Journal of Applied Mathematics and Physics, 2024, 12, 4291-4300
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Multiple Periodic Solution for a Class of Damped Vibration System

Zihan Zhang, Yuanhao Wang, Guanggang Liu*

School of Mathematical Sciences, Liaocheng University, Liaocheng, China
Email: *lgg112@163.com

How to cite this paper: Zhang, Z.H., Wang, Y.H. and Liu, G.G. (2024) Multiple Periodic Solution for a Class of Damped Vibration System. *Journal of Applied Mathematics and Physics*, **12**, 4291-4300.
https://doi.org/10.4236/jamp.2024.1212263

Received: November 15, 2024
Accepted: December 28, 2024
Published: December 31, 2024

Abstract

In this paper, we consider a class of damped vibration systems with super-quadratic potential. We obtain at least two nonzero periodic solutions through the minimization method and Morse theory. Moreover, if the potential is even with respect to the spatial variables, by applying a minimax type theorem, we can obtain a stronger multiplicity result with the number of solutions linked to the Morse index at zero.

Keywords

Damped Vibration Problems, Morse Theory, Critical Group, Morse Index, Minimization Method

1. Introduction

Consider the following system

$$\begin{cases} \ddot{z}+\left(p(t)I_N+M\right)\dot{z}+\left(\dfrac{1}{2}p(t)M-L(t)\right)z=\nabla G(t,z) & \text{a.e. } t\in[0,T] \\ z(0)-z(T)=\dot{z}(0)-\dot{z}(T)=0, \end{cases} \tag{1}$$

where $T>0$, $p\in L^1([0,T];R)$ and satisfies $\int_0^T p(t)\mathrm{d}t=0$, I_N denotes the N dimensional identity matrix, $L(t)=\left[a_{ij}(t)\right]$ is a $N\times N$ symmetric matrix-valued function defined on $[0,T]$ with $a_{ij}\in L^\infty([0,T])$ for all $i,j=1,2,\cdots,N$), and $M=\left[b_{ij}\right]$ is a $N\times N$ antisymmetric constant matrix, and $G\in C^2([0,T]\times\mathbb{R}^N,\mathbb{R})$. This type of system often appears in nature, such as damped harmonic motion and electric circuits with capacitance.

Recently, extensive research has been conducted on the system (1.1). If $M=0$ and $p(t)\equiv 0$, the system (1.1) reduces to the well-known second order Hamiltonian system, for which numerous results have been established via variational methods (see [1]-[11]). If $M = 0$ and $p(t) \neq 0$, in [12] [13], Wu *et al.* have constructed a variational framework for (1.1) and derived several existence results through variational methods. If $M \neq 0$ and $p(t) \neq 0$, in [14], Li *et al.* established a proper variational set for (1.1) and got several existence results for (1.1) with the super-quadratic potential via some critical point theorems. Since then, many authors have studied this general case using variational methods under various growth conditions (see [15]-[21] and references therein). In [21], Zhang explored (1.1) with super-quadratic and sub-quadratic potential and got infinitely many periodic solutions. In [15] [16], Chen studied (1.1) with asymptotically quadratic and super-quadratic potential. By employing a variant of the fountain theorem, he also discovered infinitely many nonzero periodic solutions. In [17], Chen and Schechter studied a class of damped vibration systems with general nonlinearities at infinity. By using a critical point theorem related to the symmetric mountain pass lemma they obtained infinitely many periodic solutions. In [18], Jiang *et al.* studied the damped vibration systems under a new super-quadratic condition. By using a fountain theorem they obtained a sequence of periodic solutions with the corresponding energy tending to infinity.

In this paper, we investigate the multiplicity of nonzero periodic solutions for (1.1), where the potential G exhibits super-quadratic at infinity. By employing Morse theory alongside a minimization method, we demonstrate the existence of two nonzero periodic solutions. Notably, our approach, which utilizes Morse theory, is infrequently applied in existing literature and does not necessitate symmetry in the nonlinearity. In the previous results, to obtain the multiple nonzero solutions of (1.1), the authors always assumed that the potential G is even with respect to the spatial variable z. However, to obtain the existence of two nonzero periodic solutions of (1.1), we do not need the symmetric assumption on the potential G. That is because compared to previous methods, the Morse theory can give a more detailed topological information about the critical points of the associated functional. Furthermore, if the nonlinearity G is even in z, we can achieve a more robust multiplicity result, with the number of solutions being influenced by the Morse index at zero. This advancement distinguishes our findings from previous multiplicity results.

Now we outline the idea of the proof. Firstly, we demonstrate that the functional associated with (1.1) is bounded below and satisfies the compactness condition. Utilizing the minimization method, we then get a minimizer of this functional. Secondly, we prove that the minimum value of the functional is negative, which guarantees the minimizer obtained is a nonzero periodic solution. Thirdly, arguing by contradiction, we further derive an additional nonzero periodic solution through the application of Morse theory. Finally, when the potential G is even in z, we achieve a stronger multiplicity result by employing a minimax-type critical point theorem. This result links the number of solutions to the Morse index at zero. The following assumptions are considered in our analysis:

(H_1) $G(t,z) \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t and $G(t,0) = 0$;

(H_2) $\nabla G(t,z) = o(|z|)$ as $|z| \to 0$ uniformly in $t \in [0,T]$;

(H_3) $\lim_{|z|\to\infty} \frac{G(t,z)}{|z|^2} = \infty$ uniformly in $t \in [0,T]$.

According to hypothesis (H_2), it is evident that $z = 0$ is a periodic solution of (1.1). Our aim is to find the nonzero periodic solution. We denote by i_0^- and ν_0 the Morse index and nullity of the functional associated with (1.1) at zero (see Section 2).

Theorem 1.1. Under the assumptions $(H_1)-(H_3)$ hold, if $i_0^- \ge 1$ and $\nu_0 = 0$, then (1.1) possesses at least two nonzero periodic solutions.

Theorem 1.2. Under the assumptions $(H_1)-(H_3)$ hold, if $i_0^- \ge 1$ and $G(t,-z) = G(t,z)$ for any $(t,z) \in \mathbb{R} \times \mathbb{R}^N$, then (1.1) possesses at least i_0^- pairs of nonzero periodic solutions.

Example 1.3. Let $G(t,z) = (a + b\sin t)|z|^\alpha$, where $a > b > 0$, $a > 2$. It is not difficult to see that $G(t,z)$ satisfies the conditions $(H_1)-(H_3)$.

The organization of the remainder is arranged as follows. In Section 2, the variational structure of (1.1) along with some important results are introduced. In Section 3, we provide the proof of results.

2. Preliminaries

In this section, we introduce the variational structure of (1.1) along with some preliminary results. Let H_T^1 be the usual Hilbert space with the inner product

$$\langle z,w \rangle_0 = \int_0^T \left[(z(t), w(t)) + (\dot{z}(t), \dot{w}(t)) \right] \mathrm{d}t, \quad \forall z, w \in H_T^1,$$

and the corresponding norm

$$\|z\|_0 = \left(\int_0^T \left(|z(t)|^2 + |\dot{z}(t)|^2 \right) \mathrm{d}t \right)^{\frac{1}{2}}, \quad \forall z \in H_T^1.$$

For simplicity, we denote $H = H_T^1$. Let

$$P(t) = \int_0^t p(z)\,\mathrm{d}z,$$

define

$$\langle z,w \rangle = \int_0^T \mathrm{e}^{P(t)} \left[(z(t), w(t)) + (\dot{z}(t), \dot{w}(t)) \right] \mathrm{d}t, \quad \forall z, w \in H,$$

and

$$\|z\| = \sqrt{\langle z,z \rangle}, \quad \forall z \in H.$$

It is evident that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent on H. In fact, due to $p \in L^1([0,T];\mathbb{R})$ and $\int_0^T p(t)\,\mathrm{d}t = 0$, we have $P(t)$ is a continuous T-period function. Thus there are two constants $c_1 > 0$ and $c_2 > 0$ satisfying

$$c_1^2 \le \mathrm{e}^{P(t)} \le c_2^2. \tag{2.1}$$

Hence

$$c_1 \|z\|_0 \le \|z\| \le c_2 \|z\|_0, \quad \forall z \in H.$$

Define the functional ψ on H by

$$\psi(z)=\frac{1}{2}\int_0^T e^P\left(|\dot{z}|^2+(Mz,\dot{z})+(L(t)z,z)\right)dt+\int_0^T e^P G(t,z)dt.$$

By (H_1), it is known that the functional $\psi\in C^2(H,\mathbb{R})$ and the critical points of ψ are the periodic solutions of (1.1) (see [14]).

Define the operator K on H by

$$\langle Kz,w\rangle=\int_0^T e^P\left[(M\dot{z},w)+\frac{1}{2}p(t)(Mz,w)\right]dt+\int_0^T e^P\left((I_N-L(t))z,w\right)dt,\quad \forall z,w\in H.$$

Clearly, K is a self-adjoint compact operator on H (see [14]). Then we can rewrite ψ as

$$\psi(z)=\frac{1}{2}\langle(I-K)z,z\rangle+\int_0^T e^P G(t,z)dt,$$

where I is the identity operator. According to the operator $I-K$, H can be decomposed as

$$H=H^-\oplus H^0\oplus H^+,$$

where H^+, H^-, H^0 are the positive definite, negative definite and null subspaces of H respectively. Then there exists some constant $\gamma>0$ satisfying

$$\langle(I-K)z,z\rangle\le-\gamma\|z\|^2,\quad \forall z\in H^-, \tag{2.2}$$

and

$$\langle(I-K)z,z\rangle\ge\gamma\|z\|^2,\quad \forall z\in H^+. \tag{2.3}$$

Clearly, H^- and H^0 are finite dimensional and we denote

$$i_0^-=\dim H^-,\quad \nu_0=\dim H^0.$$

Subsequently, for $z\in H$ we present $z=z^++z^0+z^-$ with $z^+\in H^+$, $z^0\in H^0$ and $z^-\in H^-$.

Now we state some fundamental knowledge about Morse theory which is instrumental in proving our main result, for more details, see [5] [22] and [23]. Let E be a Hilbert space and $\Psi\in C^2(E,\mathbb{R})$. Set $K(\Psi)=\{z\mid\Psi'(z)=0\}$. We call the functional Ψ satisfies (PS) condition if for any sequence $\{z_j\}\subset E$ where $\{\Psi(z_j)\}$ is bounded and $\Psi'(z_j)\to0$ as $j\to\infty$, there exists a convergent subsequence. Let (A,B) be a topological pair and $H_*(A,B)$ be the singular homology group of (A,B) over a field $\mathcal{F}$. Suppose $z\in K(\Psi)$ is isolated and $\Psi(z)=c$. Define the critical groups of Ψ at z as

$$C_p(\Psi,z)=H_p\left(\Psi^c\cap U,\Psi^c\cap U\setminus\{z\}\right),\quad p=0,1,\cdots,$$

where $\Psi^c=\{z\in E\mid\Psi(z)\le c\}$ and U is a closed neighborhood of z. For $z\in K(\Psi)$, the Morse index of Ψ at z represents the dimension of the negative subspace of E according to the spectrum of $\Psi''(z)$. $z\in K(\Psi)$ is called non-degenerate if $\Psi''(z)$ is invertible.

Remark 2.1. If z is an isolated minimum point of Ψ with $\Psi(z)=c$, then by the definition we have $C_p(\Psi,z)=H_P(\{z\},\phi)=\delta_{p,0}\mathcal{F}$, $p=0,1,\cdots$, see [5].

The following results are from [23].

Proposition 2.2. Let E be a Hilbert space and $\Psi \in C^2(E,\mathbb{R})$, $z \in K(\Psi)$ is non-degenerate and has Morse index k, then

$$C_p(\Psi,z) = \delta_{p,k}\mathcal{F}, \quad p = 0,1\cdots.$$

Proposition 2.3. Let E be a Hilbert space and $\Psi \in C^2(E,\mathbb{R})$ satisfies the (PS) condition over E. Suppose $K(\Psi) \cap \Psi^{-1}([a,b]) = \{z_1, z_2, \cdots, z_l\}$, where a,b are regular values of Ψ. Then

$$\sum_{p=0}^{\infty}\left(\sum_{i=1}^{l} rankC_p(\Psi,z_i)\right)t^p = \sum_{p=0}^{\infty} rankH_p(\Psi^b,\Psi^a)t^p + (1+t)P(t),$$

where $P(t)$ is a polynomial with nonnegative coefficients.

To prove Theorem 1.2, we introduce a theorem from [23] (see also [24] [25]). For $a > 0$, let $S_a = \{z \in E \mid \|z\| = a\}$.

Proposition 2.4. Let E be a Hilbert space and $\Psi \in C^2(E,\mathbb{R})$ is an even functional and satisfies the (PS) condition with $\Psi(0) = 0$. Suppose that $E = Y \oplus Z$, and X is a subspace of E with $\dim X = l > m = \dim Y$. If there exist $r > 0$ and $\eta > 0$ satisfying

$$\inf \Psi(Z) > -\infty, \quad \sup \Psi(S_r \cap X) \le -\eta, \tag{2.4}$$

then Ψ possesses at least $l-m$ pairs of nontrivial critical points.

3. Proof of the Main Result

In order to prove our main results, we first prove some lemmas.

Lemma 3.1. Under the assumptions $(H_1 - H_3)$, ψ is bounded below on H.

Proof. By (H_3) we know for any $A > 0$, there is a positive constant C_A satisfying

$$G(t,z) \ge A|z|^2 - C_A \tag{3.1}$$

Choose

$$A > \frac{1}{4}\|M\|^2 + \frac{1}{2}\|L(t)\|_{L^\infty} + \frac{1}{4} \tag{3.2}$$

where $\|M\|$ is the norm of M and $\|L(t)\|_{L^\infty}$ is the L^∞ norm of $L(t)$ on $[0,T]$. Then by (3.1), (3.2) and the mean value inequality, for each $z \in H$ we have

$$\begin{aligned}
\psi(z) &= \frac{1}{2}\int_0^T e^P\left(|\dot{z}|^2 + (Mz,\dot{z}) + (L(t)z,z)\right)dt + \int_0^T e^P G(t,z)\,dt \\
&\ge \frac{1}{2}\int_0^T e^P\left(|\dot{z}|^2 - \|M\||z||\dot{z}| - \|L(t)\|_{L^\infty}|z|^2\right)dt + \int_0^T e^P\left(A|z|^2 - C_A\right)dt \\
&\ge \frac{1}{2}\int_0^T e^P\left(|\dot{z}|^2 - \frac{1}{2}\|M\||z|^2 - \frac{1}{2}|\dot{z}|^2 - \|L(t)\|_{L^\infty}|z|^2\right)dt + \int_0^T e^P\left(A|z|^2 - C_A\right)dt \\
&\ge \int_0^T e^P\left(\frac{1}{4}|\dot{z}|^2 + \left(A - \frac{1}{4}\|M\|^2 - \frac{1}{2}\|L(t)\|_{L^\infty}\right)|z|^2\right)dt - C_A' \\
&\ge \frac{1}{4}\int_0^T e^P\left(|\dot{z}|^2 + |z|^2\right)dt - C_A'
\end{aligned} \tag{3.3}$$

Consequently, $\Psi(z)$ is bounded below on H.

Lemma 3.2. Under the assumptions $(H_1 - H_3)$, ψ satisfies the (PS) condition.

Proof. Suppose $\{z_n\} \subset H$ is a sequence with the property $|\psi(z_n)| \le C$ for some constant $C > 0$ and $\psi'(z_n) \to 0$ as $n \to \infty$. Since $|\psi(z_n)| \le C$, then by (3.3) we see that the sequence $\{z_n\}$ is bounded in H Therefore there exists a $z \in H$ satisfying up to a subsequence $z_n \rightharpoonup z$ in H and $z_n(t) \to z(t)$ uniformly in $[0,T]$ as $n \to \infty$. Since $\psi'(z_n) \to 0$ as $n \to \infty$, then we can conclude that

$$\left\langle \psi'(z_n) - \psi'(z), z_n^+ - z^+ \right\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Note that by (2.3),

$$\begin{aligned} &\left\langle \psi'(z_n) - \psi'(z), z_n^+ - z^+ \right\rangle \\ &= \left\langle (I-K)(z_n^+ - z^+), z_n^+ - z^+ \right\rangle + \int_0^T \mathrm{e}^P \left(\nabla G(t, z_n) - \nabla G(t, z) \right)(z_n^+ - z^+) \mathrm{d}t \\ &\ge \gamma \left\| z_n^+ - z^+ \right\|^2 + \int_0^T \mathrm{e}^P \left(\nabla G(t, z_n) - \nabla G(t, z) \right)(z_n^+ - z^+) \mathrm{d}t. \end{aligned} \tag{3.5}$$

Recall that $z_n(t) \to z(t)$ uniformly in $[0,T]$, we have

$$\int_0^T \mathrm{e}^{P(t)} \left(\nabla G(t, z_n) - \nabla G(t, z) \right)(z_n^+ - z^+) \mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Then by (3.4)-(3.6), we have $z_n^+ \to z^+$ in H as $n \to \infty$. Since both H^- and H^0 are finite dimensional, we also have $z_n^- \to z^-$ and $z_n^0 \to z^0$ in H as $n \to \infty$. Then we know $z_n \to z$ in H as $n \to \infty$ and the (PS) condition is proved.

Lemma 3.3. Under the assumptions $(H_1 - H_3)$, if $i_0^- \ge 1$ then there exist $\gamma > 0$ and $\eta > 0$ such that

$$\sup_{z \in S_r \cap H^-} \psi(z) \le -\eta.$$

Proof. By (H_2), for $0 < \varepsilon < \frac{\gamma}{4}$ there is a positive δ such that for all z satisfying $|z| \le \delta$,

$$G(t, z) \le \varepsilon |z|^2. \quad (3.7)$$

By (3.7) and the Sobolev inequality $\|z\|_{L^\infty} \le C_2 \|z\|$, for $z \in H^-$ with $\|z\| \le \frac{\delta}{C_2}$, we obtain

$$\begin{aligned} \psi(z) &= \frac{1}{2} \left\langle (I-K) z, z \right\rangle + \int_0^T \mathrm{e}^P G(t, z) \mathrm{d}t \\ &\le -\frac{\gamma}{2} \|z\|^2 + \int_0^T \mathrm{e}^P \left(\varepsilon |z|^2 \right) \mathrm{d}t \\ &\le \left(-\frac{\gamma}{2} + \varepsilon \right) \|z\|^2 \le -\frac{\gamma}{4} \|z\|^2. \end{aligned} \tag{3.8}$$

Let $r \le \frac{\delta}{C_2}$ and $\eta = \frac{\gamma r^2}{4}$. By (3.8), for $z \in H^-$ with $\|z\| = r$ we have

$$\psi(z) \le -\eta.$$

Hence

$$\sup_{z \in S_r \cap H^-} \psi(z) \le -\eta.$$

Proof of theorem 1.1. From Lemma 3.1, ψ is bounded below on H, then there is a sequence $\{z_n\} \subset H$ satisfying

$$\psi(z_n) \to \inf_{z \in H} \psi(z) \text{ and } \psi'(z_n) \to 0 \text{ as } n \to \infty.$$

According to Lemma 3.2, ψ satisfies the (PS) condition, then up to a subsequence we have $z_n \to z_0$ for some $z_0 \in H$. Thus

$$\psi(z_0) = \inf_{z \in H} \psi(z) \text{ and } \psi'(z_0) = 0.$$

Now we show that $\psi(z_0) < 0$. In fact, by Lemma 3.3,

$$\psi(z_0) = \inf_{z \in H} \psi(z) \le \inf_{z \in S_r \cap H^-} \psi(z) \le \sup_{z \in S_r \cap H^-} \psi(z) \le -\eta < 0$$

Note that $\psi(0) = 0$, we conclude that z_0 is a nonzero periodic solution of (1.1).

Now we demonstrate that the problem (1.1) has another nonzero periodic solution. We use an indirect argument. If z_0 is the only nonzero periodic solutions of (1.1), then ψ has exactly two critical points 0 and z_0. Choose $a < \psi(z_0) = \inf_{z \in H} \psi(z)$ and $b > \psi(0) = 0$, then a and b are two regular values of ψ. By Proposition 2.3, we get

$$\sum_{p=0}^{\infty} \left(rankC_p(\psi, z_0) + rankC_p(\psi, 0) \right) t^p = \sum_{p=0}^{\infty} rankH_p(\psi^b, \psi^a) t^p + (1+t) P(t), \quad (3.9)$$

where $P(t)$ is a polynomial with nonnegative coefficients. Since z_0 is the minimizer of ψ, then by Remark 2.1 we have

$$C_p(\psi, z_0) = \delta_{p,0} \mathcal{F}, \quad p = 0, 1, 2, \cdots. \quad (3.10)$$

Thus

$$rankC_p(\psi, z_0) = \begin{cases} 1, & p = 0, \\ 0, & p \ge 1. \end{cases} \quad (3.11)$$

Recall that $i_0^- = \dim H^-$ and $\nu_0 = \dim H^0$. By (H_2) we see that i_0^- and ν_0 are the Morse index and nullity of ψ at zero. Since $i_0^- \ge 1$ and $\nu_0 = 0$, by Proposition 2.2 we obtain

$$C_p(\psi, 0) = \delta_{p, i_0^-} \mathcal{F}, \quad p = 0, 1, 2, \cdots. \quad (3.12)$$

Thus

$$rankC_p(\psi, 0) = \begin{cases} 1, & p = i_0^-, \\ 0, & p \ne i_0^-. \end{cases} \quad (3.13)$$

Note that there is no critical point on $\psi^{-1}([b, +\infty))$, then by the deformation theorem we see that ψ^b is a deformation retract of the whole space H. Then by the property of the singular homology, we have

$$H_p\left(\psi^b,\psi^a\right)=H_p\left(H,\psi^a\right). \tag{3.14}$$

Recall that $a<\inf_{z\in H}\psi(z)$, then $\psi(a)=\phi$. From (3.14) we have

$$H_p\left(\psi^b,\psi^a\right)=H_p\left(H,\phi\right)=\delta_{p,0}\mathcal{F},\quad p=0,1,2,\cdots. \tag{3.15}$$

Thus

$$rankH_p\left(\psi^b,\psi^a\right)=\begin{cases}1, & p=0,\\ 0, & p\geq 1.\end{cases} \tag{3.16}$$

Then by (3.9), (3.11), (3.13) and (3.16), we have

$$t^0+t^{i_0^-}=t^0+(1+t)P(t). \tag{3.17}$$

Take $t=-1$ in (3.17), then we obtain $(-1)^{i_0^-}=0$, which is a contradiction. Hence, (1.1) has at least two nonzero periodic solutions.

Proof of theorem 1.2. Recall that $\psi\in C^2(H,\mathbb{R})$ and $\psi(0)=0$ from (H_1). Given $G(t,-z)=G(t,z)$ for any $(t,z)\in\mathbb{R}\times\mathbb{R}^N$, we observe that ψ is an even functional. From Lemma 3.2, ψ satisfies the (PS) condition. Let $Y=\{0\}$, $Z=H$ and $X=H^-$, then $\dim Y=0<i_0^-=\dim X$. By Lemma 3.1, we see that

$$\inf\Psi(Z)>-\infty.$$

From Lemma 3.3, there exist positive constants r and η satisfying

$$\sup\Psi(S_r\cap X)\leq-\eta.$$

Hence, using Proposition 2.4, (1.1) possesses at least i_0^- pairs of nonzero periodic solutions.

4. Conclusion

In this paper, the existence of multiple nonzero periodic solutions of the damped vibration system has been considered. We first prove that the associated functional satisfies the (PS) condition and is bounded from below. Then, by using the minimization method, we obtain a minimizer which is a nonzero periodic solution of the system. Based on the topological feature of the minimizer, we obtain another nonzero periodic solution by using the Morse theory. Note that this result does not require the potential to be symmetric with respect to the spatial variable. Our result generalizes some known results in literature. Furthermore, when the potential G is even in the spatial variable, by using a multiple critical point theorem, we have established a stronger multiplicity result with the number of periodic solutions related to the Morse index at zero.

Acknowledgements

The authors would like to thank the reviewer for the valuable comments, which have helped to improve the quality of this paper.

Funding

This research was supported by National Natural Science Foundation of China

(11901270) and Shandong Provincial Natural Science Foundation (ZR2019BA019).

Authors' Contributions

Zihan Zhang: Conception and design of study, writing original draft, writing review and editing.

Yuanhao Wang: Writing review and editing.

Guanggang Liu: Conception and design of study, writing review and editing.

The authors declare no conflicts of interest regarding the publication of this paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Antonacci, F. (1997) Existence of Periodic Solutions of Hamiltonian Systems with Potential Indefinite in Sign. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **29**, 1353-1364. https://doi.org/10.1016/s0362-546x(96)00190-3

[2] Berger, M.S. (1970) On Periodic Solutions of Second Order Hamiltonian Systems (I). *Journal of Mathematical Analysis and Applications*, **29**, 512-522. https://doi.org/10.1016/0022-247x(70)90065-x

[3] Zheng, J.M. and Cheng, J.X. (2010) Periodic Solutions for a Class of Second Order Hamiltonian Systems. *Acta Mathematica Sinica*, **53**, 721-726.

[4] Li, L. and Schechter, M. (2016) Existence Solutions for Second Order Hamiltonian Systems. *Nonlinear Analysis*: *Real World Applications*, **27**, 283-296. https://doi.org/10.1016/j.nonrwa.2015.08.001

[5] Mawhin, J., Willem, M. and Mawhin, J. (1989) Critical Point Theory and Hamiltonian Systems.

[6] Meng, F. and Zhang, F. (2008) Periodic Solutions for Some Second Order Systems. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **68**, 3388-3396. https://doi.org/10.1016/j.na.2007.03.031

[7] Pipan, J. and Schechter, M. (2014) Non-Autonomous Second Order Hamiltonian Systems. *Journal of Differential Equations*, **257**, 351-373. https://doi.org/10.1016/j.jde.2014.03.016

[8] Tang, C. and Wu, X. (2010) Some Critical Point Theorems and Their Applications to Periodic Solution for Second Order Hamiltonian Systems. *Journal of Differential Equations*, **248**, 660-692. https://doi.org/10.1016/j.jde.2009.11.007

[9] Tang, C. and Wu, X. (2014) Periodic Solutions for a Class of New Superquadratic Second Order Hamiltonian Systems. *Applied Mathematics Letters*, **34**, 65-71. https://doi.org/10.1016/j.aml.2014.04.001

[10] Tang, X.H. and Jiang, J. (2010) Existence and Multiplicity of Periodic Solutions for a Class of Second-Order Hamiltonian Systems. *Computers & Mathematics with Applications*, **59**, 3646-3655. https://doi.org/10.1016/j.camwa.2010.03.039

[11] Zou, W. and Li, S. (2002) Infinitely Many Solutions for Hamiltonian Systems. *Journal of Differential Equations*, **186**, 141-164. https://doi.org/10.1016/s0022-0396(02)00005-0

[12] Wu, X., Chen, S. and Teng, K. (2008) On Variational Methods for a Class of Damped Vibration Problems. *Nonlinear Analysis: Theory, Methods & Applications*, **68**, 1432-1441. https://doi.org/10.1016/j.na.2006.12.043

[13] Wu, X. and Chen, J. (2009) Existence Theorems of Periodic Solutions for a Class of Damped Vibration Problems. *Applied Mathematics and Computation*, **207**, 230-235. https://doi.org/10.1016/j.amc.2008.10.020

[14] Li, X., Wu, X. and Wu, K. (2010) On a Class of Damped Vibration Problems with Super-Quadratic Potentials. *Nonlinear Analysis: Theory, Methods & Applications*, **72**, 135-142. https://doi.org/10.1016/j.na.2009.06.044

[15] Chen, G. (2014) Infinitely Many Nontrivial Periodic Solutions for Damped Vibration Problems with Asymptotically Linear Terms. *Applied Mathematics and Computation*, **245**, 438-446. https://doi.org/10.1016/j.amc.2014.07.114

[16] Chen, G. (2015) Periodic Solutions of Superquadratic Damped Vibration Problems. *Applied Mathematics and Computation*, **270**, 794-801. https://doi.org/10.1016/j.amc.2015.08.095

[17] Chen, G. and Schechter, M. (2019) Multiple Periodic Solutions for Damped Vibration Systems with General Nonlinearities at Infinity. *Applied Mathematics Letters*, **90**, 69-74. https://doi.org/10.1016/j.aml.2018.10.014

[18] Jiang, S., Xu, H. and Liu, G. (2023) Multiplicity of Periodic Solutions for a Class of New Super-Quadratic Damped Vibration Problems. *Journal of Dynamical and Control Systems*, **29**, 1287-1297. https://doi.org/10.1007/s10883-022-09638-6

[19] Khaled, K. (2021) New Results on Periodic Solutions for Second Order Damped Vibration Systems. *Ricerche di Matematica*, **72**, 709-721. https://doi.org/10.1007/s11587-021-00567-3

[20] Liu, G. and Shi, S. (2020) Existence and Multiplicity of Periodic Solutions for Damped Vibration Problems with Nonlinearity of Linear Growth. *Applied Mathematics Letters*, **107**, Article ID: 106502. https://doi.org/10.1016/j.aml.2020.106502

[21] Zhang, X. (2013) Infinitely Many Solutions for a Class of Second-Order Damped Vibration Systems. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 15, 1-18. https://doi.org/10.14232/ejqtde.2013.1.15

[22] Bartsch, T. and Li, S. (1997) Critical Point Theory for Asymptotically Quadratic Functionals and Applications to Problems with Resonance. *Nonlinear Analysis: Theory, Methods & Applications*, **28**, 419-441. https://doi.org/10.1016/0362-546x(95)00167-t

[23] Chang, K. (1993) Infinite Dimensional Morse Theory. In: Chang, K., Ed., *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, 1-82. https://doi.org/10.1007/978-1-4612-0385-8_1

[24] Liu, Z., Su, J. and Wang, Z. (2008) A Twist Condition and Periodic Solutions of Hamiltonian Systems. *Advances in Mathematics*, **218**, 1895-1913. https://doi.org/10.1016/j.aim.2008.03.024

[25] Liu, Z., Su, J. and Wang, Z. (2008) Solutions of Elliptic Problems with Nonlinearities of Linear Growth. *Calculus of Variations and Partial Differential Equations*, **35**, 463-480. https://doi.org/10.1007/s00526-008-0215-0