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A Singular Limit in a Non-Local Reaction-Diffusion Equation in Periodic Media

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Abstract

In this paper, we provide an asymptotic analysis of a nonlocal reaction-diffusion equation and with a non-local stable operator of order $\alpha\in(0,1)$. Firstly, we prove the existence and uniqueness of positive and bounded solutions for the stationary equation. Finally, we perform a long time-long range scaling in order to prove that the stable state invades the unstable state with a speed which is exponential in time.

Keywords

Nonlocal Fractional Operator, Exponential Speed of Propagation, Asymptotics

1. Introduction

We concentrate on the following equation:

$$\begin{cases}\partial_t u(t,x)+L^{\alpha}(u)(t,x)=\mu(x)u(t,x)-u(t,x)^2+\tau\big(J*u(t,x)-u(t,x)\big), & (t,x)\in(0,+\infty)\times\mathbb{R}^N\\ u(x,0)=u_0(x)\in C_c^{\infty}(\mathbb{R}^N,\mathbb{R}^+).\end{cases} \tag{1.1}$$

In the above setting, $\tau>0$ is a constant, μ is a *l*-periodic function, $\alpha\in(0,1)$ is given and the term L^{α} denotes a fractional elliptic operator which is defined as follows:

$$L^{\alpha}(u)(t,x):=-PV\int_{\mathbb{R}^N}\big(u(t,x+h)-u(t,x)\big)\beta\left(x,\frac{h}{|h|}\right)\frac{\mathrm{d}h}{|h|^{N+2\alpha}}, \tag{1.2}$$

where PV denotes the principal value and $\beta:\mathbb{R}^N\times S^{d-1}\to\mathbb{R}$ is a *l*-periodic smooth function such that for all $(x,\theta)\in\mathbb{R}^N\times S^{d-1}$

$$\beta(x,\theta)=\beta(x,-\theta) \text{ and } 0<b\le\beta(x,\theta)\le B,$$

with b and B positive constants. When β is constant, we recover the classical fractional Laplacian $(-\Delta)^{\alpha}$.

Let $l_1,\cdots,l_N$ be N given real numbers. As stated below, the statement about a function $g:\mathbb{R}^N\to\mathbb{R}$ is l-periodic means that $g(x_1,\cdots,x_k+l_k,\cdots,x_N)\equiv g(x_1,\cdots,x_k,\cdots,x_N)$ for all $k=1,\cdots,N$. Let C_l be the period cell defined by

$$C_l=[0,l_1]\times[0,l_2]\times\cdots\times[0,l_N].$$

We assume

$$J*u(t,x)=\int_{\mathbb{R}^N}J(y)u(t,x-y)\mathrm{d}y$$

is called the nonlocal dispersal operator with $J(\cdot)$ being a C^1 nonnegative convolution kernel, and for all $x\in\mathbb{R}^N$

$$J(x)\to\frac{1}{|x|^{N+2\alpha}}\ \text{as}\ x\to\infty,\quad \int_{\mathbb{R}^N}J(y)\mathrm{d}y=1\ \text{and}\ J(ax)\le a^{-k}J(x). \tag{1.3}$$

where $a>0$ and $k>d$.

The main goal of this paper is to describe the spread of the front associated with Equation (1.1). It is demonstrated that the stable state advances into the unstable state at an exponential speed.

Equation (1) was initially presented by Fisher in 1937 [1] and by Kolmogorov, Petrovskii, and Piscunov in 1938 [2] in the specific context of a homogeneous environment ($\mu=1$) and standard diffusion ($L^{\alpha}=\Delta$), corresponding to $\alpha=1$, $\beta=1$ and $\tau=0$. Subsequently, Aronson and Weinberger[3] demonstrated a result akin to ours for the scenario proposed by Fisher and Kolmogorov, Petrovskii, and Piscunov in [1]. In this instance, the propagation occurs at a uniform speed, regardless of direction. Freidlin and Gartner, in their 1985 work [4], examined this question with a standard Laplacian in a heterogeneous environment (μ periodic). Employing a probabilistic approach, they established that the propagation speed varies based on the chosen direction $e\in S^{N-1}$, albeit remaining constant within that direction. Alternative proofs using PDE techniques are available in [5] and [6]. Turning to the fractional Laplacian and a constant environment, Cabre and Roquejoffre demonstrated in [7] that the front position grows exponentially over time (see also [8] for heuristic and numerical studies predicting this behavior, and [9] for an alternative proof). Further, in [10], Cabre, Coulon, and Roquejoffre investigated the speed of propagation in a periodic environment modeled by Equation (1.1), but with the fractional Laplacian replacing the L operator and $\tau=0$. Notably, in the fractional case, the speed of propagation becomes independent of direction. They proved that the propagation speed increases exponentially with time, with a specific exponent determined by a periodic principal eigenvalue.

In recent times, nonlocal population models have garnered considerable attention (refer to [11] [12]). Initially, the population's dispersal mechanism was considered nonlocal, implying that standard Gaussian diffusion could be replaced by a process influenced by Levy flights. Such nonlocal dispersal strategies have been observed in nature (see [13]). Moreover, the potential nonlocal characteristic of the species growth rate is motivated by real-world situations where populations benefit not just from nearby resources but also from those within their "influence zone" (analogous to the "giraffe's neck" effect) (refer to [14]). This nonlocal aspect can be modeled through convolution with an integrable kernel. It is noteworthy, from a mathematical perspective, that the two types of nonlocal operators are fundamentally distinct; the fractional Laplacian tends to decrease the differentiability of the function, whereas convolution exerts a smoothing effect.

The aim of this work is to furnish an alternative proof for this property by utilizing an asymptotic methodology known as "approximation of geometric optics." Our focus lies in examining the long-term behavior of the solution u. We show that, within the set defined by $\{|x| < e^{ct}\}$, as t approaches infinity, u converges to a stationary state u^+, whereas outside this domain, u tends towards zero. The core concept in this approach involves executing a long-time, long-range rescaling to capture the effective behavior of the solution (refer to [9] [15] [16]). This paper is closely related to [9] where the authors Meleard and Mirrahimi have introduced such an "approximation" for a model with the fractional Laplacian and a simpler reaction term $(u-u^2)$.

A significant contribution of this paper lies in the innovative rescaling techniques we introduce, which facilitate the description of the population's asymptotic dynamics. We employ two rescaling methods. The first one, focusing on long-range and long-term perspectives, hinges on the propagation speed. The underlying concept is to observe the population from a distant vantage point, akin to homogenization, where we disregard intricate details but capture the population's propagation. Consequently, we propose a novel asymptotic formalism tailored for reaction-diffusion equations involving the fractional Laplacian. This formalism extends the classical approach known as "geometric optics approximation," which is extensively developed for reaction-diffusion equations with the traditional Laplacian.

Next, under the assumptions on β, the operator $L^\alpha - \tau J - (\mu(x) - \tau) Id$ admits a principal eigenpair (ϕ_1, λ_1) that is

$$\begin{cases} L^\alpha \phi_1(x) - \tau(J * \phi_1 - \phi_1) - \mu \phi_1(x) = \lambda_1 \phi_1(x),\ x \in \mathbb{R}^N, \\ \phi_1 \text{ periodic}, \phi_1 > 0, \|\phi_1\|_\infty = 1 \end{cases} \tag{1.4}$$

We will provide a detailed proof later on. To assure the existence of a bounded, positive and periodic steady solution u^+ for (1.1), we will assume that the principal eigenvalue λ_1 is negative:

$$\lambda_1 < 0 \tag{\textbf{H1}}$$

Note that such stationary solution is unique in the class of positive and periodic stationary solutions.

The first result of this paper ensures the existence and the uniqueness of a positive bounded stationary state u^+ of (1.1), the stationary problem for (1.1) is written as:

$$L^{\alpha}\left(u^{+}\right)=\left(\mu(x)-u^{+}\right)u^{+}+\tau\left(J*u^{+}-u^{+}\right),\ \ x\in\mathbb{R}^{N} \tag{1.5}$$

Theorem 1. *Under the assumption* (*H1*), *there exists a unique positive and bounded solution* u^{+} *of* (*1.5*). *Moreover it is l-periodic.*

Next, We introduce the following rescaling

$$(t,x)\mapsto\left(\frac{t}{\varepsilon},|x|^{\frac{1}{\varepsilon}}\frac{x}{|x|}\right). \tag{1.6}$$

We perform this rescaling because one expects that for t large enough, u is close to the stationary state u_{+} in the following set $\left\{(t,x)\in\mathbb{R}^{+}\times\mathbb{R}^{N}\mid |x|<\mathrm{e}^{\frac{|\lambda_1|t}{N+2\alpha}}\right\}$ and u is close to 0 in the set $\left\{(t,x)\in\mathbb{R}^{+}\times\mathbb{R}^{N}\mid |x|>\mathrm{e}^{\frac{|\lambda_1|t}{N+2\alpha}}\right\}$. The change of variable (1.6) will therefore respect the geometries of these sets. We then rescale the solution of (1.1) as follows

$$u_{\varepsilon}(t,x)=u\left(|x|^{\frac{1}{\varepsilon}}\frac{x}{|x|},\frac{t}{\varepsilon}\right)$$

and a new steady state

$$u_{+,\varepsilon}(x)=u_{+}\left(|x|^{\frac{1}{\varepsilon}}\frac{x}{|x|}\right).$$

We prove:

Theorem 2. *Assuming* (*H1*), *let u be the solution of* (*1.1*). *Then*

(*i*) $u_{\varepsilon}\to 0$, *locally uniformly in*

$\mathcal{A}=\left\{(t,x)\in\mathbb{R}^{N}\times(0,\infty)\mid |\lambda_1|t<(N+2\alpha)\log|x|\right\}$,

(*ii*) $\dfrac{u_{\varepsilon}}{u_{+,\varepsilon}}\to 1$, *locally uniformly in*

$\mathcal{B}=\left\{(t,x)\in\mathbb{R}^{N}\times(0,\infty)\mid |\lambda_1|t>(N+2\alpha)\log|x|\right\}$.

In section 2, we prove the existence and uniqueness of the solution of Equation (1.1), and introduce some main results and technical tools to prepare the following theorems. In section 3, We prove the existence and uniqueness of the nontrivial bounded stationary solution of Equation (1.1) by its eigenvalues and eigenfunction properties. In section 4, we provide a sub and a super-solution of Equation (1.1) and demonstrate Theorem 2.

2. Preliminary Results

We initially present a well-established principle concerning the fractional heat kernel.

Proposition 2.1. *There exists a positive constant* C_1 *larger than 1 such that the heat kernal* $p_{\alpha}(t,x-y)$ *associated to the operator* $\partial t+L^{\alpha}$ *verifies the following inequalities for* $t>0$:

$$C_1^{-1}\times\min\left(t^{-\frac{N}{2\alpha}},\frac{t}{|x-y|^{N+2\alpha}}\right)\le p_\alpha(t,x-y)\le C_1\times\min\left(t^{-\frac{N}{2\alpha}},\frac{t}{|x-y|^{N+2\alpha}}\right). \quad (2.1)$$

The proof of this proposition is given in [17]. Next, we give the Existence and uniqueness of the solution and comparison principle of (1.1).

In this section, we initially revisit the concept of a mild solution pertaining to the nonhomogeneous linear problem

$$\begin{cases} u_t + L^\alpha u = h(t) & \text{in } [0,T] \\ u(0)=u_0, \end{cases} \quad (2.2)$$

where $T>0, u_0\in X$, and $h\in\left(C([0,T]);X\right)$ are given. The mild solution of (3.5) is:

$$u(t)=T_t u_0+\int_0^t T_{t-s}h(s)\mathrm{d}s$$

for all $t\in[0,T]$ and $T_t u_0(x)=\int_{\mathbb{R}^N} p_\alpha(t,x-y)u_0(y)\mathrm{d}y$ (p_α is given in Proposition 2.1). One easily checks that $u\in C([0,T];X)$. Next, the existence and uniqueness theorem and proof of solution (1.1) are given:

Proposition 2.2. *Let* $G:[0,\infty)\times X\to X$, $G(t,u)=F(t,u)+K(t,u)$, $K(t,u)=J(u(x))=J*u$ *satisfying*

$$\begin{aligned} &F\in C^1([0,+\infty)\times X;X),\ K(t,u)\in C([0,+\infty)\times X;X) \\ &F(t,\cdot) \text{ and } K(t,\cdot) \text{ is Lipschitz in } X \text{ uniformly in } t\ge 0 \end{aligned} \quad (2.3)$$

and $u\in C\left([0,\infty);C_{u,b}(\mathbb{R}^N)\right)$. *Given any* $T>0$, *Our focus is on the nonlinear problem*

$$\begin{cases} u_t + L^\alpha u = G(t,u) & \text{in } [0,T] \\ u(0)=u_0, \end{cases} \quad (2.4)$$

Then, for a given $u_0\in C_{u,b}(\mathbb{R}^N)$, *there is a unique unique global mild solution* u *to* (*2.4*) *with* $u(0,\cdot)=u_0$.

Proof. We can obtain it by referring to reference [18].

We now claim the following comparison principle. We present a more general version of the comparison theorem. In theorem proving, we think of τu as a part of $f(x,u)$.

Lemma 2.3.(comparison principle) *Let* $\alpha\in(0,1), \tau\ge 0$, *The function* $f:\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}$ *is of class* $C^{0,\alpha}$ *Lipschitz-continuous with respect to u. If* $u,v\in C\left([0,\infty);C_{u,b}(\mathbb{R}^N)\right)$ *are such that* $\forall(t,x)\in[0,\infty)\times\mathbb{R}^N$, $u_t+L^\alpha u\le f(x,u)+\tau(J*u)$, and $v_t+L^\alpha v\ge f(x,v)+\tau(J*v)$, and satisfy $u(0,\cdot)\le v(0,\cdot)$ on $\mathbb{R}^N$, then $u\le v$ on $[0,\infty)\times\mathbb{R}^N$.

Proof. We can obtain it by referring to reference [19] Theorem 2.4.

Next, we will give a property about nonlocal dispersal operator $J*u$. We'll use that later on when we prove theorem 2.

Proposition 2.4 *Let* $\hat{k}$ *be positive constants, let* $J(\cdot)$ *be a* C^1 *nonnegative* *bounded function and satisfies (1.3). For all* $x \in \mathbb{R}^N$, $g(x) = \frac{1}{1+|x|^{N+2\alpha}}$. *Then there exists a positive constant C, which does not depend on x, such that, for all* $x \in \mathbb{R}^N$:

(*i*)

$$\left| J * g(x) - g(x) \right| \le C g(x). \tag{2.5}$$

(ii) for all $a > 0$,

$$\left| J * g(ax) - g(ax) \right| \le C a^{\hat{k}} g(ax) \tag{2.6}$$

Proof of (*i*). Let $\delta < \frac{1}{2}$ be a positive constant and *N* be an enough large positive constant. By the compactness argument, we only have to prove it for $|x| > 1$. We computer

$$\begin{aligned}
\left| \frac{J * g(x) - g(x)}{g(x)} \right| &= \left| \int_{\mathbb{R}^N} \left(\frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right) J(h) \mathrm{d}h \right| \\
&\le \int_{\mathbb{R}^N \setminus [B(-x,\delta|x|)]} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| J(h) \mathrm{d}h \\
&\quad + \int_{B(-x,\delta|x|) \cap |x| \ge M} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| J(h) \mathrm{d}h \\
&\quad + \int_{B(-x,\delta|x|) \cap |x| \le M} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| J(h) \mathrm{d}h \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let us begin by approximating I_1.

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^N \setminus [B(-x,\delta|x|)]} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| J(h) \mathrm{d}h \\
&\le \int_{\mathbb{R}^N \setminus [B(-x,\delta|x|)]} \left| \frac{C}{\delta^{N+2\alpha}} - 1 \right| J(h) \mathrm{d}h \\
&\le \frac{C_1 + 1}{\delta^{N+2\alpha}} = \frac{C_2}{\delta^{N+2\alpha}}
\end{aligned}$$

For I_2, we write:

$$\begin{aligned}
I_2 &= \int_{B(-x,\delta|x|) \cap |x| \ge M} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| \frac{J(h) \mathrm{d}h}{|h|^{N+2\alpha}} \\
&\le B \int_{B(-x,\delta|x|) \cap |x| \ge M} \left| \frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} - 1 \right| \frac{\mathrm{d}h}{|h|^{N+2\alpha}} \\
&\le B \int_{B(-x,\delta|x| \cap |x| \ge M} \frac{|x|^{N+2\alpha} - |x+h|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}} \frac{\mathrm{d}h}{|h|^{N+2\alpha}}
\end{aligned}$$

$$\leq B\int_{B(-x,\delta|x|)\cap|x|\geq M}\frac{|x|^{N+2\alpha}+|x+h|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}}\frac{\mathrm{d}h}{|h|^{N+2\alpha}}$$

$$\leq B\int_{B(-x,\delta|x|)\cap|x|\geq M}\frac{|x|^{N+2\alpha}+|\delta x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}}\frac{\mathrm{d}h}{|h|^{N+2\alpha}}$$

$$\leq C\int_{B(-x,\delta|x|)\cap|x|\geq M}\frac{1}{1+|x+h|^{N+2\alpha}}\frac{|x|^{N+2\alpha}}{|h|^{N+2\alpha}}\mathrm{d}h.$$

But we know that $h\in B(-x,\delta|x|)\cap|x|\geq M$, using that $\delta<\frac{1}{2}<|x|$, we deduce that

$$|x|(1-\delta)\leq|h|\leq(1+\delta)|x|\Rightarrow\left|\frac{x}{h}\right|\leq\left|\frac{1}{1-\delta}\right|.$$

Thus, we deduce

$$I_2\leq\frac{C}{(1-\delta)^{N+2\alpha}}\int_{B(-x,\delta|x|)\cap|x|\geq M}\frac{1}{1+|x+h|^{N+2\alpha}}\mathrm{d}h$$

$$\leq\frac{C}{(1-\delta)^{N+2\alpha}}\int_0^{\delta|x|}\frac{r^{d-1}}{1+r^{N+2\alpha}}\mathrm{d}r$$

$$\leq\frac{C}{(1-\delta)^{N+2\alpha}}\int_0^{\infty}\frac{r^{d-1}}{1+r^{N+2\alpha}}\mathrm{d}r.$$

To control I_3, we write I_3 in the following form:

$$I_3=C\left|\int_{B(-x,\delta|x|)\cap|x|\leq M}\left(\frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}}+\frac{1+|x|^{N+2\alpha}}{1+|x-h|^{N+2\alpha}}-2\right)J(h)\mathrm{d}h\right|.$$

Next, we define

$$f(x,h):=\frac{1+|x|^{N+2\alpha}}{1+|x+h|^{N+2\alpha}}.$$

Since for all $x\in\mathbb{R}^N$, the map that $(h\mapsto f(x,h))$ is $C^{1+2\alpha}$, we know that I_3 is well defined. we deduce that the map $(x\mapsto I_3(x))$ is continuous and so we conclude that I_3 is bounded independently of *x*. Hence, $I_3\leq C$. Combining the above inequalities, we obtain that there exists a constant C such that for all $x\in\mathbb{R}^N$,

$$|J*g(x)-g(x)|\leq Cg(x).$$

Proof of (*ii*). For convenience, we write $(J*u(x)-u(x))=\hat{J}(u)(x)$. Using the above inequality, we can conclude with a change of variable $h=ay$ (where $a>0$):

$$\left|\hat{J}(g(ax))\right|=|J*g(ax)-g(ax)|$$

$$=\left|\int_{\mathbb{R}^N}g(ax)-g(ax+ay)J(y)\mathrm{d}y\right|$$

$$=\left|a^{d-N}\int_{\mathbb{R}^N}\{g(ax)-g(ax+h)\}J(h)\mathrm{d}h\right|$$

$$=a^{\hat{k}}\left|\hat{J}(g)(ax)\right|$$

$$\leq Ca^{\hat{k}}g(ax).$$

where $\hat{k} = d - N > 0$. Finally, we obtain

$$\left|\hat{J}\left(g\left(ax\right)\right)\right| \le Ca^{\hat{k}} g\left(ax\right).$$

Now we use this proposition 2.1 and proposition 2.4 to demonstrate that beginning with a positive compactly supported initial date leads to a solution with algebraic tails.

Propossition 2.5. *There exist two positive constants* c_m *and* c_M *depending on* u_0, N, α *, and* μ *such that for all* $x \in \mathbb{R}^N$ *, we have*

$$\frac{c_m}{1+|x|^{N+2\alpha}} \le u\left(1,x\right) \le \frac{c_M}{1+|x|^{N+2\alpha}}. \tag{2.7}$$

Proof. Defining $M := \max\left(\max u_0, \max|\mu|, 1\right)$, Thanks to the comparison principle, we have the following inequalities, for all $(t,x) \in (0,+\infty)\times\mathbb{R}^N$

$$0 \le u\left(t,x\right) \le M.$$

We begin with the proof that $u\left(1,x\right) \le \dfrac{c_M}{1+|x|^{N+2\alpha}}$.

Let $\bar{u}$ be the solution of:

$$\begin{cases} \partial_t \bar{u}\left(t,x\right) + L^\alpha\left(\bar{u}\right)\left(t,x\right) = k\bar{u} & \text{for all } (t,x) \in (0,+\infty)\times\mathbb{R}^N, \\ \bar{u}\left(x,0\right) = u_0\left(x\right) & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{2.8}$$

where $k > 0$. Thanks to Proposition 1, we can solve (2.8) and find

$$\bar{u}\left(t,x\right) = \mathrm{e}^{kt}\int_{\mathbb{R}^N} p_\alpha\left(t,x-y\right)u_0\left(y\right)\mathrm{d}y,$$

Thus for any $0 < t < 2$, we obtain

$$\bar{u}\left(t,x\right) \le \int_{\operatorname{supp}(u_0)} C \times u_0\left(y\right)\min\left(t^{-\frac{N}{2\alpha}}, \frac{t}{|x-y|^{N+2\alpha}}\right)\mathrm{d}y$$

A simple substitution is made for equation (2.1), for any $0 < t < 2$, we can obtain

$$\bar{u}\left(t,x\right) \le \int_{\operatorname{supp}(u_0)} C \times u_0\left(y\right)\left(\frac{t}{t^{\frac{N}{2\alpha}+1} + |x-y|^{N+2\alpha}}\right)\mathrm{d}y.$$

Thanks to the dominated convergence theorem, for any $0 < t < 2$, we have:

$$\frac{1+|x|^{N+2\alpha}}{t}\int_{\operatorname{supp}(u_0)} C \times u_0\left(y\right)\left(\frac{t}{t^{\frac{N}{2\alpha}+1} + |x-y|^{N+2\alpha}}\right)\mathrm{d}y$$
$$\xrightarrow[|x|\to\infty]{} \int_{\operatorname{supp}(u_0)} C \times u_0\left(y\right)\mathrm{d}y.$$

Therefore we conclude by a compactness argument that for any $x \in \mathbb{R}^N$, $0 < t < 2$, then there exist a constant c_2 depending on u_0, N, α such that:

$$\overline{u}(t,x) \le \frac{c_2 t e^{kt}}{1+|x|^{N+2\alpha}}, \tag{2.9}$$

Using the same approach, we can obtain that for any $x \in \mathbb{R}^N, 0<t<2$, then there exist a constant c_1, c_2 depending on u_0, d, α such that:

$$\frac{c_1 t e^{kt}}{1+|x|^{N+2\alpha}} \le \overline{u}(t,x) \le \frac{c_2 t e^{kt}}{1+|x|^{N+2\alpha}}, \tag{2.10}$$

Next, we will proceed to prove for all $x \in \mathbb{R}^N, 0<t<2$, such that

$$\tau(J*\overline{u}-\overline{u})+(\mu-\overline{u})\overline{u} \le k\overline{u} \tag{2.11}$$

From (2.5) of Proposition 2.4 and (2.10), we can obtain that

$$\begin{aligned}
\tau(J*\overline{u}-\overline{u})+(\mu-\overline{u})\overline{u} &\le \tau(J*\overline{u})+M\overline{u} \\
&\le \tau e^{kt} c_2 t(1+C)g(x)+M\overline{u} \\
&= \tau e^{kt}\frac{c_2}{c_1}c_1 t(1+C)g(x)+M\overline{u} \\
&\le \tau\frac{c_2}{c_1}(C+1)\overline{u}+M\overline{u}=k\overline{u}
\end{aligned}$$

takeing $k=\left(\tau\frac{c_2}{c_1}(C+1)\right)+M$.

Thanks to the comparison principle and (2.11), we have the following inequalities, for all $x \in \mathbb{R}^N$

$$u(x,1) \le \overline{u}(1,x) \le \frac{c_M}{1+|x|^{N+2\alpha}}, \tag{2.12}$$

where the last c_M in a new constant depending only on u_0 and α, N, C, μ.

In the same way, we can get

$$\frac{c_m}{1+|x|^{N+2\alpha}} \le u(1,x),$$

where the last c_m in a new constant depending only on u_0 and α, N, C, μ. By combining the above inequality and (2.12) together, we finally obtain

$$\frac{c_m}{1+|x|^{N+2\alpha}} \le u(1,x) \le \frac{c_M}{1+|x|^{N+2\alpha}}. \tag{2.13}$$

We next provide an important lemma that will be useful all along the article. Since, the same kind of result can be found in the appendix A of [8] and [9], we do not provide the proof of this lemma. Note that here, the lemma is stated with less regularity on χ such than in [9]. Nevertheless, there is no difficulty to adapt the proof. Before introducing the lemma. We first introduce the computation of L^α of a product of functions:

$$L^\alpha(fg)(t,x)=f(t,x)L^\alpha g(t,x)+g(t,x)L^\alpha f(t,x)-\tilde{K}(f,g)(t,x), \tag{2.14}$$

with

$$\tilde{K}(f,g)(t,x) := C'PV\int_{\mathbb{R}^N} \frac{\left(f(t,x)-f(t,x+h)\right)\left(g(t,x)-g(t,x+h)\right)}{|h|^{N+2\alpha}}\beta\left(x,\frac{h}{|h|}\right)\mathrm{d}y.$$

The calculation of L^{α} leads to the calculation of $J*u-u$, we denote $\hat{J}(u)=J*u-u$, then

$$-\hat{J}(fg)(t,x) = f(t,x)J\left(g(t,x)\right)+g(t,x)J\left(f(t,x)\right)-\bar{K}(f,g)(t,x) \quad (2.15)$$

with

$$\bar{K}(f,g)(t,x) := C''\int_{\mathbb{R}^N}\left(f(t,x)-f(t,x+h)\right)\left(g(t,x)-g(t,x+h)\right)J(h)\mathrm{d}h. \quad (2.16)$$

Lemma 2.6. *Let* γ *and d be two positive constants in* $(0,\alpha)$ *such that* $2\alpha-\gamma<1$ *and* $d-N>\gamma$. *Let* $\chi\in C^{\alpha}\left(\mathbb{R}^N\right)$ *be a periodic positive function and* $g(x)=\frac{1}{1+|x|^{N+2\alpha}}$. *Then there exists a positive constant C, such that we have for all* $x\in\mathbb{R}^N$:

1. for all $a>0$,

$$\left|L^{\alpha}g(ax)\right|\le a^{2\alpha}Cg(ax),$$

2. for all $a\in(0,1]$,

$$\left|\tilde{K}\left(g(a\cdot),\chi\right)(x)\right|\le\frac{Ca^{2\alpha-\gamma}}{1+\left(a|x|\right)^{N+2\alpha}}=Ca^{2\alpha-\gamma}g\left(a|x|\right),$$

3. for all $a\in(0,1]$,

$$\left|\bar{K}\left(g(a\cdot),\chi\right)(x)\right|\le Ca^{\bar{m}}g\left(a|x|\right).$$

where $\bar{m}=d-N-\gamma$.

Proof. For the proof of (i) (ii)(iii), we refer to Appendix A of reference [9].

3. Existence and Uniqueness of the Stationary State u^+

The conditions for existence, uniqueness, and the long-term behavior of solution for (1.1) are determined by the principal eigenvalue, denoted as λ_1, of the operator *B*, which is defined by

$$B\phi = L^{\alpha}\phi-\mu\phi-\tau\left(J*\phi-\phi\right)$$

with periodicity conditions. Firstly, we show that the operator B has a positive periodic first eigenfunction. We mainly prove it through Krein-Rutman Theorem and Lax-Milgram Theorem.

Proposition 3.1. *The operator B has a unique positive periodic eigenfunction with eigenvalue* λ_1, *that is*

$$\begin{cases} B\phi_1(x)=\lambda_1\phi_1(x), & x\in\mathbb{R}^N, \\ \phi_1 \text{ is } l\text{-periodic}, \ \phi_1>0, \ \|\phi_1\|_{\infty}=1. \end{cases} \quad (3.1)$$

The eigenvalue λ_1 is unique both algebraically and geometrically, and it represents the lowest point in the spectrum for *B*.

Proof. Before we prove it, let's do a sign change. Let

$$PV\frac{\beta\left(x,\frac{h}{|h|}\right)}{h^{N+2\alpha}}=K(x,h) \tag{3.2}$$

Obviously, K is positive, l-periodic and C^2 in x, and symmetic in h and is singular at $h=0$ and there exists $C_K>0$ for all $x,h\in\mathbb{R}^N$

$$C_K^{-1}\le K(x,h)|h|^{N+2\alpha}\le C_K. \quad (3.3)$$

To tackle the issue of periodicity, we have formulated this eigenfunction on the torus C_l. A smooth function u, which has a period l in $\mathbb{R}^N$, can be represented in an equivalent manner as a smooth function on C_l. Moreover,

$$\begin{aligned} L^\alpha u &= \int_{\mathbb{R}^N}\left(u(x)-u(x+h)\right)K(x,h)\mathrm{d}h \\ &= \int_{C_l}\left(u(t,x)-u(t,x+h)\right)\tilde{K}(x,h)\mathrm{d}h \end{aligned} \tag{3.4}$$

where

$$\tilde{K}(x,h)=\sum_{k=0}^{\infty}K(x,h+kl)$$

please note that, considering (3.3), the sum mentioned above converges for all $x\in C_l$ and $h\in C_l\setminus 0$.

It also follows from (3.3) and the definition of K, for a constant $C>0$, and all $x,h\in\mathbb{R}^N$

$\tilde{K}(x,h)$ is bounded from below, symmetric in h and $C^{-1}\le\tilde{K}(x,h)|h|^{N+2\alpha}\le C.$

For $\forall u,v\in H^s(C_l)=\{u\,|\,u\in H^s(C_l),\,u\equiv 0 \text{ on } \mathbb{R}^N\setminus C_l\}$, we have

$$\begin{aligned}\langle Bu,v\rangle &= \iint_{C_l}\left(u(x)-u(x+h)\right)\left(v(x)-v(x+h)\right)\tilde{K}(x,h)\mathrm{d}x\mathrm{d}h \\ &\quad -\tau\int_{C_l}(J*u)(x)v(x)\mathrm{d}x+\tau\int_{C_l}u(x)v(x)\mathrm{d}x-\int_{C_l}\mu u(x)v(x)\mathrm{d}x\end{aligned}$$

and we define

$$A[u,v]=\langle Bu,v\rangle$$

where

$$A:H^s(C_l)\times H^s(C_l)\to\mathbb{R} \tag{3.5}$$

is a bilinear mapping. For $u\in H^s(C_l)$, and we define

$$\|u\|_{\dot{H}^s(C_l)}=\left(\iint_{C_l}\left(u(x)-u(x+h)\right)^2\tilde{K}(x,h)\mathrm{d}x\mathrm{d}h\right)^{\frac{1}{2}}.$$

where $\|u\|_{\dot{H}^s(C_l)}$ and $\|u\|_{H^s(C_l)}$ denote, respectively, the homogeneous and inhomogeneous Sobolev norm, this is,

$$\|u\|_{H^s(C_l)}=\|u\|_{\dot{H}^s(C_l)}+\|u\|_{L^2(C_l)} \text{ and } \|u\|_{L^2(C_l)}=\left(\int_{C_l}|u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

For all $u,v\in H^s(C_l)$, for any given $\mu'>0$ we find

$$
\begin{aligned}
\left|\left\langle (B+\mu)u,v\right\rangle\right| &\le \iint_{C_l}\left(u(x)-u(x+h)\right)\left(v(x)-v(x+h)\right)\tilde{K}(x,h)\,\mathrm{d}x\mathrm{d}h \\
&\quad +\tau\int_{C_l}(J*u)(x)v(x)\,\mathrm{d}x+(\tau+\mu)\int_{C_l}u(x)v(x)\,\mathrm{d}x \\
&\quad +\int_{C_l}\mu(x)u(x)v(x)\,\mathrm{d}x \\
&\le \|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}+\left(2\tau+\|\mu\|_{L^\infty}+\mu'\right)\|u\|_{L^2}\|v\|_{L^2} \\
&\le \bar{k}\|u\|_{H^s}\|v\|_{H^s}
\end{aligned}
$$

where $\bar{k}>0$ is a constant.

For $u\in H^s(C_l)$, we show that u satisfies a Garding-type inequality, we find

$$
\begin{aligned}
\left\langle (B+\mu)u,u\right\rangle &\ge \iint_{C_l}\left(u(x)-u(x+h)\right)^2\tilde{K}(x,h)\,\mathrm{d}x\mathrm{d}h-\tau\int_{C_l}(J*u)(x)u(x)\,\mathrm{d}x \\
&\quad +\tau\int_{C_l}u(x)^2\,\mathrm{d}x-\int_{C_l}\mu(x)u(x)^2\,\mathrm{d}x+\mu\int_{C_l}u(x)^2\,\mathrm{d}x \\
&\ge \|u\|_{\dot{H}^s}^2-\tau\|u\|_{L^2}^2+\tau\|u\|_{L^2}^2-\|\mu\|_{L^\infty}\|u\|_{L^2}^2+\mu'\|u\|_{L^2}^2 \\
&\ge \|u\|_{\dot{H}^s}^2+\left(\mu'-\|\mu\|_{L^\infty}\right)\|u\|_{L^2}^2
\end{aligned}
$$

Subsequently, for a sufficiently large constant $\mu'>\|\mu\|_{L^\infty}$, it is evident that the operator $B+\mu'$ is bijective. Moreover, the operator $B+\mu'$ is positive compact operator, fulfilling the conditions of the Lax-Milgram theorem with in $H^s(C_l)$. Consequently, due to the Sobolev embedding theorem, it possesses a compact inverse that is defined everywhere

$$
\tilde{A}=(B+\mu')^{-1}:H^s(\mathbb{R}^N)\to \mathrm{Dom}(L^\alpha). \tag{3.6}
$$

According to the Krein-Rutman Theorem, there exists a positive eigenfunction ϕ of $\tilde{A}$ with eigenvalue $r(\tilde{A})$. Hence,

$$
L^\alpha\phi-\mu\phi-\tau(J*\phi-\phi)=\left(\frac{1}{r(\tilde{A})}-\mu'\right)\phi.
$$

This corresponds exactly to Equation (1.4) with $\lambda_1=r(\tilde{A})^{-1}-\mu'$. Evidently, λ_1 is the smallest eigenvalue in the spectrum of B.

Next, to prove Theorem 1.

Proof of Theorem 1. Let's first define $(\mu(x)-u(x))(u(x))=f(x,u)$ for any $x\in\mathbb{R}^N$. We will demonstrate the existence of a positive and periodic solution in the stationary equation. It is well known that stationary solution are provided by

$$
L^\alpha u=f(x,u)+\tau(J*u-u),\quad x\in\mathbb{R}^N. \tag{3.9}
$$

Let ϕ represent the only positive solution of

$$
\begin{cases}
B\phi(x)=\lambda_1\phi(x), & x\in\mathbb{R}^N,\\
\phi \text{ is } l\text{-periodic}, & \phi>0,\ \|\phi\|_\infty=1
\end{cases}
$$

with $\lambda_1<0$. For $\varepsilon>0$ sufficiently small, it follows that

$$
f(x,\varepsilon\phi)\ge\varepsilon\mu\phi+\frac{\lambda_1}{2}\varepsilon\phi\ \text{ in } \mathbb{R}^N. \tag{3.10}
$$

Hence, it can be concluded that

$$\varepsilon L^{\alpha}\phi-\tau\varepsilon\left(J*\phi-\phi\right)-f\left(x,\varepsilon\phi\right)\leq\frac{\lambda_1}{2}\varepsilon\phi\leq 0\ \text{ in } \mathbb{R}^N$$

Futhermore, $\varepsilon\phi$ serves as a subsolution of equation (3.9) with periodicity conditions. Additionally, if $M\geq\mu(x)\geq 0$ for any $x\in\mathbb{R}^N$, then the constant M acts as an upper solution of (3.9) with periodicity conditions, and for ε sufficiently small, $\varepsilon\phi\leq M$ in $\mathbb{R}^N$. Consequently, by employing an iterative method, it is established that there exists a periodic classical solution u to (3.9) fulfilling the inequality $\varepsilon\phi\leq u\leq M$ in $\mathbb{R}^N$. Define the order interval

$$\left[\underline{u},\overline{u}\right]=\left\{u\in C\left(\mathbb{R}^N\right):\varepsilon\phi=\underline{u}\left(x\right)\leq u\left(x\right)\leq\overline{u}\left(x\right)=M,\forall x\in\mathbb{R}^N\right\}.$$

We define $g(x,u)=\tau(J*u-u)+f(x,u)$. For any given $v\in\left[\underline{u},\overline{u}\right]$, We may assume that $M'\geq\left(\tau+\text{lip}_u\left(f\right)\right)>0$ is sufficiently large, using the variational method, we see that boundary value problem of linear equation

$$L^{\alpha}u+M'u=g\left(x,v\right)+M'v,\ \ x\in\mathbb{R}^N. \tag{3.11}$$

admits a unique solution $u\in C^{1+\alpha}\left(\mathbb{R}^N\right)$ which is denoted by $u=T(v)$.

We prove that T is monotone in the order interval $\left[\underline{u},\overline{u}\right]$, *i.e.*, $T(v)\leq T(w)$ if $v,w\in\left[\underline{u},\overline{u}\right]$ and $v\leq w$. We assume that $u\geq v$, Since $f(x,u)$ satisfies Lipschitz continuity with respect to u,v, we find

$$g\left(x,u\right)-g\left(x,v\right)\geq -M'\left(u-v\right).$$

In fact, writing $z=T(w)$, then $h=z-u$ satisfies

$$L^{\alpha}h+M'h=g\left(x,w\right)-g\left(x,v\right)+M'\left(w-v\right)\geq 0,\ \ x\in\mathbb{R}^N \tag{3.12}$$

The maximum principle [20] yields $h\geq 0$, *i.e.*, $T(v)=u\leq z=T(w)$. Defined

$$\underline{u}_1=T\left(\underline{u}\right),\ \ \underline{u}_{i+1}=T\left(\underline{u}_i\right),\ \ \overline{u}_1=T\left(\overline{u}\right),\ \ \overline{u}_{i+1}=T\left(\overline{u}_i\right),$$

By the monotonicity of T, $\underline{u}_1\leq\overline{u}_1$. We claim that the sequences $\left\{\underline{u}_i\right\}$ and $\left\{\overline{u}_i\right\}$ satisfy

$$\underline{u}\leq\underline{u}_i\leq\underline{u}_{i+1}\leq\overline{u}_{i+1}\leq\overline{u}_i\leq\overline{u},\ \ \forall i\geq 1. \tag{3.13}$$

Let $v=\overline{u}-\overline{u}_1$. Since $\overline{u}$ satisfies

$$L^{\alpha}\overline{u}+M'\overline{u}\geq g\left(x,\overline{u}\right)+M'\overline{u},\ \ x\in\mathbb{R}^N. \tag{3.14}$$

we see that v satisfies

$$L^{\alpha}v+M'v\geq g\left(x,\overline{u}\right)-f\left(x,\overline{u}\right)=0,\ \ x\in\mathbb{R}^N. \tag{3.15}$$

The Maximum principle leads to $v\geq 0$, *i.e.*, $\overline{u}_1\leq\overline{u}$. Similary, $\underline{u}\leq\underline{u}_1$. Thus, $\underline{u}\leq\underline{u}_1\leq\overline{u}_1\leq\overline{u}$. Applying the monotonicity of T we can deduce (3.13) inductively.

As $\underline{u}_i,\overline{u}_i\in\left[\underline{u},\overline{u}\right]$, We have $\left\|f\left(\cdot,\underline{u}_i(\cdot)\right),f\left(\cdot,\overline{u}_i(\cdot)\right)\right\|_{\infty}\leq C$. Take $q>N$, then the L^q theory asserts $\left\|\underline{u}_i,\overline{u}_i\right\|_{W_p^2\left(\mathbb{R}^N\right)}\leq C_1$. This leads to $g\left(\cdot,\underline{u}_i(\cdot)\right),g\left(\cdot,\overline{u}_i(\cdot)\right)\in C^{\alpha}\left(\mathbb{R}^N\right)$, and $\left\|g\left(\cdot,\underline{u}_i(\cdot)\right),g\left(\cdot,\overline{u}_i(\cdot)\right)\right\|_{\alpha}\leq C_2$. Therefore

$$\left|\underline{u}_i,\overline{u}_i\right|\leq C_3.$$

by the Schauder theory. In the above, positive constants C,C_1,C_2 and C_3 are all independent of *i*. Noting that $C^{1+\alpha}(\mathbb{R}^N) \hookrightarrow C^2(\mathbb{R}^N)$ is compact, there are subsequences of $\{\underline{u}_i\}$ and $\{\overline{u}_i\}$ converging to $\tilde{u}$ and $\hat{u}$ in $C(\mathbb{R}^N)$, respectively. As these sequences are monotone and bounded, by the uniqueness of limits, $\underline{u}_i \to \tilde{u}$ and $\overline{u}_i \to \hat{u}$ in $C(\mathbb{R}^N)$. Taking $i \to \infty$ in $\underline{u}_{i+1} = T(\underline{u}_i)$ and $\overline{u}_{i+1} = T(\overline{u}_i)$, we see that $\tilde{u}$ and $\hat{u}$ are solutions of (3.9).

Next, we prove the uniqueness of the stationary solution.

We first prove that for all given $M' \ge \tau + \mathrm{Lip}_u(f)$, (3.9) has a unique positive solution in $[\varepsilon\phi, M']$. It is easy to know (3.9) that there are maximal solution u' and minimal solution v' in $[\varepsilon\phi, M']$ and $v', u' \in [\varepsilon\phi, M']$. We know that $\forall u(x) \in [\varepsilon\phi, M']$, we have $u(x) \le u'(x)$ for $x \in \mathbb{R}^N$. According to symmetry and (3.9), we can obtain

$$\begin{aligned}\int_{\mathbb{R}^N} u(x) f(x,u')\mathrm{d}x &= \int_{\mathbb{R}^N} L^\alpha u'(x) u(x)\mathrm{d}x - \tau\int_{\mathbb{R}^N} (J*u'-u')(x)u(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} L^\alpha u(x) u'(x)\mathrm{d}x - \tau\int_{\mathbb{R}^N} (J*u-u)(x)u'(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} u'(x) f(x,u)\mathrm{d}x\end{aligned}$$

hence

$$\int_{\mathbb{R}^N} u(x)u'(x)\left(\frac{f(x,u')}{u'} - \frac{f(x,u)}{u}\right)\mathrm{d}x = 0. \tag{3.16}$$

because $u \le u'$ and $\dfrac{f(x,u)}{u}$ is monotonically decreasing about *u*, we can obtain that $\dfrac{f(x,u')}{u'} - \dfrac{f(x,u)}{u} \le 0$. For formula (3.16) to hold, we have $u(x) \equiv u'(x)$ for all $x \in \mathbb{R}^N$. By the arbitrariness of $u(x)$, we have $u'(x) \equiv v'(x)$ for all $x \in \mathbb{R}^N$.

Next, Let us define that $u_1(x), u_2(x)$ are any two bounded positive solutions of (3.9), then there exists $\breve{M} > 0$ such that $\max\{u_1(x), u_2(x)\} \le \breve{M}$. We have $u_1 \le \breve{M}$ and we define the same mapping T such that

$$u_1 = T(u_1) \le T(\breve{M}) = v_1$$

$$u_1 \le T(v_1) = v_2$$

proceed in turn, we have

$$u_1 \le v_k,\ k = 1,2,3,\cdots$$

as these sequences are monotone and bounded, by the uniqueness of limits $k \to \infty$, $v_k \to \overline{u}$. Here $\overline{u}$ is the only positive solution of (3.9) in $[\varepsilon\phi, \breve{M}]$. Hence, by (3.16) we can obtain that $u_1 \equiv \overline{u}$ for $x \in \mathbb{R}^N$. In the same way, we can get that $u_2 \equiv \overline{u}$ for $x \in \mathbb{R}^N$. Additionally, if *u* is a positive solution to equation (3.9), then the function defined by $x \to u(x_1, x_2, \cdots, x_i + l_i, x_N)$ for each $1 \le i \le N$, is also a positive solution. Consequently, *u* exhibits periodicity. This establishes of Theorem 1.

4. The Proof of Theorem 2

In this section, we will provide the proof of Theorem 2. Let us redefine (1.1) in accordance with the rescaling specified by (1.6).

Notation We rescale the solution of (1.1) as follows:

$$u_\varepsilon(t,x) = u\left(\frac{t}{\varepsilon}, |x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}\right).$$

Next, the equation becomes

$$\begin{aligned} &\varepsilon\partial_t u_\varepsilon(t,x) + L_\varepsilon^\alpha u_\varepsilon(t,x) \\ &= u_\varepsilon(t,x)\left(\mu_\varepsilon(t,x) - u_\varepsilon(t,x)\right) + \tau\left(\int_{\mathbb{R}^N} J(y) u_\varepsilon(t,x-y)\,\mathrm{d}y - u_\varepsilon(t,x)\right), \end{aligned} \tag{4.1}$$

where $L_\varepsilon^\alpha(u_\varepsilon)(t,x) = L^\alpha(u)\left(\frac{t}{\varepsilon}, |x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}\right)$. In the same way, we define

$$\tilde{K}_\varepsilon(u_\varepsilon, \chi_\varepsilon)(t,x) := \tilde{K}(u,\chi)\left(|x|^{\frac{1}{\varepsilon}-1} x, \frac{t}{\varepsilon}\right).$$

Moreover, according to the inequalities (2.7), we can consider $u(x,1)$ as our initial data instead of $u(x,0)$. So we can replace the $u_0(x)$ by:

$$\frac{c_m}{1+|x|^{N+2\alpha}} \le u_0(x) \le \frac{c_M}{1+|x|^{N+2\alpha}} \Rightarrow \frac{c_m}{1+|x|^{\frac{N+2\alpha}{\varepsilon}}} \le u_{0,\varepsilon}(x) \le \frac{c_M}{1+|x|^{\frac{N+2\alpha}{\varepsilon}}}.$$

Next we are going to provide sub and super-solutions to (1.1).

Theorem 3. *We assume (H1) and we choose positive constants* $C_m < \frac{|\lambda_1|}{\max \phi_1}$ *and* $C_M > \frac{|\lambda_1|}{\min \phi_1}$ *and* δ *such that*

$$0 < \delta \le \min\left(\sqrt{C_M \min \phi_1 - |\lambda_1|}, \sqrt{|\lambda_1| - C_m \max \phi_1}\right).$$

Then there exists a positive constant $\varepsilon_0 < \delta$ *such that for all* $\varepsilon \in]0, \varepsilon_0[$ *we have:*

(*i*) $f_\varepsilon^M(t,x) = \phi_{1,\varepsilon}(x) \times \dfrac{C_M}{1+\mathrm{e}^{-\frac{t}{\varepsilon}\left(|\lambda_1|+\varepsilon^2\right)-\frac{\delta}{\varepsilon}|x|}\dfrac{N+2\alpha}{\varepsilon}}$ *is a super-solution of (4.1),*

(*ii*) $f_\varepsilon^m(t,x) = \phi_{1,\varepsilon}(x) \times \dfrac{C_m \mathrm{e}^{-\frac{\delta}{\varepsilon}}}{1+\mathrm{e}^{-\frac{t}{\varepsilon}\left(|\lambda_1|-\varepsilon^2\right)-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}}$ *is a sub-solution of (4.1).*

(*iii*) *Moreover, if we assume (H1) and* $C_m < \frac{c_m}{\max|\phi_1|}$ *and* $C_M > \frac{c_M}{\min|\phi_1|}$ *where* c_m *and* c_M *are given by (2.13) then for all* $(t,x) \in \mathbb{R}^d \times [0,+\infty[$,

$$\phi_{1,\varepsilon}(x) \times \frac{C_m \mathrm{e}^{\frac{-\delta}{\varepsilon}-\varepsilon t}}{1+\mathrm{e}^{-\frac{|\lambda_1|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\epsilon}}} \le u_\varepsilon(t,x) \le \phi_{1,\varepsilon}(x) \times \frac{C_M \mathrm{e}^{\varepsilon t}}{1+\mathrm{e}^{-\frac{|\lambda_1|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}}.$$

Proof. Since the proofs of (i) and (ii) follow from similar arguments, we will only provide the proof of (i) and (iii). Proof of (i). We define:

$$\psi(t,x) := \frac{C_M}{1+\mathrm{e}^{-t\left(|\lambda_1|+\varepsilon^2\right)-\frac{\delta}{\varepsilon}}|x|^{N+2\alpha}}. \tag{4.2}$$

Then, noticing that ϕ_1 is independent of t, we first bound $\partial_t\psi_\varepsilon$ from below,

$$\begin{aligned}
\partial_t\psi_\varepsilon(t,x) &= \frac{C_M\frac{|\lambda_1|+\varepsilon^2}{\varepsilon}\mathrm{e}^{-t\frac{|\lambda_1|+\varepsilon^2}{\varepsilon}-\frac{\delta}{\varepsilon}|x|^{\frac{N+2\alpha}{\varepsilon}}}}{\left(1+\mathrm{e}^{-t\frac{|\lambda_1|+\varepsilon^2}{\varepsilon}-\frac{\delta}{\varepsilon}}|x|\frac{N+2\alpha}{\varepsilon}\right)^2} \\
&= \frac{\psi_\varepsilon(t,x)}{\varepsilon}\left[\left(|\lambda_1|+\varepsilon^2\right)\frac{\mathrm{e}^{-t\frac{|\lambda_1|+\varepsilon^2}{\varepsilon}-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}}{1+\mathrm{e}^{-t\frac{|\lambda_1|+\varepsilon^2}{\varepsilon}-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}}\right] \\
&\geq \frac{\psi_\varepsilon(t,x)}{\varepsilon}\left[|\lambda_1|+\varepsilon^2-\psi_\varepsilon(t,x)\phi_{1,\varepsilon}(x)\right]
\end{aligned}$$

The last inequality is obtained from the definition of C_M and ε. Actually, for such C_M and ε, we have, for any positive non-null constant A, the following relation:

$$\frac{A\left(|\lambda_1|+\varepsilon^2\right)}{1+A} \geq |\lambda_1|+\varepsilon^2-\frac{C_M\min\phi_1}{1+A} \tag{4.3}$$

because,

$$\begin{aligned}
|\lambda_1|+\varepsilon^2-\frac{C_M\min\phi_1}{1+A} &= \frac{(1+A)\left(|\lambda_1|+\varepsilon^2\right)-C_M\min\phi_1}{1+A} \\
&= \frac{A\left(|\lambda_1|+\varepsilon^2\right)-\left(C_M\min\phi_1-|\lambda_1|-\varepsilon^2\right)}{1+A} \\
&\leq \frac{A\left(|\lambda_1|+\varepsilon^2\right)}{1+A}
\end{aligned}$$

We also compute $L_\varepsilon^\alpha\left(f_\varepsilon^M\right)(t,x)$ as a fractional Laplacian of a product of functions,

$$L_\varepsilon^\alpha\left(f_\varepsilon^M\right)(t,x) = \phi_{1,\varepsilon}(x)L_\varepsilon^\alpha\psi_\varepsilon(t,x)+\psi_\varepsilon(t,x)L_\varepsilon^\alpha\phi_{1,\varepsilon}(x)-\tilde{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x) \tag{4.4}$$

with $\tilde{K}$ given in section 2. Similarly, we denote $u-J*u=\hat{J}(u)$, then

$$\hat{J}\left(f_\varepsilon^M\right)(t,x) = \phi_{1,\varepsilon}(x)\hat{J}\psi_\varepsilon(t,x)+\psi_\varepsilon(t,x)\hat{J}\phi_{1,\varepsilon}(x)-\bar{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x). \tag{4.5}$$

with $\bar{K}$ given in section 2. Replacing this in Equation (4.1) and using the three previous results (4.3), (4.4) and (4.5), we find:

$$\begin{aligned}
&\varepsilon\partial_t f_\varepsilon^M(t,x)+L_\varepsilon^\alpha f_\varepsilon^M(t,x)-f_\varepsilon^M(t,x)\left[\mu_\varepsilon(x)-f_\varepsilon^M(t,x)\right]+\tau\hat{J}\left(f_\varepsilon^M\right)(t,x) \\
&\geq f_\varepsilon^M(t,x)\left(|\lambda_1|+\varepsilon^2-f_\varepsilon^M(t,x)\right)+\phi_{1,\varepsilon}(x)L_\varepsilon^\alpha\psi_\varepsilon(t,x)+\psi_\varepsilon(t,x)L_\varepsilon^\alpha\phi_{1,\varepsilon}(x) \\
&\quad -\tilde{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x)-\mu_\varepsilon(x)f_\varepsilon^M(t,x)+f_\varepsilon^M(t,x)^2+\tau\phi_{1,\varepsilon}(x)\hat{J}\psi_\varepsilon(t,x) \\
&\quad +\tau\psi_\varepsilon(t,x)\hat{J}\phi_{1,\varepsilon}(x)-\tau\bar{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x) \\
&= \varepsilon^2 f_\varepsilon^M(t,x)+\phi_{1,\varepsilon}(x)L_\varepsilon^\alpha\psi_\varepsilon(t,x)-\tilde{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x) \\
&\quad +\tau\phi_{1,\varepsilon}(x)\hat{J}\psi_\varepsilon(t,x)-\tau\bar{K}_\varepsilon\left(\psi_\varepsilon,\phi_{1,\varepsilon}\right)(t,x),
\end{aligned}$$

where we have used (1.4) and (H1) for the last equality. In order to control $L_\varepsilon^\alpha \psi_\varepsilon(t,x)$ and $\tilde{K}_\varepsilon(\psi_\varepsilon,\phi_{1,\varepsilon})(t,x)$, we will utilize Lemma 2.6.

For $L_\varepsilon^\alpha \psi_\varepsilon(t,x)$ noticing that $\psi_\varepsilon(t,x) = C_M g\left(\mathrm{e}^{\frac{-t(|\lambda_1|+\varepsilon^2)-\delta}{\varepsilon(N+2\alpha)}} |x|^{\frac{1}{\varepsilon}-1} x\right)$, and thanks to the point (i) of Lemma 2.6 we obtain:

$$-C\mathrm{e}^{-2\alpha\frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon}} \psi_\varepsilon(t,x) \le L_\varepsilon^\alpha \psi_\varepsilon(t,x).$$

But, comparing the growths, there exists $\varepsilon_1' > 0$ such that for $\varepsilon < \varepsilon_1'$ and for all $t \ge 0$:

$$C_M \times C\mathrm{e}^{-2\alpha\frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon(N+2\alpha)}} - \frac{\varepsilon^2}{5} \le 0,$$

hence:

$$-\frac{\varepsilon^2}{5}\psi_\varepsilon(t,x) \le L_\varepsilon^\alpha \psi_\varepsilon(t,x).$$

By the same way, comparing the growths, there exists $\varepsilon_2' > 0$ such that for $\varepsilon < \varepsilon_2'$ and for all $t \ge 0$:

$$C_M \times C\mathrm{e}^{-\hat{k}\frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon(N+2\alpha)}} - \frac{\varepsilon^2}{5} \le 0.$$

hence:

$$-\frac{\varepsilon^2}{5}\psi_\varepsilon(t,x) \le \hat{J}_\varepsilon \psi_\varepsilon(t,x).$$

Now we deal with $\tilde{K}_\varepsilon(\psi_\varepsilon,\phi_{1,\varepsilon})(t,x)$ in a similar fashion. Thanks to Lemma 2.6, we find:

$$\begin{aligned}\tilde{K}_\varepsilon(\psi_\varepsilon,\phi_{1,\varepsilon})(t,x) &= \tilde{K}(\psi_1\phi_1)\left(|x|^{\frac{1}{\varepsilon}}\frac{x}{|x|},\frac{t}{\varepsilon}\right)\\ &\le C\mathrm{e}^{-\frac{(2\alpha-\gamma)\left[t(|\lambda_1|+\varepsilon^2)+\delta\right]}{\varepsilon}}\psi\left(|x|^{\frac{1}{\varepsilon}}\frac{x}{|x|},\frac{t}{\varepsilon}\right)\\ &= C\mathrm{e}^{-\frac{(2\alpha-\gamma)\left[t(|\lambda_1|+\varepsilon^2)+\delta\right]}{\varepsilon}}\psi_\varepsilon(t,x).\end{aligned}$$

Then, noticing that for any choice of $\alpha, 2\alpha-\gamma$ is strictly positive, we deduce there exists $\varepsilon_3' > 0$ such that for all $\varepsilon < \varepsilon_3'$:

$$C\mathrm{e}^{-(2\alpha-\gamma)\frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon}} - \frac{\varepsilon^2 \min\phi_1}{5} \le 0.$$

We deduce that

$$\tilde{K}_\varepsilon(\psi_\varepsilon,\phi_{1,\varepsilon})(t,x) \le \frac{\varepsilon^2}{5}\psi_\varepsilon(t,x)\min\phi_1 \le \frac{\varepsilon^2}{5}\psi_\varepsilon(t,x)\phi_{1,\varepsilon}(x)$$

In the same way, We deduce there exists $\varepsilon'_4 > 0$ such that for all $\varepsilon < \varepsilon'_4$:

$$\bar{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(t,x) \le \frac{\varepsilon^2}{5}\psi_\varepsilon(t,x)\min\phi_1 \le \frac{\varepsilon^2}{5}\psi_\varepsilon(t,x)\phi_{1,\varepsilon}(x).$$

We set:

$$\varepsilon_0 = \min(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4).$$

Then, we conclude that for $\varepsilon \le \varepsilon_0$, we have:

$$\begin{aligned}
&\varepsilon\partial_t f_\varepsilon^M(t,x) + L_\varepsilon^\alpha f_\varepsilon^M(t,x) - f_\varepsilon^M(t,x)\left[\mu_\varepsilon(x) - f_\varepsilon^M(t,x)\right] + \tau\hat{J}\left(f_\varepsilon^M\right)(t,x) \\
&\ge \varepsilon^2 f_\varepsilon^M(t,x) + \phi_{1,\varepsilon}(x)L_\varepsilon^\alpha\psi_\varepsilon(t,x) - \tilde{K}_\varepsilon(\psi,\phi_1)(t,x) \\
&\quad + \tau\phi_{1,\varepsilon}(x)\hat{J}_\varepsilon^\alpha\psi_\varepsilon(t,x) - \tau\bar{K}_\varepsilon(\psi,\phi_1)(t,x) \\
&\ge \varepsilon^2 f_\varepsilon^M(t,x) - \frac{\varepsilon^2}{5}\phi_{1,e}(x)\psi_\varepsilon(t,x) - \frac{\varepsilon^2}{5}\phi_{1,\varepsilon}(x)\psi_\varepsilon(t,x) \\
&\quad - \frac{\varepsilon^2}{5}\phi_{1,\varepsilon}(x)\psi_\varepsilon(t,x) - \frac{\varepsilon^2}{5}\phi_{1,\varepsilon}(x)\psi_\varepsilon(t,x) \\
&\ge \frac{\varepsilon^2}{5} f_\varepsilon^M(t,x) \\
&\ge 0.
\end{aligned}$$

Therefore f_ε^M is a super-solution of (4.1) and this concludes the proof of the point (i).

Proof of (*iii*). From (2.13), since $\max|\phi_1|C_m < c_m$ and $c_M < C_M \min|\phi_1|$, we have:

$$f_\varepsilon^m(x,0) = \frac{\phi_{1,\varepsilon}(x)\times C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}} = \frac{\phi_{1,\varepsilon}(x)\times C_m}{e^{\frac{\delta}{\varepsilon}} + ||x|^{\frac{N+2\alpha}{\varepsilon}}} \le \frac{c_m}{1+|x|^{\frac{N+2\alpha}{\varepsilon}}} \le u_\varepsilon(x,0) \le f_\varepsilon^M(t,x).$$

Then, according to Lemma 2.3, we obtain:

$$\phi_{1,\varepsilon}(x)\times\frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1|-\varepsilon^2)-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}} \le u_\varepsilon(t,x) \le \phi_{1,\varepsilon}(x)\times\frac{C_M}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1|+\varepsilon^2)-\frac{\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}},$$

and hence

$$\phi_{1,\varepsilon}(x)\times\frac{C_m e^{\frac{-\delta}{\varepsilon}-\varepsilon t}}{1 + e^{-\frac{|\lambda|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}} \le u_\varepsilon(t,x) \le \phi_{1,\varepsilon}(x)\times\frac{C_M e^{\varepsilon t}}{1 + e^{-\frac{|\lambda_1|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}}. \tag{4.6}$$

Thanks to the inequalities stated in (4.6), we are now in a position to prove Theorem 2. In order to accomplish this, we will adopt the approach outlined by [9] and [16].

Proof of Theorem 2. First, we perform a Hopf-Cole transformation

$$n_\varepsilon(t,x) := \varepsilon\log u_\varepsilon(t,x) \text{ and } n_{+,\varepsilon}(x) := \varepsilon\log u_{+,\varepsilon}(x). \tag{4.7}$$

Taking the logarithm in (4.6) and multiplying by ε, we find:

$$-\varepsilon^2 t+\varepsilon\log C_m\phi_{1,\varepsilon}-\varepsilon\log\left(1+\mathrm{e}^{-\frac{|\lambda_1|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}\right)-\delta\le n_\varepsilon(t,x)$$

$$\text{and } n_\varepsilon(t,x)\le\varepsilon^2 t+\varepsilon\log C_M\phi_{1,\varepsilon}-\varepsilon\log\left(1+\mathrm{e}^{-\frac{|\lambda_1|t+\delta}{\varepsilon}}|x|^{\frac{N+2\alpha}{\varepsilon}}\right).$$

Define

$$\underline{n}(t,x)=\liminf_{\varepsilon\to0} n_\varepsilon(t,x),\ \ \overline{n}(t,x)=\limsup_{\varepsilon\to0} n_\varepsilon(t,x),\ \text{ for all } (t,x)\in\mathbb{R}^N\times(0,+\infty).$$

Letting $\varepsilon\to0$, we obtain

$$\min\left(0,|\lambda_1|t+\delta-(N+2\alpha)\log|x|\right)-\delta\le\underline{n}(t,x)\le\overline{n}(t,x)\le\min\left(0,|\lambda_1|t+\delta-(N+2\alpha)\log|x|\right).$$

We then let $\delta\to0$ and we obtain

$$n(t,x):=\underline{n}(t,x)=\overline{n}(t,x)=\min\left(0,|\lambda_1|t-(N+2\alpha)\log|x|\right).$$

We deduce that n_ε converges locally uniformly in $\mathbb{R}^N\times[0,+\infty)$ to n since the above limits are locally uniform in ε.

Proof of (*i*). For any compact set K in $\mathcal{A}$, there exists a positive constant a such that for all $(x_0,t_0)\in K$, we have $n(x_0,t_0)<-a$. It is thus immediate from (4.7) that u_ε converges uniformly to 0 in $K\subset\mathcal{A}$. This concludes the proof of (i).

Proof of (*ii*). We can obtain it by referring to reference [20].

5. Conclusion

In this paper, we provide an asymptotic analysis of a nonlocal reaction-diffusion equation in periodic media and with a nonlocal stable operator of order $\alpha\in(0,1)$. The objective of this work is to provide an alternative proof of this property using an asymptotic approach known as "approximation of geometric optics". We will be interested in the long-time behavior of the solution u. Our focus lies in examining the long-term behavior of the solution u. We show that, within the set defined by $\left\{|x|<\mathrm{e}^{ct}\right\}$, as t approaches infinity, u converges to a stationary state u^+, whereas outside this domain, u tends towards zero. The core concept in this approach involves executing a long-time, long-range rescaling to capture the effective behavior of the solution (refer to [10] [12]).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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