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Research on θ-Intuitionistic Fuzzy Homomorphism Theory

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Abstract

Fuzzy homomorphism is an important research content of fuzzy group theory, different fuzzy mappings will produce different fuzzy homomorphisms. In this paper, the fuzzy homomorphism of groups is generalized. Firstly, the θ-intuitionistic fuzzy mapping is defined, and the θ-intuitionistic fuzzy homomorphism of groups is obtained. The properties of intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups are studied under the θ-intuitionistic fuzzy homomorphism of groups, and the fundamental theorem of θ-intuitionistic fuzzy homomorphism is proved.

Keywords

θ-Intuitionistic Fuzzy Homomorphism, Intuitionistic Fuzzy Subgroup, Intuitionistic Fuzzy Normal Subgroup

1. Introduction

In 1965, Zadeh proposed the concept of fuzzy subsets. It is a class of objects with continuous membership grades. Its set feature is to use a membership function to give each object a membership grade between 0 and 1 [1]. In 1971, Rosenfeld proposed the concept of fuzzy subgroups, which is of pioneering significance for the study of fuzzy algebraic structure [2]. In 2001, Yao obtained the concept of fuzzy homomorphism of groups by defining the concept of fuzzy mapping. He studied the properties of fuzzy subgroups and fuzzy normal subgroups under fuzzy homomorphism and proved the basic theorem of fuzzy homomorphism [3]. Different fuzzy mappings will produce different fuzzy homomorphisms. In 2014, Hao defined θ-fuzzy mappings and discussed the fuzzy homomorphisms generated by θ-fuzzy mappings on groups [4]. In 2018, Addis introduced the concept of fuzzy kernels of fuzzy homomorphisms on groups, proved that any fuzzy normal subgroup is a fuzzy kernel of some fuzzy epimorphism, and finally gave and proved the fuzzy version of the fundamental theorem of homomorphism and those isomorphism theorems [5].

In 1986, Atanassov proposed the concept of intuitionistic fuzzy subsets, which is a generalization of fuzzy subsets proposed by Zadeh [6]. In 1989, Biswas proposed the concept of intuitionistic fuzzy subgroups [7]. After that, the concepts of intuitionistic fuzzy normal subgroups, intuitionistic fuzzy cosets and intuitionistic fuzzy quotient groups are introduced and their related properties are studied [8]. In 2011, Sharma studied the related conclusions about intuitionistic fuzzy subgroups under group homomorphism [9]. In 2020, Adamu studied the properties of intuitionistic fuzzy multigroups under group homomorphism [10], Muhammad et al. proposed the concept of complex intuitionistic fuzzy subgroups and studied the homomorphic image and preimage of complex intuitionistic fuzzy subgroups under group homomorphism [11]. It can be seen that group homomorphism plays an important role in intuitionistic fuzzy theory. In recent years, the application of intuitionistic fuzzy information in multi-attribute group decision-making has also been an important research direction [12] [13]. Since the fuzzy homomorphism of groups has been widely studied in fuzzy group theory, there are few studies on the intuitionistic fuzzy homomorphism of groups. This paper attempts to generalize the fuzzy mapping, define the θ-intuitionistic fuzzy mapping, and further study the θ-intuitionistic fuzzy homomorphism of groups.

2. Preliminaries

Let G, G_1, G_2 denote an arbitrary group with binary multiplication, whose identities are e, e_1, e_2 respectively. The concepts of intuitionistic fuzzy subsets, intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups are given below.

Definition 2.1 [6] Let X be a non-empty set. An intuitionistic fuzzy subset A of X is $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, Where $\mu_A(x) \in [0,1]$ define the degree of membership element $x \in X$ and $\nu_A(x) \in [0,1]$ define the degree of non-membership of element $x \in X$, these functions must be satisfied the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2.2 [6] An intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ of G is an intuitionistic fuzzy subgroup of G, if the following conditions hold:

1) $\mu_A(xy) \ge \mu_A(x) \wedge \mu_A(y)$;
2) $\mu_A(x^{-1}) \ge \mu_A(x)$;
3) $\nu_A(xy) \le \nu_A(x) \vee \nu_A(y)$;
4) $\nu_A(x^{-1}) \le \nu_A(x), \forall x, y \in G$.

Equivalently, an intuitionistic fuzzy subset A of G is an intuitionistic fuzzy subgroup of G if $\mu_A(xy^{-1}) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \le \nu_A(x) \vee \nu_A(y)$ holds for $\forall x, y \in G$.

Definition 2.3 [7] Let A be an intuitionistic fuzzy subgroup of G, if for

$\forall x, y \in G$, $\mu_A\left(xyx^{-1}\right) \geq \mu_A(y)$, $v_A\left(xyx^{-1}\right) \leq v_A(y)$, then A is said to be an intuitionistic fuzzy normal subgroup of G.

Definition 2.4 [8] Let A be an intuitionistic fuzzy subgroup of G. Let $x \in G$ be a fixed element. Then $xA = \left\{\left\langle g, \mu_{xA}(g), v_{xA}(g)\right\rangle \mid g \in G\right\}$ where $\mu_{xA}(g) = \mu_A\left(x^{-1}g\right)$ and $v_{xA}(g) = v_A\left(x^{-1}g\right)$ for $\forall g \in G$ is called intuitionistic fuzzy left coset of G determined by A and x. Similarly, $Ax = \left\{\left\langle g, \mu_{Ax}(g), v_{Ax}(g)\right\rangle \mid g \in G\right\}$ where $\mu_{Ax}(g) = \mu_A\left(gx^{-1}\right)$ and $v_{Ax}(g) = v_A\left(gx^{-1}\right)$ for $\forall g \in G$ is called intuitionistic fuzzy right coset of G determined by A and x.

Theorem 2.5 [8] Let A be an intuitionistic fuzzy normal subgroup of *G*, then $G/A = \{xA \mid x \in G\}$ is a group with respect to the operation $(xA)(yA) = (xy)A$, and the identity of G/A is A, the identity of xA is $x^{-1}A$.

3. θ-Intuitionistic Fuzzy Homomorphism

The classical mapping f from G_1 to G_2 can be considered as a special relation from G_1 to G_2, that is f is a special subset of $G_1 \times G_2$. Therefore, the intuitionistic fuzzy mapping can be considered as the intuitionistic fuzzy relation from G_1 to G_2, that is the intuitionistic fuzzy subset of $G_1 \times G_2$. The concept of θ-intuitionistic fuzzy mapping is given below.

Let $\theta = (\theta_1, \theta_2)$, where $\theta_1, \theta_2 \in [0,1]$ and $0 \leq \theta_1 + \theta_2 \leq 1$, f is an intuitionistic fuzzy relation from G_1 to G_2. The following conditions are considered:

(A) $\forall x \in G_1, \exists y \in G_2$, such that $\mu_f(x,y) \geq \theta_1$, $v_f(x,y) \leq \theta_2$;

(B) $\forall y \in G_2, \exists x \in G_1$, such that $\mu_f(x,y) \geq \theta_1$, $v_f(x,y) \leq \theta_2$;

(C) $\forall x \in G_1, y_1, y_2 \in G_2$, from $\mu_f(x,y_1) \geq \theta_1$, $v_f(x,y_1) \leq \theta_2$; $\mu_f(x,y_2) \geq \theta_1$, $v_f(x,y_2) \leq \theta_2$, it follows that $y_1 = y_2$;

(D) $\forall x_1, x_2 \in G_1, y \in G_2$, from $\mu_f(x_1,y) \geq \theta_1$, $v_f(x_1,y) \leq \theta_2$; $\mu_f(x_2,y) \geq \theta_1$, $v_f(x_2,y) \leq \theta_2$, it follows that $x_1 = x_2$.

Definition 3.1 Let f be an intuitionistic fuzzy relation from G_1 to G_2. If Conditions (A), (C) are satisfied, then f is called a θ-intuitionistic fuzzy mapping from G_1 to G_2; if Conditions (A), (B), (C) are satisfied, then f is called a θ-intuitionistic fuzzy surjection; if Conditions (A), (C), (D) are satisfied, then f is called θ-intuitionistic fuzzy injective; if Conditions (A)-(D) are satisfied, then f is called a θ-intuitionistic fuzzy bijection.

When f is a θ-intuitionistic fuzzy mapping from G_1 to G_2, for $\forall x \in G_1$, there exists a unique $y \in G_2$ such that $\mu_f(x,y) \geq \theta_1, v_f(x,y) \leq \theta_2$ holds, denote this y as y_x.

Definition 3.2 Let f be a θ-intuitionistic fuzzy mapping from G_1 to G_2, if for $\forall x_1, x_2 \in G_1, y \in G_2$, $\mu_f(x_1x_2, y) = \sup\left\{\mu_f(x_1,y_1) \wedge \mu_f(x_2,y_2) \mid y = y_1y_2\right\}$, $v_f(x_1x_2, y) = \inf\left\{v_f(x_1,y_1) \vee v_f(x_2,y_2) \mid y = y_1y_2\right\}$, then f is called a θ-intuitionistic fuzzy homomorphism from G_1 to G_2, for short θ-intuitionistic fuzzy homomorphism.

If a θ-intuitionistic fuzzy homomorphism f from G_1 to G_2 is a θ-intuitionistic fuzzy surjective, then f is called a θ-intuitionistic fuzzy epimorphism from G_1 to G_2; if a θ-intuitionistic fuzzy homomorphism f from G_1 to G_2 is a θ-intuitionistic fuzzy bijection, then f is called a θ-intuitionistic fuzzy isomorphism from G_1 to G_2.

If there exists a θ-intuitionistic fuzzy epimorphism from G_1 to G_2, then G_1 and G_2 are θ-intuitionistic fuzzy homomorphism. If there exists a θ-intuitionistic fuzzy isomorphism from G_1 to G_2, then G_1 and G_2 are said to be θ-intuitionistic fuzzy isomorphism.

Theorem 3.3 Let $f: G_1 \to G_2$ be a θ-intuitionistic fuzzy homomorphism, then

1) $\mu_f(x_1, y_1) \geq \theta_1$, $v_f(x_1, y_1) \leq \theta_2$; $\mu_f(x_2, y_2) \geq \theta_1$, $v_f(x_2, y_2) \leq \theta_2 \Rightarrow \mu_f(x_1x_2, y_1y_2) \geq \theta_1$, $v_f(x_1x_2, y_1y_2) \leq \theta_2$;

2) $\mu_f(e_1, e_2) \geq \theta_1, v_f(e_1, e_2) \leq \theta_2$;

3) $\mu_f(x, y) \geq \theta_1, v_f(x, y) \leq \theta_2 \Rightarrow \mu_f(x^{-1}, y^{-1}) \geq \theta_1, v_f(x^{-1}, y^{-1}) \leq \theta_2$;

4) $y_{x_1x_2} = y_{x_1}y_{x_2}$;

5) $(y_x)^{-1} = y_{x^{-1}}$.

Proof: 1) Since $\mu_f(x_1, y_1) \geq \theta_1$, $v_f(x_1, y_1) \leq \theta_2$; $\mu_f(x_2, y_2) \geq \theta_1$, $v_f(x_2, y_2) \leq \theta_2$, we can get $\mu_f(x_1x_2, y_1y_2) \geq \mu_f(x_1, y_1) \wedge \mu_f(x_1, y_1) \geq \theta_1$, $v_f(x_1x_2, y_1y_2) \leq v_f(x_1, y_1) \vee v_f(x_2, y_2) \leq \theta_2$.

2) Let $y \in G_2$ satisfies $\mu_f(e_1, y) \geq \theta_1$, $v_f(e_1, y) \leq \theta_2$, then $\mu_f(e_1, yy) = \mu_f(e_1e_1, yy) \geq \theta_1$, $v_f(e_1, yy) = v_f(e_1e_1, yy) \leq \theta_2$, so $yy = y$, *i.e.*, y = e_2, hence, $\mu_f(e_1, e_2) \geq \theta_1$, $v_f(e_1, e_2) \leq \theta_2$.

3) Let $\mu_f(x, y) \geq \theta_1$, $v_f(x, y) \leq \theta_2$, and suppose there exists $z \in G_2$ that satisfies $\mu_f(x^{-1}, z) \geq \theta_1$, $v_f(x^{-1}, z) \leq \theta_2$, then $\mu_f(e_1, yz) = \mu_f(xx^{-1}, yz) \geq \theta_1$, $v_f(e_1, yz) = v_f(xx^{-1}, yz) \leq \theta_2$, which is equivalent to $yz = e_2$, also $z = y^{-1}$, so $\mu_f(x^{-1}, y^{-1}) \geq \theta_1$, $v_f(x^{-1}, y^{-1}) \leq \theta_2$.

4) Because of $\mu_f(x_1, y_{x_1}) \geq \theta_1$, $v_f(x_1, y_{x_1}) \leq \theta_2$, $\mu_f(x_2, y_{x_2}) \geq \theta_1$, $v_f(x_2, y_{x_2}) \leq \theta_2$, we can conclude $\mu_f(x_1x_2, y_{x_1}y_{x_2}) \geq \theta_1$, $v_f(x_1x_2, y_{x_1}y_{x_2}) \leq \theta_2$, also $\mu_f(x_1x_2, y_{x_1x_2}) \geq \theta_1$, $v_f(x_1x_2, y_{x_1x_2}) \leq \theta_2$, therefore $y_{x_1x_2} = y_{x_1}y_{x_2}$.

5) Because $\mu_f(x, y_x) \geq \theta_1$, $v_f(x, y_x) \leq \theta_2$, then $\mu_f(x^{-1}, (y_x)^{-1}) \geq \theta_1$, $v_f(x^{-1}, (y_x)^{-1}) \leq \theta_2$ can be obtained, also $\mu_f(x^{-1}, y_{x^{-1}}) \geq \theta_1$, $v_f(x^{-1}, y_{x^{-1}}) \leq \theta_2$, hence $(y_x)^{-1} = y_{x^{-1}}$.

Theorem 3.4 Let f be a θ-intuitionistic fuzzy homomorphism, then.

1) $N = \ker f = \{x \in G_1 \mid \mu_f(x, e_2) \geq \theta_1, v_f(x, e_2) \leq \theta_2\}$ is a normal subgroup of G_1, and N is called the kernel of θ-intuitionistic fuzzy homomorphism f;

2) If f is a θ-intuitionistic fuzzy epimorphism, then $G_1 / N \cong G_2$.

Proof: 1) Obviously $e_1 \in N$, let $x_1, x_2 \in N$, then $\mu_f(x_1, e_2) \geq \theta_1$, $v_f(x_1, e_2) \leq \theta_2$, $\mu_f(x_2, e_2) \geq \theta_1$, $v_f(x_2, e_2) \leq \theta_2$, therefore $\mu_f(x_1^{-1}, e_2) = \mu_f(x_1^{-1}, e_2^{-1}) \geq \theta_1$, $v_f(x_1^{-1}, e_2) = v_f(x_1^{-1}, e_2^{-1}) \leq \theta_2$; $\mu_f(x_1x_2, e_2) = \mu_f(x_1x_2, e_2e_2) \geq \theta_1$, $v_f(x_1x_2, e_2) = v_f(x_1x_2, e_2e_2) \leq \theta_2$ *i.e.*, $x_1^{-1} \in N$, $x_1x_2 \in N$, N is a subgroup of G_1. Let $x_1 \in N$, $x \in G_1$, then $\mu_f(x_1, e_2) \geq \theta_1$, $v_f(x_1, e_2) \leq \theta_2$, and there exists $y \in G_2$ such that $\mu_f(x, y) \geq \theta_1$, $v_f(x, y) \leq \theta_2$, we can get $\mu_f(xx_1, ye_2) \geq \theta_1$, $v_f(xx_1, ye_2) \leq \theta_2$. Also because $\mu_f(x^{-1}, y^{-1}) \geq \theta_1$, $v_f(x^{-1}, y^{-1}) \leq \theta_2$, so $\mu_f(xx_1x^{-1}, ye_2y^{-1}) \geq \theta_1$, $v_f(xx_1x^{-1}, ye_2y^{-1}) \leq \theta_2$, *i.e.*, $xx_1x^{-1} \in N$, so N is a normal subgroup of G_1.

2) Let $g: G_1/N \to G_2, g(xN) = y_x, \forall x \in G_1$, Since

$$x_1N = x_2N \Rightarrow x_1^{-1}x_2 \in N \Rightarrow \mu_f\left(x_1^{-1}x_2, e_2\right) \geq \theta_1, v_f\left(x_1^{-1}x_2, e_2\right) \leq \theta_2$$
$$\Rightarrow y_{x_1^{-1}x_2} = e_2 \Rightarrow \left(y_{x_1}\right)^{-1} y_{x_2} = e_2 \Rightarrow y_{x_1} = y_{x_2}$$

Then g is a mapping. In turn, since

$$y_{x_1} = y_{x_2} \Rightarrow \left(y_{x_1}\right)^{-1} y_{x_2} = e_2 \Rightarrow y_{x_1^{-1}x_2} = e_2 \Rightarrow \mu_f\left(x_1^{-1}x_2, e_2\right) \geq \theta_1, v_f\left(x_1^{-1}x_2, e_2\right) \leq \theta_2$$
$$\Rightarrow x_1^{-1}x_2 \in N \Rightarrow x_1N = x_2N$$

So, g is a injective. Moreover, because f is a θ-intuitionistic fuzzy epimorphism, g is a surjective, therefore g is a bijective. For $\forall x_1, x_2 \in G$, $g\left(\left(x_1N\right)\left(x_2N\right)\right) = g\left(x_1x_2N\right) = y_{x_1x_2} = y_{x_1}y_{x_2} = g\left(x_1N\right)g\left(x_2N\right)$, so g is an isomorphic mapping, hence $G_1/N \cong G_2$.

Theorem 3.5 Let f be a θ-intuitionistic fuzzy homomorphism, then

1) If A is an intuitionistic fuzzy subgroup of G_1, then $f(A)$ is an intuitionistic fuzzy subgroup of G_2;

2) If A is an intuitionistic fuzzy normal subgroup of G_1, and f is θ-intuitionistic fuzzy homomorphism epimorphism, then $f(A)$ is an intuitionistic fuzzy normal subgroup of G_2;

3) If B is an intuitionistic fuzzy subgroup of G_2, then $f^{-1}(B)$ is an intuitionistic fuzzy subgroup of G_1;

4) If B is an intuitionistic fuzzy normal subgroup of G_2, then $f^{-1}(B)$ is an intuitionistic fuzzy normal subgroup of G_1, where

$f(A)(y) = \left(\mu_{f(A)}(y), v_{f(A)}(y)\right)$, $f^{-1}(B) = \left(\mu_{f^{-1}(B)}(x), v_{f^{-1}(B)}(x)\right)$.

$$\mu_{f(A)}(y) = \begin{cases} \sup\left\{\mu_A(x) \mid \mu_f(x,y) \geq \theta_1\right\}, \exists x \in G_1, \mu_f(x,y) \geq \theta_1 \\ 0, \qquad \forall x \in G_1, \mu_f(x,y) < \theta_1 \end{cases}, \forall y \in G_2$$

$$v_{f(A)}(y) = \begin{cases} \inf\left\{v_A(x) \mid v_f(x,y) \leq \theta_2\right\}, \exists x \in G_1, v_f(x,y) \leq \theta_2 \\ 1, \qquad \forall x \in G_1, v_f(x,y) > \theta_2 \end{cases}, \forall y \in G_2$$

$$\mu_{f^{-1}(B)}(x) = \mu_B(y_x), v_{f^{-1}(B)}(x) = v_B(y_x)$$

Proof: 1) For $\forall y_1, y_2 \in G_2$, if there is no $x_1 \in G_1$ such that $\mu_f(x_1, y_1) \geq \theta_1$, $v_f(x_1, y_1) \leq \theta_2$, or there is no $x_2 \in G_2$ such that $\mu_f(x_2, y_2) \geq \theta_1$, $v_f(x_2, y_2) \leq \theta_2$, then $\mu_{f(A)}\left(y_1^{-1}y_2\right) \geq 0 = \mu_{f(A)}(y_1) \wedge \mu_{f(A)}(y_2)$, $v_{f(A)}\left(y_1^{-1}y_2\right) \leq 1 = v_{f(A)}(y_1) \vee v_{f(A)}(y_2)$, otherwise A is an intuitionistic fuzzy subgroup of G_1, we can get

$$\begin{aligned}\mu_{f(A)}\left(y_1^{-1}y_2\right) &= \sup\left\{\mu_A\left(x_1^{-1}x_2\right) \mid \mu_f\left(x_1^{-1}x_2, y_1^{-1}y_2\right) \geq \theta_1\right\} \\ &\geq \sup\left\{\mu_A(x_1) \wedge \mu_A(x_2) \mid \mu_f(x_1, y_1) \geq \theta_1, \mu_f(x_2, y_2) \geq \theta_1\right\} \\ &= \sup\left\{\mu_A(x_1) \mid \mu_f(x_1, y_1) \geq \theta_1\right\} \wedge \sup\left\{\mu_A(x_2) \mid \mu_f(x_2, y_2) \geq \theta_1\right\} \\ &= \mu_{f(A)}(y_1) \wedge \mu_{f(A)}(y_2)\end{aligned}$$

Similarly, it can be obtained $v_{f(A)}\left(y_1^{-1}y_2\right)\le v_{f(A)}\left(y_1\right)\vee v_{f(A)}\left(y_2\right)$, So $f(A)$ is an intuitionistic fuzzy subgroup of G_2.

2) If A is an intuitionistic fuzzy normal subgroup of G_1, it follows from 1) that $f(A)$ is an intuitionistic fuzzy subgroup of G_2. Since f is θ-intuitionistic fuzzy epimorphism, so for $\forall y_1,y_2\in G_2$, $\exists x_1,x_2\in G_1$ such that $\mu_f\left(x_1,y_1\right)\ge\theta_1$, $\mu_f\left(x_2,y_2\right)\ge\theta_1$, we can get

$$\begin{aligned}\mu_{f(A)}\left(y_1y_2y_1^{-1}\right)&=\sup\left\{\mu_A\left(x_1x_2x_1^{-1}\right)\mid\mu_f\left(x_1x_2x_1^{-1},y_1y_2y_1^{-1}\right)\ge\theta_1\right\}\\&\ge\sup\left\{\mu_A\left(x_2\right)\mid\mu_f\left(x_2,y_2\right)\ge\theta_1\right\}\\&=\mu_{f(A)}\left(y_2\right)\end{aligned}$$

Similarly, it can be obtained $v_{f(A)}\left(y_1y_2y_1^{-1}\right)\le v_{f(A)}\left(y_2\right)$, So $f(A)$ is an intuitionistic fuzzy normal subgroup of G_2.

3) For $\forall x_1,x_2\in G_1$, because B is an intuitionistic fuzzy subgroup of G_2, we have

$$\begin{aligned}\mu_{f^{-1}(B)}\left(x_1^{-1}x_2\right)&=\mu_B\left(y_{x_1^{-1}x_2}\right)=\mu_B\left(y_{x_1}^{-1}y_{x_2}\right)\\&\ge\mu_B\left(y_{x_1}\right)\wedge\mu_B\left(y_{x_2}\right)\\&=\mu_{f^{-1}(B)}(x_1)\wedge\mu_{f^{-1}(B)}\left(x_2\right)\end{aligned}$$

Similarly, it can be obtained $v_{f^{-1}(B)}\left(x_1^{-1}x_2\right)\le v_{f^{-1}(B)}\left(x_1\right)\vee v_{f^{-1}(B)}\left(x_2\right)$, So $f^{-1}(B)$ is an intuitionistic fuzzy subgroup of G_1.

4) If B is an intuitionistic fuzzy normal subgroup of G_2, it follows from 3) that $f^{-1}(B)$ is an intuitionistic fuzzy subgroup of G_1. For $\forall x_1,x_2\in G_1$, we have

$$\begin{aligned}\mu_{f^{-1}(B)}\left(x_1x_2x_1^{-1}\right)&=\mu_B\left(y_{x_1x_2x_1^{-1}}\right)=\mu_B\left(y_{x_1}y_{x_2}y_{x_1}^{-1}\right)\\&\ge\mu_B\left(y_{x_2}\right)=\mu_{f^{-1}(B)}\left(x_2\right)\end{aligned}$$

Similarly, it can be obtained $v_{f^{-1}(B)}\left(x_1x_2x_1^{-1}\right)\le v_{f^{-1}(B)}\left(x_2\right)$, So $f^{-1}(B)$ is an intuitionistic fuzzy normal subgroup of G_1.

4. Fundamental Theorem of θ-Intuitionistic Fuzzy Homomorphism

Theorem 4.1 Let A be an intuitionistic fuzzy normal subgroup of G, then G and G/A are θ-intuitionistic fuzzy homomorphism, where $\theta=\left(\theta_1,\theta_2\right)$, $\theta_1=\mu_A(e)$, $\theta_2=v_A(e)$.

Proof: Let $f:G\to G/A$, $\mu_f\left(x,y\mu_A\right)=\mu_A\left(y^{-1}x\right)$, $v_f\left(x,yv_A\right)=v_A\left(y^{-1}x\right)$, $\forall x,y\in G$. First, prove that f is an intuitionistic fuzzy relation, *i.e.*, prove that $y_1A=y_2A\Rightarrow A\left(y_1^{-1}x\right)=A\left(y_2^{-1}x\right)$. From $y_1A=y_2A$ we can get $\mu_{y_1A}(x)=\mu_{y_2A}(x)$, $v_{y_1A}(x)=v_{y_2A}(x)$, $\forall x\in G$, then $\mu_A\left(y_1^{-1}x\right)=\mu_A\left(y_2^{-1}x\right)$, $v_A\left(y_1^{-1}x\right)=v_A\left(y_2^{-1}x\right)$, *i.e.*, $A\left(y_1^{-1}x\right)=A\left(y_2^{-1}x\right)$, so f is an intuitionistic fuzzy relation. In the following, we prove that f is a θ-intuitionistic fuzzy epimorphism, for $\forall x\in G$, obviously $\mu_f\left(x,x\mu_A\right)=\mu_A\left(x^{-1}x\right)=\mu_A(e)\ge\theta_1$, $v_f\left(x,xv_A\right)=v_A\left(x^{-1}x\right)=v_A(e)\le\theta_2$ holds, so f satisfies Conditions (A) (B). If for $\forall x\in G$, $\exists y_1,y_2\in G$, satisfies $\mu_f(x,y_1\mu_A)\geq\theta_1$, $v_f(x,y_1v_A)\leq\theta_2$; $\mu_f(x,y_2\mu_A)\geq\theta_1$, $v_f(x,y_2v_A)\leq\theta_2$, this implies that $\mu_A(y_1^{-1}x)\geq\mu_A(e)$, $v_A(y_1^{-1}x)\leq v_A(e)$, $\mu_A(y_2^{-1}x)\geq\mu_A(e)$, $v_A(y_2^{-1}x)\leq v_A(e)$, We can get $\mu_A(y_1^{-1}x)=\mu_A(e)$, $v_A(y_1^{-1}x)=v_A(e)$, $\mu_A(y_2^{-1}x)=\mu_A(e)$, $v_A(y_2^{-1}x)=v_A(e)$, So $y_1A=xA$, $y_2A=xA$ *i.e.*, $y_1A=y_2A$, f satisfies Condition (C), so f is a θ-intuitionistic fuzzy surjective from G to G/A. For $\forall x_1,x_2,y\in G$, if $\exists y_1,y_2\in G$, such that $yA=(y_1A)(y_2A)=(y_1y_2)A$, then

$$\begin{aligned}\mu_f(x_1x_2,y\mu_A)&=\mu_f(x_1x_2,(y_1y_2)\mu_A)=\mu_A\left((y_1y_2)^{-1}x_1x_2\right)\\&=\mu_A\left(y_2^{-1}x_2x_2^{-1}y_1^{-1}x_1x_2\right)\\&\geq\mu_A\left(y_2^{-1}x_2\right)\wedge\mu_A\left(x_2^{-1}y_1^{-1}x_1x_2\right)\\&\geq\mu_f(x_2,y_2\mu_A)\wedge\mu_A\left(y_1^{-1}x_1\right)\\&=\mu_f(x_2,y_2\mu_A)\wedge\mu_f(x_1,y_1\mu_A)\end{aligned}$$

Similarly, it can be obtained $v_f(x_1x_2,yv_A)\leq v_f(x_2,y_2v_A)\vee v_f(x_1,y_1v_A)$, we can get

$$\mu_f(x_1x_2,y\mu_A)\geq\sup\left\{\mu_f(x_2,y_2\mu_A)\wedge\mu_f(x_1,y_1\mu_A)\mid y\mu_A=(y_1\mu_A)(y_2\mu_A)\right\}$$

$$v_f(x_1x_2,yv_A)\leq\inf\left\{v_f(x_2,y_2v_A)\vee v_f(x_1,y_1v_A)\mid yv_A=(y_1v_A)(y_2v_A)\right\}$$

Also because of

$$\begin{aligned}&\sup\left\{\mu_f(x_1,y_1\mu_A)\wedge\mu_f(x_2,y_2\mu_A)\mid y\mu_A=(y_1\mu_A)(y_2\mu_A)\right\}\\&\geq\mu_f(x_1,x_1\mu_A)\wedge\mu_f\left(x_2,(x_1^{-1}y)\mu_A\right)=\mu_A(e)\wedge\mu_A\left(y^{-1}x_1x_2\right)\\&=\mu_A\left(y^{-1}x_1x_2\right)=\mu_f(x_1x_2,y\mu_A)\end{aligned}$$

Similarly, it can be obtained

$\inf\left\{v_f(x_1,y_1v_A)\vee v_f(x_2,y_2v_A)\mid yv_A=(y_1v_A)(y_2v_A)\right\}\leq v_f(x_1x_2,yv_A)$, so

$$\mu_f(x_1x_2,y\mu_A)=\sup\left\{\mu_f(x_1,y_1\mu_A)\wedge\mu_f(x_2,y_2\mu_A)\mid y\mu_A=(y_1\mu_A)(y_2\mu_A)\right\}$$

$$v_f(x_1x_2,yv_A)=\inf\left\{v_f(x_1,y_1v_A)\vee v_f(x_2,y_2v_A)\mid yv_A=(y_1v_A)(y_2v_A)\right\}$$

Hence $f:G\to G/A$ is a θ-intuitionistic fuzzy epimorphism, G and G/A are θ-intuitionistic fuzzy homomorphism.

Theorem 4.2 Let $f:G_1\to G_2$ be a θ-intuitionistic fuzzy homomorphism, $\mu_f(e_1,e_2)\geq\mu_f(x,y)$, $v_f(e_1,e_2)\leq v_f(x,y)$, for $\forall x\in G_1,\forall y\in G_2$, then

1) A is an intuitionistic fuzzy normal subgroup of G_1, where $\mu_A(x)=\mu_f(x,e_2)\wedge\mu_f\left(x^{-1},e_2\right)\wedge\theta_1$, $v_A(x)=v_f(x,e_2)\vee v_f\left(x^{-1},e_2\right)\vee\theta_2$, $\forall x\in G_1$;

2) If f is a θ-intuitionistic fuzzy epimorphism, then G_1/A and G_2 are θ-intuitionistic fuzzy isomorphic.

Proof: 1) For $\forall x,y\in G_1$, we have

$$\begin{aligned}\mu_A(xy)&=\mu_f(xy,e_2)\wedge\mu_f\left((xy)^{-1},e_2\right)\wedge\theta_1\\&\geq\mu_f(x,e_2)\wedge\mu_f(y,e_2)\wedge\mu_f\left(y^{-1},e_2\right)\wedge\mu_f\left(x^{-1},e_2\right)\wedge\theta_1\\&=\left(\mu_f(x,e_2)\wedge\mu_f\left(x^{-1},e_2\right)\right)\wedge\left(\mu_f(y,e_2)\wedge\mu_f\left(y^{-1},e_2\right)\right)\wedge\theta_1\end{aligned}$$

$$= \left(\mu_f(x,e_2) \wedge \mu_f(x^{-1},e_2) \wedge \theta_1\right) \wedge \left(\mu_f(y,e_2) \wedge \mu_f(y^{-1},e_2) \wedge \theta_1\right)$$
$$= \mu_A(x) \wedge \mu_A(y)$$

Similarly, it can be obtained $v_A(xy) \le v_A(x) \vee v_A(y)$, and also have

$$\mu_A(x^{-1}) = \mu_f(x^{-1},e_2) \wedge \mu_f(x,e_2) \wedge \theta_1 = \mu_A(x)$$
$$v_A(x^{-1}) = v_f(x^{-1},e_2) \vee v_f(x,e_2) \vee \theta_2 = v_A(x)$$

Therefore A is an intuitionistic fuzzy subgroup of G_1. Let A be an intuitionistic fuzzy normal subgroup of G_1, and we have

$$\begin{aligned}
&\mu_A(xyx^{-1}) \\
&= \mu_f(xyx^{-1},e_2) \wedge \mu_f(xy^{-1}x^{-1},e_2) \wedge \theta_1 \\
&\ge \mu_f(x,y_x) \wedge \mu_f(y,e_2) \wedge \mu_f(x^{-1},y_x^{-1}) \wedge \mu_f(x,y_x) \\
&\quad \wedge \mu_f(y^{-1},e_2) \wedge \mu_f(x^{-1},y_x^{-1}) \wedge \theta_1 \\
&= \mu_f(x,y_x) \wedge \mu_f(x^{-1},y_x^{-1}) \wedge \mu_f(y,e_2) \wedge \mu_f(y^{-1},e_2) \wedge \theta_1 \\
&\ge \theta_1 \wedge \theta_1 \wedge \mu_f(y,e_2) \wedge \mu_f(y^{-1},e_2) \wedge \theta_1 \\
&= \mu_f(y,e_2) \wedge \mu_f(y^{-1},e_2) \wedge \theta_1 = \mu_A(y)
\end{aligned}$$

Similarly, $v_A(xyx^{-1}) \le v_A(y)$ can be obtained, hence A is an intuitionistic fuzzy normal subgroup of G_1.

2) Let $h: G_1/A \to G_2$, $\mu_h(x\mu_A, y) = \mu_f(x,y)$, $v_h(xv_A, y) = v_f(x,y)$, $\forall x \in G_1$, $\forall y \in G_2$. First, prove that h is an intuitionistic fuzzy relation, i.e., $x_1A = x_2A \Rightarrow \mu_f(x_1,y) = \mu_f(x_2,y)$, $v_f(x_1,y) = v_f(x_2,y)$. If $x_1A = x_2A$, then $\mu_A(x_2^{-1}x_1) = \mu_A(e_1)$, $v_A(x_2^{-1}x_1) = v_A(e_1)$. From $\mu_A(x_2^{-1}x_1) = \mu_A(e_1)$ we can get

$$\begin{aligned}
&\mu_f(x_2^{-1}x_1,e_2) \wedge \mu_f(x_1^{-1}x_2,e_2) = \mu_f(e_1,e_2) \wedge \mu_f(e_1^{-1},e_2) \\
&\Rightarrow \mu_f(x_2^{-1}x_1,e_2) \wedge \mu_f(x_1^{-1}x_2,e_2) = \mu_f(e_1,e_2) \\
&\Rightarrow \mu_f(x_2^{-1}x_1,e_2) = \mu_f(e_1,e_2), \mu_f(x_1^{-1}x_2,e_2) = \mu_f(e_1,e_2)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mu_f(x_1,y) = \mu_f(x_2x_2^{-1}x_1,y) &\ge \mu_f(x_2,y) \wedge \mu_f(x_2^{-1}x_1,e_2) \\
&= \mu_f(x_2,y) \wedge \mu_f(e_1,e_2) = \mu_f(x_2,y)
\end{aligned}$$

Similarly, $\mu_f(x_2,y) \ge \mu_f(x_1,y)$ can be obtained, so we have $\mu_f(x_2,y) = \mu_f(x_1,y)$. From $v_A(x_2^{-1}x_1) = v_A(e_1)$, we can get

$$\begin{aligned}
&v_f(x_2^{-1}x_1,e_2) \vee v_f(x_1^{-1}x_2,e_2) = v_f(e_1,e_2) \vee v_f(e_1^{-1},e_2) \\
&\Rightarrow v_f(x_2^{-1}x_1,e_2) \vee v_f(x_1^{-1}x_2,e_2) = v_f(e_1,e_2) \\
&\Rightarrow v_f(x_2^{-1}x_1,e_2) = v_f(e_1,e_2), v_f(x_2^{-1}x_1,e_2) = v_f(e_1,e_2)
\end{aligned}$$

Therefore

$$\begin{aligned}
v_f(x_1,y) = v_f(x_2x_2^{-1}x_1,y) &\le v_f(x_2,y) \vee v_f(x_2^{-1}x_1,e_2) \\
&= v_f(x_2,y) \vee v_f(e_1,e_2) = v_f(x_2,y)
\end{aligned}$$

Similarly, $\nu_f(x_2,y)\le\nu_f(x_1,y)$ can be obtained, so $\nu_f(x_2,y)=\nu_f(x_1,y)$, this implies that h is an intuitionistic fuzzy relation. The following we prove h that is a θ-intuitionistic fuzzy isomorphism such that $\mu_h(x\mu_A,y_x)=\mu_f(x,y_x)\ge\theta_1$, $\nu_h(x\nu_A,y_x)=\nu_f(x,y_x)\le\theta_2$ for $\forall x\in G_1$, $\exists y_x\in G_2$. If $x_1A=x_2A$, $\exists y_1,y_2\in G_2$, then $\mu_h(x_1\mu_A,y_1)\ge\theta_1$, $\nu_h(x_1\nu_A,y_1)\le\theta_2$, $\mu_h(x_2\mu_A,y_2)\ge\theta_1$, $\nu_h(x_2\nu_A,y_2)\le\theta_2$, so $\mu_f(x_1,y_1)\ge\theta_1$, $\nu_f(x_1,y_1)\le\theta_2$, $\mu_f(x_2,y_2)\ge\theta_1$, $\nu_f(x_2,y_2)\le\theta_2$ and $\mu_f(x_1^{-1},y_1^{-1})\ge\theta_1$, $\nu_f(x_1^{-1},y_1^{-1})\le\theta_2$. From $x_1A=x_2A$ we get $e_1\mu_A=(x_1^{-1}x_2)\mu_A$, $e_1\nu_A=(x_1^{-1}x_2)\nu_A$, therefore $\mu_h(e_1\mu_A,y_1^{-1}y_2)=\mu_h((x_1^{-1}x_2)\mu_A,y_1^{-1}y_2)$, $\nu_h(e_1\nu_A,y_1^{-1}y_2)=\nu_h((x_1^{-1}x_2)\nu_A,y_1^{-1}y_2)$, so we have

$$\begin{aligned}\mu_f(e_1,y_1^{-1}y_2)&=\mu_f(x_1^{-1}x_2,y_1^{-1}y_2)\ge\mu_f(x_1^{-1},y_1^{-1})\wedge\mu_f(x_2,y_2)\\&\ge\theta_1\wedge\theta_1=\theta_1\end{aligned}$$

and

$$\begin{aligned}\nu_f(e_1,y_1^{-1}y_2)&=\nu_f(x_1^{-1}x_2,y_1^{-1}y_2)\le\nu_f(x_1^{-1},y_1^{-1})\vee\nu_f(x_2,y_2)\\&\le\theta_2\vee\theta_2=\theta_2\end{aligned}$$

Then $y_1^{-1}y_2=e_2$, $y_1=y_2$, so h is a θ-intuitionistic fuzzy mapping, also because f is a θ-intuitionistic fuzzy epimorphism, so h is a θ-intuitionistic fuzzy surjection. If $\exists x_1,x_2\in G_1$, such that $\mu_h(x_1\mu_A,y)\ge\theta_1$, $\nu_h(x_1\nu_A,y)\le\theta_2$, $\mu_h(x_2\mu_A,y)\ge\theta_1$, $\nu_h(x_2\nu_A,y)\le\theta_2$, *i.e.*, $\mu_f(x_1,y)\ge\theta_1$, $\nu_f(x_1,y)\le\theta_2$, $\mu_f(x_2,y)\ge\theta_1$, $\nu_f(x_2,y)\le\theta_2$, then $\mu_f(x_1^{-1},y^{-1})\ge\theta_1$, $\nu_f(x_1^{-1},y^{-1})\le\theta_2$, $\mu_f(x_2^{-1},y^{-1})\ge\theta_1$, $\nu_f(x_2^{-1},y^{-1})\le\theta_2$, therefore we have

$$\begin{aligned}\mu_f(x_1^{-1}x_2,e_2)&=\mu_f(x_1^{-1}x_2,y^{-1}y)\ge\mu_f(x_1^{-1},y^{-1})\wedge\mu_f(x_2,y)\\&\ge\theta_1\wedge\theta_1=\theta_1\end{aligned}$$

and

$$\begin{aligned}\nu_f(x_1^{-1}x_2,e_2)&=\nu_f(x_1^{-1}x_2,y^{-1}y)\le\nu_f(x_1^{-1},y^{-1})\vee\nu_f(x_2,y)\\&\le\theta_2\vee\theta_2=\theta_2\end{aligned}$$

Similarly, we can get $\mu_f(x_2^{-1}x_1,e_2)\ge\theta_1$, $\nu_f(x_2^{-1}x_1,e_2)\le\theta_2$, so. $\mu_A(x_1^{-1}x_2)=\mu_f(x_1^{-1}x_2,e_2)\wedge\mu_f(x_2^{-1}x_1,e_2)\wedge\theta_1=\theta_1$, $\nu_A(x_1^{-1}x_2)=\nu_f(x_1^{-1}x_2,e_2)\vee\nu_f(x_2^{-1}x_1,e_2)\vee\theta_2=\theta_2$. Consequently, $\mu_A(e_1)=\mu_f(e_1,e_2)\wedge\mu_f(e_1^{-1},e_2)\wedge\theta_1=\theta_1$, $\nu_A(e_1)=\nu_f(e_1,e_2)\vee\nu_f(e_1^{-1},e_2)\vee\theta_2=\theta_2$, so $\mu_A(x_1^{-1}x_2)=\mu_A(e_1)$, $\nu_A(x_1^{-1}x_2)=\nu_A(e_1)$, then we can get $x_1A=x_2A$, this implies that h is θ-intuitionistic fuzzy injective, so h is θ-intuitionistic fuzzy bijective. For $\forall x_1,x_2\in G_1$, $y\in G_2$, we have

$$\begin{aligned}\mu_h((x_1\mu_A)(x_2\mu_A),y)=\mu_h((x_1x_2)\mu_A,y)&=\mu_f(x_1x_2,y)\\&=\sup\{\mu_f(x_1,y_1)\wedge\mu_f(x_2,y_2)\mid y=y_1y_2\}\\&=\sup\{\mu_h(x_1\mu_A,y_1)\wedge\mu_h(x_2\mu_A,y_2)\mid y=y_1y_2\}\end{aligned}$$

and

$$\begin{aligned} v_h\left(\left(x_1 v_A\right)\left(x_2 v_A\right), y\right) &= v_h\left(\left(x_1 x_2\right) v_A, y\right) = v_f\left(x_1 x_2, y\right) \\ &= \inf\left\{v_f\left(x_1, y_1\right) \vee v_f\left(x_2, y_2\right) \mid y = y_1 y_2\right\} \\ &= \inf\left\{v_h\left(x_1 v_A, y_1\right) \vee v_h\left(x_2 v_A, y_2\right) \mid y = y_1 y_2\right\} \end{aligned}$$

Hence $h: G_1 / A \to G_2$ is a θ-intuitionistic fuzzy isomorphism, G_1 / A and G_2 are θ-intuitionistic fuzzy isomorphism.

5. Summary

In this paper, we first define θ-intuitionistic fuzzy mapping, then give the definition of θ-intuitionistic fuzzy homomorphism of groups, and further study the related properties of intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups under the θ-intuitionistic fuzzy homomorphism of groups. Finally, the fundamental theorem of θ-intuitionistic fuzzy homomorphism is obtained. Since different intuitionistic fuzzy mappings also produce different intuitionistic fuzzy homomorphisms, we can try to define other intuitionistic fuzzy mappings to study the intuitionistic fuzzy homomorphisms of groups.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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