbb{R}$ by:

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2-(2\omega+\phi_u)\phi_u u^2\right]\mathrm{d}x-\lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x$$
$$-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x. \tag{2.7}$$

For any $u,\tilde{v}\in E$, we also have:

$$\langle I'_\lambda(u),\tilde{v}\rangle=\int_{\mathbb{R}^3}\left[\nabla u\nabla\tilde{v}+V(x)u\tilde{v}-(2\omega+\phi_u)\phi_u u\tilde{v}\right]\mathrm{d}x-\lambda\int_{\mathbb{R}^3}f(u)\tilde{v}\mathrm{d}x. \tag{2.8}$$

Let $\mathcal{S}:=\left\{u\in H_r^1(\mathbb{R}^3)\setminus\{0\}:I'(u)=0\right\}$ be the critical points set. It is easy to see that any critical point u of I satisfies the following Pohožaev equality:

$$\begin{aligned}P(u)&:=\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\left[3V(x)+(\nabla V(x),x)-5\omega\phi_u-2\phi_u^2\right]u^2\,\mathrm{d}x\\&\quad+\frac{3\beta}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x-6\int_{\mathbb{R}^3}F(u)\mathrm{d}x\\&=0.\end{aligned} \tag{2.9}$$

For convenience, we also defined:

$$\begin{aligned}P_\lambda(u)&:=\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\left[3V(x)+(\nabla V(x),x)-5\omega\phi_u-2\phi_u^2\right]u^2\,\mathrm{d}x\\&\quad+\frac{3\beta}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,\mathrm{d}x-6\lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x\\&=0.\end{aligned} \tag{2.10}$$

Let:

$$\begin{aligned} G(u) &:= \langle I'(u), u\rangle - \frac{1}{2}P(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\left[V(x) + (\nabla V(x), x) - \omega\phi_u\right]u^2 \mathrm{d}x \\ &\quad - \frac{3\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4 \mathrm{d}x + \int_{\mathbb{R}^3}\left[3F(u) - f(u)u\right]\mathrm{d}x. \end{aligned} \tag{2.11}$$

Then, $G(u) = 0$ for any $u \in \mathcal{S}$. We also define:

$$\begin{aligned} G_\lambda(u) &:= \langle I'_\lambda(u), u\rangle - \frac{1}{2}P_\lambda(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\left[V(x) + (\nabla V(x), x) - \omega\phi_u\right]u^2 \mathrm{d}x \\ &\quad - \frac{3\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla \phi_u|^4 \mathrm{d}x + \lambda\int_{\mathbb{R}^3}\left[3F(u) - f(u)u\right]\mathrm{d}x. \end{aligned} \tag{2.12}$$

Lemma 2.2. *Assume that* (F1) - (F3) *hold. Then, there exist some constants* ζ_λ, $\alpha_\lambda > 0$, $t_\lambda > 0$ *and* $v_\lambda = t_\lambda u$, $\lambda \in \left[\frac{1}{2}, 1\right]$ (*see* [12]) *such that*:

1) $\inf_{\|u\| \le \zeta_\lambda} I_\lambda(u) \ge 0$ *and* $\inf_{\|u\| = \zeta_\lambda} I_\lambda(u) \ge \alpha_\lambda$;

2) $\|v_\lambda\|_E > \zeta_\lambda$ *and* $I_\lambda(v_\lambda) < 0$.

Proof. 1) From (F1) and (F2), for $C_1 = \frac{1}{6r_2^2}$, there exists $C_2 > 0$ such that:

$$F(t) \le C_1|t|^2 + C_2|t|^6, \ \text{for all } t \in \mathbb{R}. \tag{2.13}$$

By (2.1), (2.3), (2.7), (2.13), $-\omega \le \phi_u \le 0$ and Hölder inequality, we have:

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2\right]\mathrm{d}x - \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2 \mathrm{d}x \\ &\quad - \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4 \mathrm{d}x - \lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2 \mathrm{d}x \\ &\quad + \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4 \mathrm{d}x - \lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^3}\left(C_1|u|^2 + C_2|u|^6\right)\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|_E^2 - C_1\lambda\|u\|_{L^2}^2 - C_2\lambda\|u\|_{L^6}^6 \\ &\ge \frac{1}{2}\|u\|_E^2 - \frac{1}{6}\|u\|_E^2 - \lambda C_2 r_6^6\|u\|_E^6 \\ &\ge \frac{1}{3}\|u\|_E^2 - \lambda C_2 r_6^6\|u\|_E^6. \end{aligned} \tag{2.14}$$

From (2.14), there exist ζ_λ, $\alpha_\lambda > 0$ such that $\inf_{\|u\| \le \zeta_\lambda} I_\lambda(u) \ge 0$ and $\inf_{\|u\| = \zeta_\lambda} I_\lambda(u) \ge \alpha_\lambda$.

2) From (F2) and (F3), there exists $C_3, C_4 > 0$ such that:

$$F(t) \geq C_3 |t|^{\mu} - C_4 |t|^2, \text{ for all } t \in \mathbb{R}. \tag{2.15}$$

From (2.15), for $u \in E \setminus \{0\}$, we get:

$$\begin{aligned}
I_\lambda(t_\lambda u) &= \frac{t_\lambda^2}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x) u^2 - (2\omega + \phi_u) \phi_u u^2 \right] \mathrm{d}x - \frac{1}{8\pi} \int_{\mathbb{R}^3} \left| \nabla \phi_{t_\lambda u} \right|^2 \mathrm{d}x \\
&\quad - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} \left| \nabla \phi_{t_\lambda u} \right|^4 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(t_\lambda u) \mathrm{d}x \\
&\leq \frac{t_\lambda^2}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x) u^2 \right] \mathrm{d}x + \omega^2 t_\lambda^2 \int_{\mathbb{R}^3} u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(t_\lambda u) \mathrm{d}x \\
&\leq \frac{t_\lambda^2}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x) u^2 \right] \mathrm{d}x + \omega^2 t_\lambda^2 \int_{\mathbb{R}^3} u^2 \mathrm{d}x - C_3 t_\lambda^{\mu} \lambda \int_{\mathbb{R}^3} |u|^{\mu} \mathrm{d}x + C_4 t_\lambda^2 \lambda \int_{\mathbb{R}^3} u^2 \mathrm{d}x \\
&\to -\infty, \text{ as } t_\lambda \to \infty.
\end{aligned} \tag{2.16}$$

Hence, from (2.16), we can let $v_\lambda = t_\lambda u$ with $t_\lambda > 0$ large enough such that $\|v_\lambda\|_E > \zeta_\lambda$ and $I_\lambda(v_\lambda) < 0$.□

Lemma 2.3. *Assume that* (V1) *and* (F1) - (F3) *hold. Let* $\{u_n\} \subset E$ *be a bounded* $(PS)_c$*-sequence for* I *with* $c \in (0, \infty)$, *then* $\{u_n\}$ *has a strongly convergent subsequence in E.*

Proof. Consider a sequence $\{u_n\}$ in *E*, which satisfies:

$$I(u_n) \to c, \quad I'(u_n) \to 0. \tag{2.17}$$

We may assume that, for any $n \in \mathbb{N}$, there exists a $u \in E$ such that:

- $u_n \rightharpoonup u$ in *E*;
- $u_n \to u$ in $L^p(\mathbb{R}^3)$, for $2 < p < 6$;
- $u_n \to u$ a.e. in $\mathbb{R}^3$.

By (2.6), we easily get:

$$\begin{aligned}
\|u_n - u\|_E &= \langle I'(u_n) - I'(u), u_n - u \rangle + 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) \mathrm{d}x + \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \mathrm{d}x.
\end{aligned} \tag{2.18}$$

It is clear that:

$$(I'(u_n) - I'(u), u_n - u) \to 0, \text{ as } n \to \infty. \tag{2.19}$$

From (F1) and (F2), there exist $C_6, C_7 > 0$ such that:

$$|f(t)| \leq C_5 |t| + C_6 |t|^5, \ \forall t \in \mathbb{R}. \tag{2.20}$$

By (2.20), one has:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[f(u_n) - f(\bar{u}) \right](u_n - \bar{u}) \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \left[C_5 (|u_n| + |\bar{u}|) + C_6 (|u_n|^5 + |\bar{u}|^5) \right] |u_n - \bar{u}| \mathrm{d}x \\
&\leq C_5 (\|u_n\|_{L^2} + \|\bar{u}\|_{L^2}) \|u_n - \bar{u}\|_{L^2} + C_6 (\|u_n\|_{L^2}^5 + \|\bar{u}\|_{L^2}^5) \|u_n - \bar{u}\|_{L^2}.
\end{aligned} \tag{2.21}$$

By Lemma 2.1, Sobolev inequality and Hölder inequality, it easily gains that:

$$\begin{aligned}
&\left|2\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n-\phi_u u\right)\left(u_n-u\right)\mathrm{d}x\right| \\
&\le\left|2\int_{\mathbb{R}^3}\phi_u\left(u_n-u\right)\left(u_n-u\right)\mathrm{d}x\right|+\left|2\int_{\mathbb{R}^3}\left(\phi_{u_n}-\phi_u\right)u_n\left(u_n-u\right)\mathrm{d}x\right| \\
&\le\left\|\phi_u\right\|_{L^6}\cdot\left\|u_n-u\right\|_{L^3}\cdot\left\|u_n-u\right\|_{L^2}+\left\|\phi_{u_n}-\phi_u\right\|_{L^6}\cdot\left\|u_n-u\right\|_{L^3}\cdot\left\|u_n\right\|_{L^2} \\
&\le\left\|\phi_u\right\|_{L^6}\cdot\left\|u_n-u\right\|_{L^3}\cdot\left\|u_n-u\right\|_{L^2}+C_5\left\|\phi_{u_n}-\phi_u\right\|_{D_r^{1,2}}\cdot\left\|u_n-u\right\|_{L^3}\cdot\left\|u_n\right\|_{L^2}.
\end{aligned} \tag{2.22}$$

From Lemma 2.1 and the boundeness of $\{u_n\}$, there exists a positive constant C_6 such that:

$$\left\|\phi_{u_n}^2u_n\right\|_{L^{\frac{3}{2}}}\le\left\|\phi_{u_n}\right\|_{L^6}^2\left\|u_n\right\|_{L^3}\le C_6. \tag{2.23}$$

Hence, from (2.23), the sequence $\{\phi_{u_n}^2u_n\}$ is bounded in $L^{\frac{3}{2}}(\mathbb{R}^3)$, so that:

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\left(\phi_{u_n}^2u_n-\phi_u^2u\right)\left(u_n-u\right)\mathrm{d}x\right|&\le\left\|\phi_{u_n}^2u_n-\phi_u^2u\right\|_{L^{\frac{3}{2}}}\left\|u_n^i-u\right\|_{L^3} \\
&\le\left(\left\|\phi_{u_n}^2u_n\right\|_{L^{\frac{3}{2}}}+\left\|\phi_u^2u\right\|_{L^{\frac{3}{2}}}\right)\left\|u_n-u\right\|_{L^3}.
\end{aligned} \tag{2.24}$$

Since $u_n\to u$ in $L^p(\mathbb{R}^3)$, for any $2<p<6$, from (2.21), (2.22) and (2.24), one has:

$$\int_{\mathbb{R}^3}\left[f\left(u_n\right)-f\left(u\right)\right]\left(u_n-u\right)\mathrm{d}x\to 0 \text{ as } n\to\infty. \tag{2.25}$$

$$2\omega\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n-\phi_u u\right)\left(u_n-u\right)\mathrm{d}x\to 0, \text{ as } n\to\infty, \tag{2.26}$$

$$\int_{\mathbb{R}^3}\left(\phi_{u_n}^2u_n-\phi_u^2u\right)\left(u_n-u\right)\mathrm{d}x\to 0, \text{ as } n\to\infty. \tag{2.27}$$

From (2.18), (2.19), (2.25), (2.26) and (2.27), we have $\left\|u_n-u\right\|_E\to 0$, that is, $u_n\to u$ in E. □

Lemma 2.4. [12] *Let* $(X,\|.\|)$ *be a Banach space and let* $J\subset\mathbb{R}^+$ *be an interval. Consider the family of* C^1*-functionals on X with* $\lambda\in J$:

$$\Phi_\lambda(u)=A(u)-\lambda B(u),$$

with $B(u)$ nonnegative and either $A(u)\to+\infty$ or $B(u)\to+\infty$, as $\|u\|\to+\infty$, and such that $\Phi_\lambda(0)=0$. For any $\lambda\in J$, we set:

$$\Gamma_\lambda=\left\{\gamma\in C\left([0,1],X\right):\gamma(0)=0,\Phi_\lambda\left(\gamma(1)\right)\le 0\right\}.$$

If for every $\lambda\in J$, the set Γ_λ is nonempty and

$$c_\lambda:=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}\Phi_\lambda\left(\gamma(t)\right)>0$$

then for almost every $\lambda\in J$, there is a sequence $\{(u_\lambda)_n\}\subset X$ such that:

1) $\{(u_\lambda)_n\}$ *is bounded*;

2) $\Phi_\lambda\left(\left(u_\lambda\right)_n\right)\to c_\lambda$;

3) $\Phi'_\lambda\left(\left(u_\lambda\right)_n\right)\to 0$ *in the dual* X^{-1} *of X.*

Lemma 2.5. Γ_λ *is nonempty, where* Γ_λ *is given by Lemma 2.4.*

Proof. From (2.16) and $u\in E\setminus\{0\}$, we can choose $T>0$ such that $I_\lambda(Tu)<0$. Let $\gamma_s(t)=Ttu$, $t\in[0,1]$, such that, $\gamma_s(t)\in C([0,1],E)$, $\gamma_s(0)=0$, $I_\lambda(\gamma_s(1))<0$, and $\max_{t\in[0,1]} I_\lambda(\gamma_s(t))<\infty$, for any $\lambda\in J$. This means that Γ_λ is nonempty. □

Lemma 2.6. $c_\lambda>0$, *where* c_λ *is given by Lemma 2.4.*

Proof. For any $\gamma\in\Gamma_\lambda$ and any $\lambda\in J$, we have $\gamma(0)=0$ and $I_\lambda(\gamma(1))<0$. From Lemma 2.2, we get that $\|\gamma(1)\|>\zeta_\lambda$. By continuity, we deduce that there exists $t_\gamma\in(0,1)$ such that $\|\gamma(t_\gamma)\|=\zeta_\lambda$. From Lemma 2.2, we have $I_\lambda(\gamma(t_\gamma))\ge\alpha_\lambda$. Therefore, we have:

$$\infty>c_\lambda\ge\inf_{\gamma\in\Gamma} I_\lambda\left(\gamma\left(t_\gamma\right)\right)\ge\alpha_\lambda>0.$$

□

Lemma 2.7. *Assume that* (V1), (V2) *and* (F1) - (F3) *hold. Then, there exists a sequence* $\{u_n\}$ *satisfying*:

$$I(u_n)\to c_1,\quad I'(u_n)\to 0. \tag{2.28}$$

is a bounded $(PS)_{c_1}$ *-sequence with* $c_1\in(0,\infty)$.

Proof. From (2.3) and (2.7), we get that:

$$\begin{aligned} I_\lambda(u)&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2\,dx\\ &\quad+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,dx-\lambda\int_{\mathbb{R}^3}F(u)\,dx. \end{aligned}\tag{2.29}$$

From Lemma 2.2, we see that I_λ has mountain pass geometry. We can define the Mountain Pass level c_λ by:

$$c_\lambda:=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \tag{2.30}$$

where

$$\Gamma_\lambda=\left\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=v_\lambda\right\}.$$

From Lemma 2.6, we have the estimate $c_1\in(0,\infty)$, set $X=E$, $J=\left[\frac{1}{2},1\right]$, $\Phi_\lambda=I_\lambda$,

$$A(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2\,dx+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,dx,$$

$$B(u)=\int_{\mathbb{R}^3}F(u)\,dx.$$

It is easy to know that $B(u)\ge 0$ for every $u\in E$ and $A(u)\to\infty$ when $\|u\|\to\infty$. Thus, from Lemma 2.4 and Lemma 2.5, for almost every $\lambda\in J$, there is a sequence $\{(u_\lambda)_n\} \subset E$ such that:

1) $\{(u_\lambda)_n\}$ is bounded in *E*;

2) $I_\lambda\left((u_\lambda)_n\right) \to c_\lambda$;

3) $I'_\lambda\left((u_\lambda)_n\right) \to 0$ in the dual E^{-1} of *E*.

Since $c_\lambda \in (0,\infty)$, there exists $u_\lambda \in E$ satisfying:

$$I'_\lambda(u_\lambda) = 0, \quad I_\lambda(u_\lambda) = c_\lambda$$

for almost every $\lambda \in J$. We can choose a suitable $\lambda_n \to 1$ and u_{λ_n} such that:

$$I'_{\lambda_n}\left(u_{\lambda_n}\right) = 0, \quad I_{\lambda_n}\left(u_{\lambda_n}\right) = c_{\lambda_n} \to c_1. \tag{2.31}$$

We still denote u_{λ_n} by u_n. From (2.8) and $I'_{\lambda_n}(u_n) = 0$, we have:

$$\left\langle I'_{\lambda_n}(u_n), u_n \right\rangle = \int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + V(x)u_n^2 - \left(2\omega + \phi_{u_n}\right)\phi_{u_n} u_n^2 \right] dx - \lambda \int_{\mathbb{R}^3} f(u_n)u_n dx = 0, \tag{2.32}$$

and from (2.12), one has that:

$$\begin{aligned} G_{\lambda_n}(u_n) &= \left\langle I'_{\lambda_n}(u_n), u_n \right\rangle - \frac{1}{2} P_{\lambda_n}(u_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \left[V(x) + \left(\nabla V(x), x\right) - \omega \phi_{u_n} \right] u_n^2 dx \\ &\quad - \frac{3\beta}{16\pi} \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^4 dx + \lambda \int_{\mathbb{R}^3} \left[3F(u_n) - f(u_n)u_n \right] dx \\ &= 0. \end{aligned} \tag{2.33}$$

Next, we will prove that $\{u_n\}$ is bounded in *E*.

Case (1): $4 \le \mu < 6$. By (V1), (F3), (2.1), (2.3), (2.28), (2.29), (2.31), (2.32), $-\omega \le \phi_u \le 0$ and Hölder inequality, we have:

$$\begin{aligned} \mu c_1 + o(1) &\ge \mu I_{\lambda_n}(u_n) - \left\langle I'_{\lambda_n}(u_n), u_n \right\rangle \\ &= \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + V(x)u_n^2 \right] dx + \left(\frac{\mu}{2} + 1\right) \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx + 2\int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx \\ &\quad + \frac{\mu}{8\pi} \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^2 dx + \frac{3\beta\mu}{16\pi} \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^4 dx + \lambda_n \int_{\mathbb{R}^3} \left[f(u_n)u_n - \mu F(u_n) \right] dx \\ &\ge \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + V(x)u_n^2 \right] dx + 2\int_{\mathbb{R}^3} \left(\phi_{u_n}^2 u_n^2 + \omega \phi_{u_n} u_n^2 \right) dx + \frac{\mu}{8\pi} \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^2 dx \\ &\quad + \frac{3\beta\mu}{16\pi} \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^4 dx + \lambda_n \int_{\mathbb{R}^3} \left[f(u_n)u_n - \mu F(u_n) \right] dx \\ &= \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + V(x)u_n^2 \right] dx + \left(\frac{\mu}{8\pi} - \frac{1}{2\pi}\right) \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^2 dx \\ &\quad + \left(\frac{3\beta\mu}{16\pi} - \frac{\beta}{2\pi}\right) \int_{\mathbb{R}^3} \left|\nabla \phi_{u_n}\right|^4 dx + \lambda_n \int_{\mathbb{R}^3} \left[f(u_n)u_n - \mu F(u_n) \right] dx \\ &\ge \left(\frac{\mu}{2} - 1\right) \|u_n\|_E^2. \end{aligned} \tag{2.34}$$

When $4 \le \mu < 6$, from (2.34), we know that $\{u_n\}$ is bounded in *E*.

Case (2): $2 < \mu < 4$. By (V1), (V2), (F3), (2.28), (2.29), (2.31), (2.32), (2.33) and $-\omega \le \phi_u \le 0$, we have:

$$\begin{aligned} c_1 + o(1) &\ge I_{\lambda_n}(u_n) + \frac{\mu-4}{6-\mu}\left\langle I'_{\lambda_n}(u_n), u_n \right\rangle + \frac{2-\mu}{6-\mu} G_{\lambda_n}(u_n) \\ &= \frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[(\mu-2)V(x) + 2(3-\mu)\omega\phi_{u_n} + (4-\mu)\phi_{u_n}^2\right]|u_n|^2\,\mathrm{d}x \\ &\quad + \frac{2\lambda_n}{6-\mu}\int_{\mathbb{R}^3}\left[f(u_n)u_n - \mu F(u_n)\right]\mathrm{d}x + \frac{\mu-2}{2(6-\mu)}\int_{\mathbb{R}^3}(\nabla V(x), x)|u_n|^2\,\mathrm{d}x \\ &\quad + \frac{2\beta\mu}{16\pi(6-\mu)}\int_{\mathbb{R}^3}\left|\nabla\phi_{u_n}\right|^4\mathrm{d}x. \end{aligned} \tag{2.35}$$

We will prove the boundedness of $\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x$, to do this, we have two cases to consider.

Subcase (i): $3 \le \mu < 4$. In this case, we have:

$$(4-\mu)s^2 + 2(3-\mu)\omega s \ge 0, \quad \forall -\omega \le s \le 0. \tag{2.36}$$

From (2.35), (2.36), (V2) and (F3), we have that $\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x$ is bounded.

Subcase (ii): $\mu \in (2,3)$ and $\omega \in \left(0, \sqrt{(\mu-2)(4-\mu)V_0} \big/ (3-\mu)\right)$. For $s \in [-\omega, 0]$, by a direct computation, we have that:

$$\begin{aligned} &(4-\mu)s^2 + 2(3-\mu)\omega s + (\mu-2)V_0 \\ &\ge -\frac{(3-\mu)^2}{4-\mu}\omega^2 + (\mu-2)V_0 = \frac{(\mu-2)(4-\mu)V_0 - (3-\mu)^2\omega^2}{4-\mu} > 0. \end{aligned} \tag{2.37}$$

Thus, from (2.35), (2.37), (V1), (V2) and (F3), we get:

$$\begin{aligned} c_n + o(1) &\ge \frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[(\mu-2)V(x) + 2(3-\mu)\omega\phi_{u_n} + (4-\mu)\phi_{u_n}^2\right]|u_n|^2\,\mathrm{d}x \\ &\quad + \frac{\mu-2}{2(6-\mu)}\int_{\mathbb{R}^3}(\nabla V(x), x)|u_n|^2\,\mathrm{d}x + \frac{2\beta\mu}{16\pi(6-\mu)}\int_{\mathbb{R}^3}\left|\nabla\phi_{u_n}\right|^4\mathrm{d}x \\ &\quad + \frac{2}{6-\mu}\int_{\mathbb{R}^3}\left[f(u_n)u_n - \mu F(u_n)\right]\mathrm{d}x \\ &\ge \frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[(\mu-2)V(x) + 2(3-\mu)\omega\phi_{u_n} + (4-\mu)\phi_{u_n}^2\right]|u_n|^2\,\mathrm{d}x \\ &\ge \frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[(\mu-2)V_0 + 2(3-\mu)\omega\phi_{u_n} + (4-\mu)\phi_{u_n}^2\right]|u_n|^2\,\mathrm{d}x \\ &\ge \frac{(4-\mu)(\mu-2)V_0 - (3-\mu)^2\omega^2}{(6-\mu)(4-\mu)}\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x. \end{aligned} \tag{2.38}$$

It follows from (2.38) that $\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x$ is bounded when $\mu \in (2,3)$. From Case (1) and Subcase (i), we have that $\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x$ is also bounded. Hence, by Lemma 2.1, there exists a positive constant C_7 such that:

$$\left|\int_{\mathbb{R}^3}\omega\phi_{u_n}|u_n|^2\,\mathrm{d}x\right| \le \omega^2\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x \le C_7 \tag{2.39}$$

and

$$\left|\int_{\mathbb{R}^3}\phi_{u_n}^2|u_n|^2\,\mathrm{d}x\right| \le \omega^2\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x \le C_7. \tag{2.40}$$

From (2.35), (2.39) and (2.40), we know that $\int_{\mathbb{R}^3} V(x)|u_n|^2 \mathrm{d}x$ is bounded when $\mu \in (2,3)$ and $\omega \in \left(0, \sqrt{(\mu-2)(4-\mu)V_0}/(3-\mu)\right)$. From Hardy inequality, we have:

$$\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{u^2}{|x|^2}\mathrm{d}x, \ \forall u \in H_r^1(\mathbb{R}^3). \tag{2.41}$$

By (2.31), (2.32), (2.33), (2.39), (2.40), (2.41) and (V2), we have:

$$\begin{aligned} a_n + o(1) &\geq I_{\lambda_n}(u_n) - \frac{1}{3}\langle I'_{\lambda_n}(u_n), u_n\rangle + \frac{1}{3}G_{\lambda_n}(u_n) \\ &= \frac{1}{3}\int_{\mathbb{R}^3}\left(\omega\phi_{u_n} + \phi_{u_n}^2\right)|u_n|^2 \mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 \mathrm{d}x \\ &\quad - \frac{1}{6}\int_{\mathbb{R}^3}(\nabla V(x), x)|u_n|^2 \mathrm{d}x \\ &\geq \frac{1-\theta}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 \mathrm{d}x - \frac{2C_7}{3}. \end{aligned} \tag{2.42}$$

From (2.42) and $\theta \in [0,1)$, we know that $\{\nabla u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. Hence, $\{u_n\}$ is bounded in E when $\mu \in (2,4)$. Therefore, $\{u_n\}$ is bounded in E.

In order to obtain infinitely many solutions of system (1.1), we shall use the following critical point theorem introduced by Bartsch in [29]. The space X is reflexive and separable, then there exist $e_i \in X$ and $f_i \in X^*$ such that $X = \overline{\langle e_i, i \in \mathbb{N}\rangle}$, $X^* = \overline{\langle f_i, i \in \mathbb{N}\rangle}$, $\langle f_i, e_i\rangle = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker symbol. Put

$$X_k = span\{e_k\}, \quad Y_k = \oplus_{i=1}^k X_i, \quad Z_k = \overline{\oplus_{i=k}^\infty X_i}. \tag{2.43}$$

Now, we state the following critical points theorem given by Bartsch.

Lemma 2.8. *Assume* $\Psi \in C^1(X, \mathbb{R})$ *satisfies the (PS) condition,* $\Psi(-u) = \Psi(u)$. *For every* $k \in \mathbb{N}$, *there exists* $\rho_k > r_k > 0$, *such that:*

1) $a_k := \max_{u \in Y_k, \|u\| = \rho_k} \Psi(u) \leq 0$;

2) $b_k := \inf_{u \in Z_k, \|u\| = r_k} \Psi(u) \to +\infty$ *as* $k \to \infty$.

Then, Ψ *has a sequence of critical values tending to* $+\infty$.

Lemma 2.9. *Assume that* (V1) *and* (F1) - (F4) *hold. For every* $k \in \mathbb{N}$, *there exists* $\rho_k > d_k > 0$, *such that:*

1) $a_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0$;

2) $b_k := \inf_{u \in Z_k, \|u\| = d_k} I(u) \to +\infty$ *as* $k \to \infty$,

where Y_k *and* Z_k *are defined by* (2.43). *Then, I has a sequence of critical values tending to* $+\infty$.

Proof. From (2.1), (2.4), (2.15) and $-\omega \leq \phi_u \leq 0$, one obtains:

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2\right]\mathrm{d}x \\ &\quad - \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla \phi_u|^2 \mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla \phi_u|^4 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2+2\omega^2u^2\right]dx-\int_{\mathbb{R}^3}\left(C_3|u|^\mu-C_4u^2\right)dx \\
&=\frac{1}{2}\|u\|_E^2+r_2^2\omega^2\|u\|_E^2-C_3r_\mu^\mu\|u\|_E^\mu+C_4r_2^2\|u\|_E^2 .
\end{aligned}
\tag{2.44}
$$

Since $\mu>2$, from (2.44), there exists $\rho_k>0$ such that:

$$a_k:=\max_{u\in Y_k,\|u\|=\rho_k} I(u)\leq 0.$$

Subsequently, for any $k\in\mathbb{N}$ and $p\in[2,6)$, we set:

$$\beta_k(p)=\sup_{u\in Z_k,\|u\|_E=1}\|u\|_p .$$

Similar to Lemma 2.8 in [27], we have $\beta_k(p)\to 0$ as $k\to\infty$. Letting $d_k=\left(\frac{1}{8C_2\beta_k^6(6)}\right)^{\frac{1}{4}}$, for any $u\in Z_k$. From (2.1), (2.5), (2.13), we get that:

$$
\begin{aligned}
I(u)&=\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2+\phi_u^2u^2\right]dx+\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx \\
&\quad+\frac{3\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx-\int_{\mathbb{R}^3}F(u)dx \\
&\geq\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+V(x)u^2\right]dx-\int_{\mathbb{R}^3}F(u)dx \\
&\geq\frac{1}{2}\|u\|_E^2-C_1\beta_k^2(2)\|u\|_E^2-C_2\beta_k^6(6)\|u\|_E^6 \\
&\geq\frac{1}{8}\|u\|_E^2 .
\end{aligned}
$$

Thus, we obtain $b_k:=\inf_{u\in Z_k,\|u\|=d_k} I(u)\geq\frac{1}{8}d_k^2\to+\infty$ as $\to\infty$. □

3. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. It follows from Lemma 2.3 and Lemma 2.7 that I satisfies the (PS) condition. By (F1), (F2) and (F4), it is easy to see that $I(0)=0$ and $I(-u)=I(u)$. By Lemma 2.9, the functional I satisfies the geometric conditions of Lemma 2.8. Hence, problem (1.1) has infinitely many nontrivial solutions $(u_n,\phi_n)\in E\times D_r^{1,2}(\mathbb{R}^3)$. This completes the proof. □

Proof of Theorem 1.2. First, we show that the set $\mathcal{S}\neq 0$. Similar to Lemma 2.7, we can prove that, there exists a sequence $\{w_n\}$ bounded in $H_r^1(\mathbb{R}^3)$ and $I(w_n)=c_1$, $I'(w_n)=0$. We claim that:

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|w_n|^2dx>0. \tag{3.1}$$

If not, from Lion's concentration compactness principle [30], we have that $w_n\to 0$ in $L^s(\mathbb{R}^3)$ for $2<s<6$. From (F1), (F2), there exists $C_8,C_9,C_{10}>0$ such that:

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}f(w_n)w_n-F(w_n)\right]dx\leq C_8\|w_n\|_{L^2}^2+C_9\|w_n\|_{L^s}^s+C_{10}\|w_n\|_{L^6}^6\leq\frac{C_1}{2}+o(1).$$

Thus,

$$\begin{aligned}c_1+o(1)&\le I(w_n)-\frac{1}{2}\langle I'(w_n),w_n\rangle\\&=-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{w_n}|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{w_n}|^4\,\mathrm{d}x+\int_{\mathbb{R}^3}\left[\frac{1}{2}f(w_n)w_n-F(w_n)\right]\mathrm{d}x\\&\le\frac{c_1}{2}+o(1).\end{aligned}$$

This contradiction shows that (3.1) holds, and so there exist $\delta>0$ and $\{y_n\}\subset\mathbb{R}^3$ such that:

$$\int_{B_2(y_n)}|w_n|^2\,\mathrm{d}x>\delta>0.$$

Let $\overline{w}_n=w_n(x+y_n)$, thus $\|\overline{w}_n\|=\|w_n\|$, $I(\overline{w}_n)=c_1$, $I'(\overline{w}_n)=0$ and

$$\int_{B_2(0)}|\overline{w}_n|^2\,\mathrm{d}x>\delta>0$$

which implies

$$\overline{w}_n\rightharpoonup\overline{w}\not\equiv0\ \text{ in } H_r^1(\mathbb{R}^3).$$

By a standard argument, we can show that $I'(\overline{w})=0$, and so $\overline{w}\in\mathcal{S}$.

Next, we will prove $0<k:=\inf_{u\in\mathcal{S}}I(u)$ is achieved. Let $\{w_n\}\subset\mathcal{S}$ be such that $I(w_n)\to k$ and $I'(w_n)=G(w_n)=0$ as $n\to\infty$. Arguing as before, we can prove that there exists $\tilde{w}\in\mathcal{S}$ such that $I(\tilde{w})\ge k$ and $I'(\tilde{w})=G(\tilde{w})=0$. If $\mu\in[4,6)$, from (V2), (F3), Lemma 2.1, (2.5), (2.6) and Fatou's Lemma, we have:

$$\begin{aligned}k&=\lim_{n\to\infty}\left[I(w_n)-\frac{1}{\mu}\langle I'(w_n),w_n\rangle\right]\\&=\lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left[|\nabla w_n|^2+V(x)|w_n|^2\right]\mathrm{d}x+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{w_n}|^4\,\mathrm{d}x\right.\\&\quad\left.+\int_{\mathbb{R}^3}\left[\frac{1}{\mu}\phi_{w_n}^2-\left(\frac{1}{2}-\frac{2}{\mu}\right)\omega\phi_{w_n}\right]|w_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\left[\frac{1}{\mu}f(w_n)w_n-F(w_n)\right]\mathrm{d}x\right\}\\&\ge\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left[|\nabla\tilde{w}|^2+V(x)|\tilde{w}|^2\right]\mathrm{d}x+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{\tilde{w}}|^4\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}\left[\frac{1}{\mu}\phi_{\tilde{w}}^2-\left(\frac{1}{2}-\frac{2}{\mu}\right)\omega\phi_{\tilde{w}}\right]|\tilde{w}|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\left[\frac{1}{\mu}f(\tilde{w})\tilde{w}-F(\tilde{w})\right]\mathrm{d}x\\&=I(\tilde{w})-\frac{1}{\mu}\langle I(\tilde{w}),\tilde{w}\rangle.\end{aligned}\tag{3.2}$$

If $\mu\in(2,4)$, from Lemma 2.1, (2.5), (2.6), (2.36), (2.37) and Fatou's Lemma, we obtain:

$$\begin{aligned}k&=\lim_{n\to\infty}\left[I(w_n)+\frac{\mu-4}{6-\mu}\langle I'(w_n),w_n\rangle+\frac{2-\mu}{6-\mu}G(w_n)\right]\\&=\lim_{n\to\infty}\left\{\frac{\mu-2}{6-\mu}\int_{\mathbb{R}^3}V(x)|w_n|^2\,\mathrm{d}x+\frac{\mu-2}{2(6-\mu)}\int_{\mathbb{R}^3}(\nabla V(x),x)|w_n|^2\,\mathrm{d}x\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[2(3-\mu)\omega\phi_{w_n}+(4-\mu)\phi_{w_n}^2\right]|w_n|^2\,\mathrm{d}x\\
&+\frac{2}{6-\mu}\int_{\mathbb{R}^3}\left[f(w_n)w_n-\mu F(w_n)\right]\mathrm{d}x+\frac{2\beta\mu}{16\pi}\int_{\mathbb{R}^3}\left|\nabla\phi_{w_n}\right|^4\mathrm{d}x\Big\}\\
&\geq\frac{\mu-2}{6-\mu}\int_{\mathbb{R}^3}V(x)|\tilde{w}|^2\,\mathrm{d}x+\frac{\mu-2}{2(6-\mu)}\int_{\mathbb{R}^3}(\nabla V(x),x)|\tilde{w}|^2\,\mathrm{d}x\\
&+\frac{1}{6-\mu}\int_{\mathbb{R}^3}\left[2(3-\mu)\omega\phi_{\tilde{w}}+(4-\mu)\phi_{\tilde{w}}^2\right]|\tilde{w}|^2\,\mathrm{d}x\\
&+\frac{2}{6-\mu}\int_{\mathbb{R}^3}\left[f(\tilde{w})\tilde{w}-\mu F(\tilde{w})\right]\mathrm{d}x+\frac{2\beta\mu}{16\pi}\int_{\mathbb{R}^3}\left|\nabla\phi_{\tilde{w}}\right|^4\mathrm{d}x\\
&=I(\tilde{w})+\frac{\mu-4}{6-\mu}\langle I'(\tilde{w}),\tilde{w}\rangle+\frac{2-\mu}{6-\mu}G(\tilde{w}).
\end{aligned}
\tag{3.3}
$$

It follows from (3.2), (3.3) and $I(\tilde{w})\geq k$ that $\tilde{w}\in\mathcal{S}$ and $I(\tilde{w})=k=\inf_{u\in\mathcal{S}}I(u)$. Hence, it follows from (2.2) and (2.5) that

$$J(u,\phi_u)=I(u)\geq I(\tilde{w})=J(\tilde{w},\phi_{\tilde{w}})=k,\ \forall u\in\mathcal{S}.$$

This proves that $(\tilde{w},\phi_{\tilde{w}})$ is a ground-state solution for system (1.1). □

4. Conclusion

In this paper, we used Pohožaev identity of (1.1) to certify the boundedness of Palais-Smale sequence of energy functional of problem (1.1) at level c_1, and then certified the boundedness of Palais-Smale sequence. By using critical point theory and the method of Nehari manifold, we obtained two existing results of infinitely many high-energy radial solutions and a ground-state solution for the system (1.1).

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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