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A Different Brachistochrone Problem with Counterintuitive Results

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Abstract

A problem similar to the famous brachistochrone problem is examined in which, instead of a smooth curve, the path consists of two straight-line sections, one slant and one horizontal. The condition for minimum sliding time is investigated, producing results that are both counterintuitive and interesting.

Keywords

Brachistochrone, Cycloid, Bead, Sliding, Minimum, Time, Counterintuitive

1. Introduction

Motion optimization and time minimization are areas of interest in many fields of science and engineering from both scientific point of view as well as application. One of the phenomena for which time minimization has been of interest to physicists and mathematicians alike is an object sliding down a ramp without friction along a path that minimizes the time.

Consider two pints in a vertical plane, *A* and *B*, separated by a vertical distance *h* and a horizontal distance $d > 0$, as shown in **Figure 1**. A thin smooth wire connects the two points together, and a bead slides without friction along the wire from point *A* to point *B*. We want to find the shape of the wire, which minimizes the sliding time of the bead. This problem, known as a *brachistochrone* problem, can be solved using calculus of variations [1]. The answer is that the wire must have the shape of an inverted cycloid with the parametric equations:

$$\begin{cases} x = \dfrac{h}{2}(\theta - \sin\theta) \\ y = \dfrac{h}{2}(1 + \cos\theta) \end{cases} \tag{1}$$

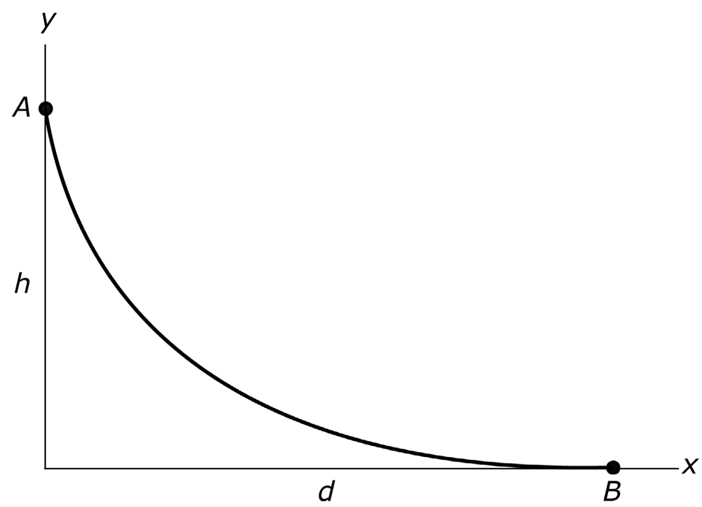

Figure 1. A wire connecting two points, A and B, in a vertical plane, along which a bead can slide without friction.

This curve is shown in **Figure 1** in the interval $0 \le x \le \pi h/2$, which corresponds to $0 \le \theta \le \pi$, and the time for the bead to slide down this wire is [2]:

$$t_c = \pi\sqrt{\frac{h}{2g}} \tag{2}$$

where $g = 9.80 \text{ m/s}^2$ is the acceleration due to gravity. However, point B does not necessarily have to be at the lowest point of the half cycloid, in which case, the sliding time would be different.

A cycloidal wire has other interesting properties as well. For example, regardless of where the bead starts its slide from rest on the wire, it reaches point B at the same time. Therefore, if a bead oscillates on a wire that has the shape of one full cycle of an inverted cycloid, its period of oscillation would be independent of its amplitude, and it is given by [2]:

$$T = 4t_c = 2\pi\sqrt{\frac{2h}{g}} \tag{3}$$

which is known as *tautochrone* oscillation [3]. This property was discovered in the 17th century by Christiaan Huygens in connection with improving the accuracy of pendulum clocks [4] [5] [6]. Yet another property of a cycloid is that when a bead slides down on a cycloidal wire without friction, the magnitude of its acceleration remains constant [7] [8].

2. A Different Brachistochrone Problem

In this work, we examine a different type of brachistochrone problem, in which the path of the sliding bead consists of two straight sections, one slant and one horizontal, as shown in **Figure 2**. We want to find the distance x which makes the total sliding time of the bead from point A to point B a minimum.

Again, the wire is frictionless and the bead starts from rest at point A. According to classical mechanics, the acceleration of the bead along the slant section of

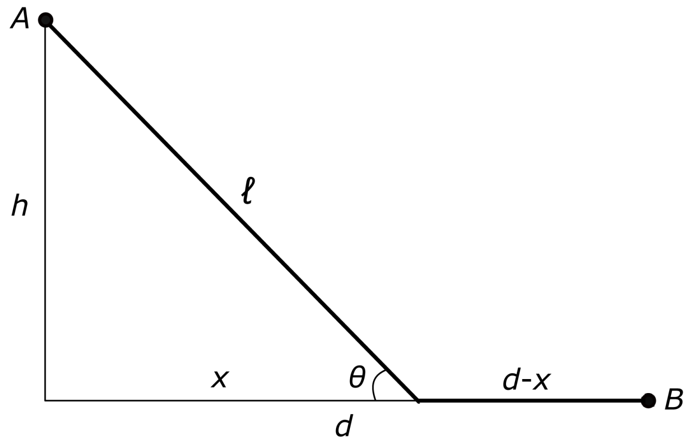

Figure 2. A frictionless wire consisting of two straight sections connects two points, *A* and *B*, in a vertical plane.

the wire is:

$$a = g\sin\theta \tag{4}$$

Here, θ is the angle of the slant section, as shown in **Figure 2**, and g is the acceleration due to gravity, as mentioned earlier. Then, we have:

$$l = \frac{1}{2}at_1^2 = \frac{1}{2}(g\sin\theta)t_1^2 \tag{5}$$

where t_1 is the sliding time along the slant section of the wire. From the geometry in **Figure 2**, we have:

$$l = \sqrt{x^2 + h^2} \tag{6}$$

and

$$\sin\theta = \frac{h}{\sqrt{x^2 + h^2}} \tag{7}$$

Substituting these quantities in Equation (5), and solving for t_1, we get:

$$t_1 = \sqrt{\frac{2(x^2 + h^2)}{gh}} \tag{8}$$

The speed of the bead at the bottom of the slant section can be calculated using either conservation of mechanical energy or the equation $v = (g\sin\theta)t_1$. Either way, we get $v = \sqrt{2gh}$. The bead then slides a distance $d - x$ along the horizontal section of the wire with this constant speed. Therefore, the sliding time for this part of the motion, t_2, is:

$$t_2 = \frac{d - x}{\sqrt{2gh}} \tag{9}$$

Therefore, the total sliding time of the bead, $t_1 + t_2$, is:

$$t = \sqrt{\frac{2(x^2 + h^2)}{gh}} + \frac{d - x}{\sqrt{2gh}} \tag{10}$$

We now differentiate this equation with respect to x to find the value of x that minimizes the total time t:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{\frac{2}{gh}}\frac{x}{\sqrt{x^2+h^2}} - \frac{1}{\sqrt{2gh}} = 0 \quad (11)$$

Solving this equation for x, we find:

$$x_m = \frac{h}{\sqrt{3}} \quad (12)$$

Where we have use the subscript "m" to indicate that this value of x corresponds to the minimum sliding time. Note that according to this result, the length of the slant path corresponding the minimum time, l_m, is:

$$l_m = \sqrt{h^2 + x_m^2} = \sqrt{h^2 + \frac{h^2}{3}} = \frac{2}{\sqrt{3}}h \quad (13)$$

these results are both interesting and counterintuitive because, for the minimum sliding time, the horizontal distance from point A where the sliding object reaches the level of point B as well as the length of the slant path, are independent of the distance d. However, the minimum time does depend on the value of d, as we shall see now. The minimum time, t_m, is obtained by substituting x_m in Equation (10), which gives:

$$t_m = \left(\sqrt{\frac{8}{3}} - \sqrt{\frac{1}{6}}\right)\sqrt{\frac{h}{g}} + \frac{d}{\sqrt{2gh}} \quad (14)$$

which shows that t_m increases linearly with d. Figure 3 shows the total sliding time of the bead as a function of x for different values of the horizontal distance d for comparison.

Finally, to compare this minimum sliding time to that for an inverted cycloid

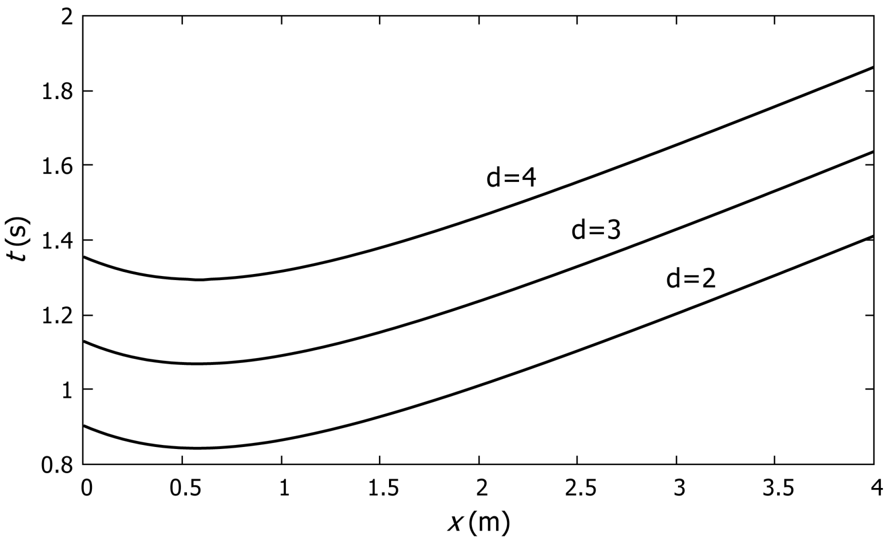

Figure 3. A graph of total sliding time of the bead for $h = 1$ m but different values of the horizontal distance d (shown in meters). In all cases, the value of x that minimizes the time is $1/\sqrt{3} = 0.5774$ m.

shown in **Figure 1**, let us find the ratio t_c/t_m with $d = h\pi/2$ in Equation (14), so that in both cases the horizontal distances between A and B is the same:

$$\frac{t_c}{t_m} = \frac{\pi\sqrt{\frac{h}{2g}}}{\left(\sqrt{\frac{8}{3}}-\sqrt{\frac{1}{6}}\right)\sqrt{\frac{h}{g}}+\frac{h\pi/2}{\sqrt{2gh}}} = \frac{\pi}{2\left(\sqrt{\frac{8}{3}}-\sqrt{\frac{1}{6}}\right)+\frac{\pi}{\sqrt{2}}} = 0.6726 \tag{15}$$

Therefore, for the same horizontal distance, the cycloid time is still shorter.

3. Summary

We have discussed a different type of brachistochrone problem, in which instead of a smooth curve, the path consists of two straight sections: one slant and one horizontal. Thus, we have examined the condition for minimum frictionless sliding time of an object along this path. The results are both interesting and counterintuitive as they show that, for the minimum sliding time, the length of the slant path is independent of the length of the horizontal path.

For the cycloidal brachistochrone curve shown in **Figure 1**, the horizontal distance d and the vertical duistance h between points A and B must be related by $d = \pi h/2$, as indicated just before Equation (2). Otherwise, points A and B will not be located at the highest and lowest points of the cycloid. Nonetheless, cycloid is still a brachistochrone curve for an object to slide down without friction from point A to point B.

Finally, we point out that the approach to this problem is, in a way, similar to that for finding the Snell's law of refraction, using Fermat's principle [9] [10].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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